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On simple normal crossing Fano varieties and logarithmic Fano varieties with large index

Kento Fujita (RIMS, Kyoto University)

Definition

$(X, D;\ L)$ logarithmic Fano n-fold w/ index r

$\overset{\text{def}}{\Leftrightarrow}$
- X smooth proj. n-fold,
- $D \subset X$ snc div. on X,
- L ample invertible sheaf on X

s.t. $-(K_X + D) \sim rL$.

$(\mathcal{X}; \mathcal{L})$ snc Fano n-fold w/ index r

$\overset{\text{def}}{\Leftrightarrow}$
- $\mathcal{X}$ conn. n-dim'l proj. scheme/$\mathbb{C}$,
- $\widehat{\mathcal{O}}_{\mathcal{X},x} \simeq \mathbb{C}[[x_1, \ldots, x_{n+1}]]/(x_1 \cdots x_k)$ for $\forall x \in \mathcal{X}$,
- $\forall$irr. cpnt. are smooth,
- $\mathcal{L}$ ample invertible sheaf on $\mathcal{X}$

s.t. $\omega_{\mathcal{X}}^{\vee} \simeq \mathcal{L}^{\otimes r}$.

Remark

- For $(\mathcal{X}; \mathcal{L})$ snc Fano n-fold w/ index r, all irr. cpnt. w/ conductor $(X, D; \mathcal{L}|_X) \subset \mathcal{X}$ are log. Fano n-folds w/ index r.
- log. Fano w/ $r = 1, n \leq 3$ has been considered by Maeda Hironobu.

----Many applications are expected:

Application(Kollár)

We can construct many `good' terminal singularities:

$(\mathcal{X}; \mathcal{L})$ snc Fano 3-fold with index 2. Then $\exists (0 \in Z)$ a germ of 4-dim'l isol. terminal sing. w/ a partial resol $(\mathcal{X} \subset W) \to (0 \in Z)$;
- W canonical sing.
- $\mathcal{X} \subset W$ Cartier div. and $\mathcal{N}_{\mathcal{X}/W} \simeq \mathcal{L}^{\vee}$
- K_Z Cartier and $Z \setminus \{0\}$ simply connected.
- the embedded dim. of $(0 \in Z)$ is $h^0(\mathcal{X}, \mathcal{L})$.

Remark

Given $(X_1, D_1; L_1), \cdots, (X_m, D_m; L_m)$ log. Fano n-folds w/ index r and certain '*gluing conditions*', we obtain $(\mathcal{X}; \mathcal{L})$ snc Fano n-fold w/ index r s.t. $\mathcal{X} = \bigcup_{1 \leq i \leq m} (X_i, D_i)$ and $L_i \simeq \mathcal{L}|_{X_i}$.

Motivation

Classify log. Fano mfds with large index.

Results

Consider $(X, D; L)$ w/ $D \neq 0$ and $r \geq 2$.

Proposition If $n < 2r$ and $\rho(X) \geq 2$, then $n = 2r - 1$ and $X \simeq \mathbb{P}_{\mathbb{P}^{r-1}}(\mathcal{O}^{\oplus r} \oplus \mathcal{O}(m)), D \simeq \mathbb{P}_{\mathbb{P}^{r-1}}(\mathcal{O}^{\oplus r})$ w/ $m \geq 0$.

Theorem If $n = 2r$ and $\rho(X) \geq 2$, then (X, D) is...

