Mirror Symmetry for Weighted Projective Lines

代数幾何学シンポジウム記録
2011年度　pp.153-153

(Joint work with Yoshihisa ISHIBASHI and Atsushi TAKAHASHI)

Osaka University D1 Yuuki SHIRAISHI

0. Introduction

There are three constructions of Frobenius manifolds (Saito's flat structures) from completely different origins and purposes; the Gromov-Witten theory, the theory of primitive forms and the invariant theory of Weyl groups.

Our main purpose is to give isomorphisms among these Frobenius manifolds by studying the Gromov-Witten theory for weighted projective lines, the theory of primitive forms for cusp singularities and the invariant theory of extended affine Weyl groups. In particular, we simplify and generalize the result given by Milanov–Tseng [3] and Rossi [4].

1. Three Constructions

Let (a_1, a_2, a_3) be a triple of positive integers such that $\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} > 1$.

A. Gromov-Witten Theory

Object

Counting numbers of stable maps from orbifold curves to $\mathcal{X} := \mathbb{P}^1_{a_1,a_2,a_3}$, where $\mathbb{P}^1_{a_1,a_2,a_3}$ is the orbifold $\mathbb{P}^1$ with 3 isotropic points of orders a_1, a_2, a_3.

Genus g potential

$$\mathcal{F}_g^{\mathcal{X}} := \sum_{n,\beta} \frac{1}{n!} \langle \mathbf{t}, \dots, \mathbf{t} \rangle_{g,n,\beta}^{\mathcal{X}} q^{\beta}, \quad \mathbf{t} = \sum_i t_i \gamma_i$$

$$\langle \gamma_1, \dots, \gamma_n \rangle_{g,n,\beta}^{\mathcal{X}} := \int_{[\overline{\mathcal{M}}_{g,n,\beta}(\mathcal{X})]^{vir}} ev_1^* \gamma_1 \wedge \dots ev_n^* \gamma_n, \quad \gamma_1, \dots, \gamma_n \in H^*_{orb}(\mathcal{X}, \mathbb{Q}),$$

$$ev_i^* : H^*_{orb}(\mathcal{X}, \mathbb{Q}) \longrightarrow H^*(\overline{\mathcal{M}}_{g,n,\beta}(\mathcal{X}, \mathbb{Q}).$$

Genus 0 potential $\mathcal{F}_0^{\mathcal{X}}$ satisfies WDVV equations.
$\Longrightarrow$ **Frobenius manifold of rank $a_1 + a_2 + a_3 - 1$ and of dimension 1.**

B. Theory of Primitive Form

Object

Universal unfolding $F(\underline{x}, \underline{t}) = f(\underline{x}, e^t) + \sum_{i=1}^{\mu-1} s_i \phi_i(\underline{x})$ of the polynomial $f(\underline{x}, e^{-t}) = x_1^{a_1} + x_2^{a_2} + x_3^{a_3} - e^{-t} x_1 x_2 x_3$, where $\{[\phi_i(\underline{x})]\}$ form $\mathbb{C}[e^{\pm t}]$-linear basis of Jaccobian ring $\mathbb{C}[e^{\pm t}][x_1, x_2, x_3]/(\partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3)$ of f.

Saito structure

We can construct (by the use of Sabbah's formalism [1])

- Relative de Rham cohomology groups $\mathcal{H}_F$,
- Gauss–Manin connection ∇ on $\mathcal{H}_F$,
- Higher residue pairings K_F on $\mathcal{H}_F$.

Also, we can show that $\zeta = [e^{-t} dx_1 \wedge dx_2 \wedge x_3]$ **is a primitive form.**
$\Longrightarrow$ **Frobenius manifold of rank $a_1 + a_2 + a_3 - 1$ and of dimension 1.**

Remark. The choice of primitive form is **not unique.**

C. Weyl Group invariant theory

Object

Extended affine Weyl group $\widetilde{W}$ of W acting on $\widetilde{\mathfrak{h}} := \mathfrak{h} \oplus \mathbb{C}$, where $W = W_{a_1,a_2,a_3}$ and $\mathfrak{h}$ are the Weyl group and the Cartan subalgebra associated to the Coxeter–Dynkin diagram T_{a_1,a_2,a_3}.

Geometry of Orbit space

Dubrovin–Zhang [2] construct a Frobenius structure on the orbit space $\widetilde{\mathfrak{h}}/\widetilde{W} := \mathrm{Spec}(\mathcal{R})$ where $\mathcal{R}$ is the ring of $\widetilde{W}$-invariant Fourier polynomials.

- **Chevalley's theorem** tells us that $\widetilde{\mathfrak{h}}/\widetilde{W} \simeq \mathbb{C}^{a_1+a_2+a_3-2} \times \mathbb{C}^*$.
- There exists a "nice" subring $\mathcal{A}$ of $\mathcal{R}$ isomorphic to a polynomial ring which defines a partial compactification $\widetilde{\mathfrak{h}}/\widetilde{W} \hookrightarrow \mathbb{C}^{a_1+a_2+a_3-2} \times \mathbb{C}$.
- Cartan matrix $(+\alpha)$ defines a pairing on the cotangent space of $\widetilde{\mathfrak{h}}/\widetilde{W}$.

Also, Dubrovin–Zhang show the existence of a special point in $\widetilde{\mathfrak{h}}$.
$\Longrightarrow$ **Frobenius manifold of rank $a_1 + a_2 + a_3 - 1$ and of dimension 1.**

2. Statements

The following theorem on the uniqusness is motivated by the one in [7].

