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Algebro-geometric characterization of Cayley polytopes

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1 Cayley polytopes

Definition 1.1. $P_0, \ldots, P_r \subset \mathbb{R}^s$: lattice polytopes, $e_1, \ldots, e_r$: the standard basis of $\mathbb{Z}^r$.

The Cayley sum $P_0 * \cdots * P_r \subset \mathbb{R}^s \times \mathbb{R}^r$ is

$$P_0 * \cdots * P_r := \mathrm{conv}\{(P_0 \times 0) \cup (P_1 \times e_1) \cup \ldots \cup (P_r \times e_r)\}$$

where $\mathrm{conv}\{\cdot\}$ means the convex hull of $\cdot$.

For a lattice polytope $P \subset \mathbb{R}^n$,

P is a Cayley polytope of length $r+1$

$\overset{\mathrm{def}}{\Longleftrightarrow}$ P is identified with $P_0 * \cdots * P_r$ by an affine isomorphism $\mathbb{Z}^n \cong \mathbb{Z}^{n-r} \times \mathbb{Z}^r$ for some $P_0, \ldots, P_r \subset \mathbb{R}^{n-r}$.

In other words, P is a Cayley polytope of length $r+1$ if and only if $\pi(P)$ is a unimodular r-simplex for some lattice projection $\pi : \mathbb{R}^n \to \mathbb{R}^r$ (i.e. the linear map induced by a surjective group homomorpfism $\mathbb{Z}^n \to \mathbb{Z}^r$).

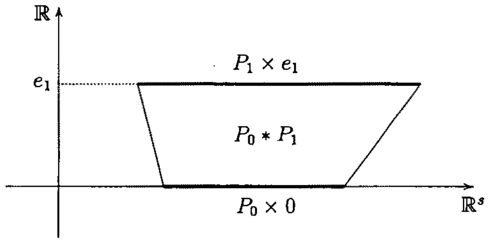

Figure 1. $r = 1$ case

2 Main theorem

Definition 2.1. For a polarized variety (X, L),

(X, L) is covered by r-planes

$\overset{\mathrm{def}}{\Longleftrightarrow}$ $\forall p \in X, \exists Z \subset X$ s.t. $p \in Z$ and $(Z, L|_Z) \cong (\mathbb{P}^r, \mathcal{O}(1))$.

When $r = 1$, we say that (X, L) is covered by lines.

For any lattice polytope $P \subset \mathbb{R}^n$ of dimension n, we can define the polarized toric variety (X_P, L_P) as

$$(X_P, L_P) := (\mathrm{Proj}\,\mathbb{C}[\Gamma_P], \mathcal{O}(1)),$$

where $\Gamma_P := \bigoplus_{k \in \mathbb{N}} \{k\} \times (kP \cap \mathbb{Z}^n)$ is a semigroup in $\mathbb{N} \times \mathbb{Z}^n$.

We give a characterization of (X_P, L_P) for a Cayley polytope P of length $r+1$. The main theorem is following;

Theorem 2.2. $P \subset \mathbb{R}^n$ *: a lattice polytope,* $\dim P = n$. *Then, P is a Cayley polytope of length $r+1 \Leftrightarrow (X_P, L_P)$ is covered by r-planes.*

3 Cayley polytopes of length 2 and Seshadri constants

Definition 3.1. (X, L) : a polarized variety, $p \in X$.
The Seshadri constant $\varepsilon(X, L; p)$ of L at p is

$$\begin{aligned}\varepsilon(X, L; p) &:= \inf\left\{\frac{C.L}{\mathrm{mult}_p(C)} \,\middle|\, C \subset X; \text{a curve containing } p\right\} \\ &= \max\{t > 0 \,|\, \mu^*L - tE \text{ is nef}\},\end{aligned}$$

where μ is the blowing up of X at p and E is the exceptional divisor.

The Seshadri constant $\varepsilon(X, L; p)$ is an invariant measuring the local positivity of the line bundle L at p.