X	D
$\mathrm{Bl}_{\mathbb{P}^{r-2}} \mathbb{P}^{2r}$	$\mathrm{Bl}_{\mathbb{P}^{r-2}} \mathbb{P}^{2r-1}$ with $\mathbb{P}^{r-2} \subset \mathbb{P}^{2r-1} \subset \mathbb{P}^{2r}$ linear
$\mathbb{P}^{r-1} \times \mathbb{P}^{r+1}$	$\mathbb{P}^{r-1} \times \mathbb{Q}^r$
	$\mathbb{P}^{r-1} \times \mathbb{P}^r \cup \mathbb{P}^{r-1} \times \mathbb{P}^r$
$\mathbb{P}_{\mathbb{P}^{r-1}}(\mathcal{O}^{\oplus r} \oplus \mathcal{O}(m_1) \oplus \mathcal{O}(m_2))$ with $0 \leq m_1 \leq m_2, 1 \leq m_2$	$\mathbb{P}_{\mathbb{P}^{r-1}}(\mathcal{O}^{\oplus r} \oplus \mathcal{O}(m_1)) \cup \mathbb{P}_{\mathbb{P}^{r-1}}(\mathcal{O}^{\oplus r} \oplus \mathcal{O}(m_2))$
$R_B \mathbb{P}_{\mathbb{P}^{r-1}}(\mathcal{O}^{\oplus r+1} \oplus \mathcal{O}(m))$ with $m \geq 0$, $B \in \lvert\mathcal{O}_{\mathbb{P}}(2)\rvert$ smooth	$R_{B \cap \mathbb{P}_{\mathbb{P}^{r-1}}(\mathcal{O}^{\oplus r+1})} \mathbb{P}_{\mathbb{P}^{r-1}}(\mathcal{O}^{\oplus r+1})$ $(\simeq \mathbb{P}^{r-1} \times \mathbb{Q}^r)$ smooth
$(r \geq 3)$ $\mathbb{P}_{\mathbb{Q}^r}(\mathcal{O}^{\oplus r} \oplus \mathcal{O}(m))$ with $m \geq 0$	$\mathbb{P}_{\mathbb{Q}^r}(\mathcal{O}^{\oplus r}) (\simeq \mathbb{P}^{r-1} \times \mathbb{Q}^r)$
$(r = 2)$ $\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(m_1, m_2))$ with $0 \leq m_1 \leq m_2$	$\mathbb{P}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}^{\oplus 2}) (\simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1)$
$\mathbb{P}_{\mathbb{P}^r}(T_{\mathbb{P}^r} \oplus \mathcal{O}(m))$ with $m \geq 1$	$\mathbb{P}_{\mathbb{P}^r}(T_{\mathbb{P}^r})$
$\mathbb{P}^r \times \mathbb{P}^r$	$D \in \lvert\mathcal{O}(1,1)\rvert$ smooth divisor $(\simeq \mathbb{P}_{\mathbb{P}^r}(T_{\mathbb{P}^r}))$
	$\mathbb{P}^{r-1} \times \mathbb{P}^r \cup \mathbb{P}^r \times \mathbb{P}^{r-1}$
$\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}^{\oplus r} \oplus \mathcal{O}(1))$	$\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}^{\oplus r-1} \oplus \mathcal{O}(1)) (\simeq \mathrm{Bl}_{\mathbb{P}^{r-2}} \mathbb{P}^{2r-1})$
	$\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}^{\oplus r}) \cup \mathbb{P}_{\mathbb{P}^{r-1}}(\mathcal{O}^{\oplus r} \oplus \mathcal{O}(1))$
$\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}^{\oplus r-1} \oplus \mathcal{O}(1) \oplus \mathcal{O}(m))$ with $m \geq 1$	$\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}^{\oplus r-1} \oplus \mathcal{O}(1)) (\simeq \mathrm{Bl}_{\mathbb{P}^{r-2}} \mathbb{P}^{2r-1})$
$\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}^{\oplus r} \oplus \mathcal{O}(m))$ with $m \geq 2$	$\mathbb{P}_{\mathbb{P}^r}(\mathcal{O}^{\oplus r}) \cup \mathbb{P}_{\mathbb{P}^{r-1}}(\mathcal{O}^{\oplus r} \oplus \mathcal{O}(m))$

Corollary We have classified log. Fano n-fold w/ index $r \geq n - 2$.

Outline of the proof $\exists R \subset NE(X)$ extremal ray; $(D \cdot R) > 0$.

This ray must be K_X-negative and long ray → See cont_R in detail.