Theorem 1. There exists a unique Frobenius manifold M with flat coordinates $t_0, t_{1,1}, \dots, t_{1,a_1-1}, t_{2,1}, \dots, t_{2,a_2-1}, t_{3,1}, \dots, t_{3,a_3-1}, t$ satisfying

(i) There exists an isomorshism of complex manifolds:

$$M \simeq \mathbb{C} \times \mathbb{C}^{(a_1-1)+(a_2-1)+(a_3-1)} \times \mathbb{C}^*$$
$$s \mapsto \left(t_0(s), t_{1,1}(s), \dots, t_{3,a_3-1}(s), e^{t(s)}\right).$$

(ii) The unit vector field e and the Euler vector field E are given by

$$e = \frac{\partial}{\partial t_0}, \ E = t_0 \frac{\partial}{\partial t_0} + \sum_{i=1}^{3} \sum_{j=1}^{a_i-1} \frac{a_i - j}{a_i} t_{i,j} \frac{\partial}{\partial t_{i,j}} + \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} - 1\right) \frac{\partial}{\partial t}.$$

(iii) Non-degenerate symmetric bilinear form η satisfies

$$\eta\left(\frac{\partial}{\partial t_0}, \frac{\partial}{\partial t}\right) = \eta\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t_0}\right) = 1,$$
$$\eta\left(\frac{\partial}{\partial t_{i_1,j_1}}, \frac{\partial}{\partial t_{i_2,j_2}}\right) = \begin{cases} 1/a_{i_1}, & \text{if } i_1 = i_2 \text{ and } j_2 = a_{i_1} - j_1. \\ 0, & \text{otherwise.} \end{cases}$$

(iv) The Frobenius potential $\mathcal{F}_0$ satisfies $E\mathcal{F}_0 = 2\mathcal{F}_0$,

$$\mathcal{F}_0|_{t_0=0} \in \mathbb{C}\left[[t_{1,1}, \dots, t_{1,a_1-1}, t_{2,1}, \dots, t_{2,a_2-1}, t_{3,1}, \dots, t_{3,a_3-1}, e^t]\right]$$

(v) Assume the condition (v). In the frame $\frac{\partial}{\partial t_0}, \frac{\partial}{\partial t_{1,1}}, \dots, \frac{\partial}{\partial t_{3,a_3-1}}, \frac{\partial}{\partial t}$ of TM, the product $\circ$ can be extended to the limit $t_0 = t_{1,1} = \dots = t_{3,a_3-1} = e^t = 0$. The $\mathbb{C}$-algebra obtained in this limit is isomorphic to

$$\mathbb{C}[x_1, x_2, x_3] \big/ \left(x_1x_2, x_2x_3, x_3x_1, a_1x_1^{a_1} - a_2x_2^{a_2}, a_2x_2^{a_2} - a_3x_3^{a_3}, a_3x_3^{a_3} - a_1x_1^{a_1}\right),$$

where $\partial/\partial t_{1,1}, \partial/\partial t_{2,1}, \partial/\partial t_{3,1}$ are mapped to x_1, x_2, x_3, respectively.

(vi) The term $t_{1,1}t_{2,1}t_{3,1}e^t$ occurs with the coeffcient 1 in $\mathcal{F}_0$.

We have another theorem on the uniqueness (see also [2] and [6]):

Theorem 2. A Frobenius manifold of rank $a_1 + a_2 + a_3 - 1$ and of dimension 1 with the following e and E is uniquely determined by the intersection form:

$$e = \frac{\partial}{\partial t_0}, \ E = t_0 \frac{\partial}{\partial t_0} + \sum_{i=1}^{3} \sum_{j=1}^{a_i-1} \frac{a_i - j}{a_i} t_{i,j} \frac{\partial}{\partial t_{i,j}} + \left(\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} - 1\right) \frac{\partial}{\partial t}.$$

3. Isomorphisms

A $\Longleftrightarrow$ B : Use Theorem 1.

- Calculate the orbifold cohomology ring of $\mathcal{X}$ and count the number of certain stable maps of degree one.
- Calculate the "limit" of Jacobian ring in B.

B $\Longleftrightarrow$ C : Use Theorem 2.

- Culculate the **intersection form** in (B) by the period mapping of the primitive form (see [5]) and by a classical geometric approach.

Reference

[1] A. Douai and C. Sabbah, *Gauss-Manin systems, Brieskorn lattices and Frobenius structures.I*, Ann. Inst. Fourier (Grenoble) 53 (2003), no. 4, 1055–1116.

[2] B. Dubrovin and Y. Zhang, *Extended affine Weyl groups and Frobenius manifolds*, Compositio Mathematica, (1998) 167–219.

[3] T. E. Milanov, H.-H. Tseng, *The space of Laurent polynomials, P1-orbifolds, and integrable hierarchies*, arXiv: 0607012v3 .

[4] P. Rossi, *Gromov-Witten theory of orbicurves, the space of tri-polynomials and Symplectic Field Theory of Seifert fibrations*, arXiv:0808.2626.

[5] K. Saito, *Period mapping associated to a primitive form*, Publ. RIMS, Kyoto Univ. **19** (1983) 1231–1264.

[6] I. Satake, *Frobenius manifolds for elliptic root systems*, Osaka J. Math. **47** (2010) 301–330.

[7] I. Satake and A. Takahashi, *Gromov–Witten invariants for mirror orbifolds of simple elliptic singularities*, arXiv:1103.0951.