Example 3.2. (1) $\varepsilon(\mathbb{P}^n, \mathcal{O}(k); p) = k$ for any $p \in \mathbb{P}^n$.
(2) $\varepsilon(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(a, b); p) = \min\{a, b\}$ for $p \in \mathbb{P}^1 \times \mathbb{P}^1$ and $a, b \in \mathbb{N}$.
(3) For a smooth cubic surface $S \subset \mathbb{P}^3$,

$$\varepsilon(S, \mathcal{O}_S(1); p) = \begin{cases} 1 & \text{if } p \in l \text{ for a line } l \subset S \\ 3/2 & \text{otherwise.} \end{cases}$$

We can also characterize Cayley polytopes of length 2 by using Seshadri constants;

Theorem 3.3. $P \subset \mathbb{R}^n$ *: a lattice polytope,* $\dim P = n$. *Then the following statements are equivalent;*

i) P is a Cayley polytope of length 2,

ii) (X_P, L_P) is covered by lines,

iii) $\varepsilon(X_P, L_P; p) = 1$ for any $p \in X_P$,

iv) $\varepsilon(X_P, L_P; 1_P) = 1$ for the identity of the torus $1_P \in (\mathbb{C}^\times)^n \subset X_P$.

Remark 3.4. In fact, ii) $\Leftrightarrow$ iii) $\Leftrightarrow$ iv)' $\varepsilon(X, L; p) = 1$ for a very general point $p \in X$, always hold if (X, L) is a polarized variety such that $|L|$ is base point free. i) $\Leftrightarrow$ ii) is nothing but Theorem 2.2 for $r = 1$.

In general, it is very difficult to compute Seshadri constants, even in toric cases. Theorem 3.3 gives an explicit description for which lattice polytope P the Seshadri constant $\varepsilon(X_P, L_P; 1_P)$ is one.

4 Dual defects

Cayley polytopes are related to dual defects;

Definition 4.1. $X \subset \mathbb{P}^N$: a projective variety. The dual variety $X^* \subset (\mathbb{P}^N)^\vee$ of X is

$$X^* := \overline{\{H \in (\mathbb{P}^N)^\vee \mid \exists p \in X_{reg},\ T_{X,p} \subset H\}},$$

where $(\mathbb{P}^N)^\vee := \{H : \text{a hyperplane in } \mathbb{P}^N\}$. The dual defect $\mathrm{def}(X)$ of X is $\mathrm{def}(X) := N - 1 - \dim X^*$.

Remark 4.2. By dimension counts, we find that the expected dimension of $\dim X^*$ is $N - 1$. It is known that $X \subset \mathbb{P}^N$ is covered by r-planes if $\mathrm{def}(X) = r$.

The following is an easy corollary of Theorem 2.2;

Corollary 4.3. $P \subset \mathbb{R}^n$ *: an n-dimensional lattice polytope s.t. $P \cap \mathbb{Z}^n$ spans the lattice $\mathbb{Z}^n$, i.e., the morphism $\phi_P : X_P \to \mathbb{P}^{\dim |L_P|}$ defined by $|L_P|$ is birational onto the image.*
Then, $\mathrm{def}(\phi_P(X_P)) = r \Rightarrow$ *P is a Cayley polytope of length $r+1$.*

Remark 4.4. (1) In Corollary 4.3, the assumption that $P \cap \mathbb{Z}^n$ spans $\mathbb{Z}^n$ is necessary. For example, let $P \subset \mathbb{R}^3$ be the convex hull of $(0,0,0), (1,1,0), (1,0,1)$ and $(0,1,1)$. Then the image $\phi_P(X_P) \subset \mathbb{P}^3$ is $\mathbb{P}^3$, so $\mathrm{def}(\phi_P(X_P)) = 3$. But P is not a Cayley polytope of length 4.
(2) The converse of Corollary 4.3 does not hold. For example, let $P \subset \mathbb{R}^2$ be the convex hull of $(0,0), (1,0), (0,1)$, and $(1,1)$. Then P is a Cayley polytope of length 2. But $\phi_P(X_P) = X_P \subset \mathbb{P}^3$ is a smooth quadric surface, whose defect is 0.