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$\mathscr{A}$-SCHEMES AND THEIR APPLICATIONS

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1 Motivation (I)

Let (**Top**) (resp. (**CHaus**)) be the category of topological spaces (resp. compact Hausdorff spaces).

Stone-Čech compactification

$$\beta : (\mathbf{Top}) \rightleftarrows (\mathbf{CHaus}) : U$$

The compact Hausdorff space βX is called the Stone-Čech compactification of the given space X.

To construct the left adjoint, we need the following lemma:

Tychonoff's Theorem

(**CHaus**) is small complete.

We may hope that similar things hold for schemes: schemes are extended concepts of commutative rings. However,

The category (**Ring**) of rings is (small) complete and co-complete.

⇕(compare)

The category (Sch) of schemes is NOT complete NOR co-complete.

This becomes an obstruction, when we try to construct a universal compactification of schemes using the above method.

Why is the category of schemes not complete?

2 Motivation (II)

There are several attempts to construct 'scheme-like topological objects' from other algebraic type, such as:

(1) Monoids ("schemes over $\mathbb{F}_1$")

(2) Semirings (Tropical geometry).

What are the common aspects between them?

3 Results

Properties of $\mathscr{A}$-schemes (I)

Let (**CohSch**) (resp. (**CohLRS**), ($\mathscr{A}$-**Sch**)) be the category of coherent schemes (resp. coherent locally ringed spaces, $\mathscr{A}$-schemes).

(1) Then, we have an adjunction

$$\mathrm{Spec} : (\mathbf{Ring}) \rightleftarrows (\mathscr{A}\text{-}\mathbf{Sch})^{\mathrm{op}} : \Gamma$$

(2) We have a fully faithful functors

$$(\mathbf{CohSch}) \xrightarrow{j} (\mathscr{A}\text{-}\mathbf{Sch}) \to (\mathbf{CohLRS}),$$

and j preserves fibre products and finite quasi-compact open patchings.

Properties of $\mathscr{A}$-schemes (II)

(1) The category ($\mathscr{A}$-**Sch**) is **small complete and co-complete.**

(2) We have the notions of **separatedness and properness** (we exclude the 'of finite type' condition). Valuative criteria exist.

(3) Fix a base $\mathscr{A}$-scheme S, and (**p.**$\mathscr{A}$-**Sch**/S) be the category of proper $\mathscr{A}$-schemes over S. Then, we have the adjunction

$$\mathrm{ZR}_S : (\mathscr{A}\text{-}\mathbf{Sch}/S) \rightleftarrows (\mathbf{p.}\mathscr{A}\text{-}\mathbf{Sch}/S) : \mathrm{U}$$

The $\mathscr{A}$-scheme $\mathrm{ZR}_S(X)$ is the **Zariski-Riemann space** associated to X.

As a corollary, we obtain the well known application for usual schemes:

Cor (Nagata embedding)

Let $X \to S$ be a separated, of finite type morphism of coherent schemes. Then, there exists an open immersion $X \to \overline{X}$ of S-schemes, where $\overline{X}$ is proper, of finite type over S.

Unfortunately, we don't have enough space to show here the precise definition of $\mathscr{A}$-schemes. The key point is to

Abandon the feature of schemes, that it is locally a spectrum of a ring.

References

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[2] Takagi, S.: *Nagata embeddings and $\mathscr{A}$-schemes*, preprint, arXiv: mathAG/1107.3414