代数幾何学シンポジウム記録
2011年度 pp.131-141

Unitarity of the KZ/Hitchin connection in genus 0

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ABSTRACT. Vector bundles of conformal blocks, on suitable moduli spaces of genus zero curves with marked points, for arbitrary simple Lie algebras and arbitrary integral levels, are shown in [**2**] to carry unitary metrics of geometric origin which are preserved by the Knizhnik-Zamolodchikov/Hitchin connection (as conjectured by Gawedzki et al. in [**4**]). The proof builds upon the work of Ramadas [**14**] who proved this unitarity statement in the case of the Lie algebra $\mathfrak{sl}_2$ (and genus 0). This is an expository paper devoted to the context and resolution (in genus 0) of such unitarity statements.

1. Introduction

Consider a finite dimensional simple Lie algebra $\mathfrak{g}$, a non-negative integer k called the level and a N-tuple $\vec{\lambda} = (\lambda_1, \dots, \lambda_N)$ of dominant weights of $\mathfrak{g}$ of level k. The mathematical theory of Tsuchiya-Kanie [**17**] and Tsuchiya-Ueno-Yamada [**18**], associates to this data a vector bundle $\mathcal{V} = \mathcal{V}_{\vec{\lambda},k}$ on $\overline{\mathfrak{M}}_{g,N}$, the moduli stack of stable N-pointed curves of genus g.

On the open part $\mathfrak{M}_{g,N}$ of smooth pointed curves, $\mathcal{V}$ carries a flat projective connection ∇, which is the restriction of a suitable Knizhnik-Zamolodchikov (KZ) connection when $g = 0$. The WZW connection [**18**] generalizes the KZ connection to all genera.

The fibers of $\mathcal{V}$ on $\mathfrak{M}_{g,N}$ can also be described in terms of sections of natural line bundles on suitable moduli stacks of parabolic principal bundles (for the simply connected group G corresponding to $\mathfrak{g}$) on N-pointed curves of genus g. These sections generalize classical theta

Partially supported by NSF grant DMS-0901249.

functions, and are hence called non-abelian or generalized theta functions (see the survey [**16**]). The connection on $\mathcal{V}$ was described from the above algebro-geometric point of view by Hitchin [**8**].

One of the basic questions, especially from an algebraic geometry view point, is whether the Hitchin/KZ connection can be realized in the cohomology of smooth projective varieties.

QUESTION 1.1.
Is there a family of smooth projective algebraic varieties $f : \Lambda \to \mathfrak{M}_{g,N}$ *and an inclusion* $i_x : \mathcal{V}_x \hookrightarrow H^M(\Lambda_x, \mathbb{C})$ *for* $x \in \mathfrak{M}_{g,N}$ *which is consistent with connections (Hitchin/KZ on one side and Gauss-Manin on the other,* $\Lambda_x = f^{-1}(x)$*)?*

(Since the Hitchin connection is only a projective connection, one may have to work over a cover of $\mathfrak{M}_{g,N}$, these subtleties will be ignored in the introduction.)

One could ask for more: (a) a characterization of the image of the inclusion, (b) same question with relaxed conditions on f (to allow families of open varieties). Perhaps one may hope that the varieties Λ_x turn out to be interesting in their own right, and that counting points of these varieties over finite fields, or their periods, give us interesting functions. More importantly for us, if the image of i_x lands in $H^{M,0}(\Lambda_x, \mathbb{C})$ with M relative dimension of f then the Hodge metric on $H^{M,0}$ will restrict to a unitary metric on $\mathcal{V}_x$ left invariant by the KZ/Hitchin connection (we will call this the Ramadas-Gawedzki method for a unitary metric).

Question 1.1 is related to a basic conjecture in the subject, with origins in physics, which asserts that $\mathcal{V}$ carries a projective unitary metric preserved (projectively) by the connection ∇. It was pointed out to us by A. Kirillov that the combined work of Kirillov and Wenzl [**10, 11, 21**] (Theorem 10.9 in [**11**] and Theorem 3.7 in [**21**]) implies this conjecture for all genera including genus 0. There is also a recent preprint of Andersen and Ueno [**1**], where this unitarity conjecture is proved for the special linear groups $G = \mathrm{SL}(n)$ for all genera. The unitary metrics obtained via these approaches are not (a priori) of Hodge-theoretic origin, and do not answer Question 1.1.

In the 90's Gawedzki and collaborators [**7, 4**] proposed a conjectural, explicit construction of the unitary metric via integration of the Schechtman-Varchenko forms [**15**]. Recently the case $\mathfrak{g} = \mathfrak{sl}_2$ and genus 0 of Gawedzki's proposal was rigorously proved by Ramadas [**14**], leading to a positive answer to Question 1.1 in this case. In fact Ramadas' map in cohomology lands in $H^{M,0}$, and hence a unitary metric can be obtained by restricting the Hodge metric.

Following Ramadas' general strategy, we prove the (geometric) unitarity conjecture, as proposed by Gawedzki et al. [4] for arbitrary simple Lie algebras $\mathfrak{g}$ in genus 0. As in Ramadas' work, the unitary metric is obtained by realizing the bundle of conformal blocks inside a Gauss-Manin system of cohomology of smooth projective varieties (thus answering Question 1.1 in the affirmative in genus 0). The map to cohomology again lands in $H^{M,0}$:

THEOREM 1.2. *The KZ/Hitchin connection on bundles of conformal blocks over configuration spaces of distinct points on $\mathbb{A}^1$ is unitary, with the unitary metric of geometric origin, for any simple Lie algebra $\mathfrak{g}$ and any integral level k.*

1.1. Acknowledgements. I would like to thank the organizers of the Kinosaki symposium for inviting me to this conference, and Professor Takeshi Abe for his kind hospitality.

2. Some basics

There are two approaches to non-abelian theta functions: via representation theory of Kac-Moody lie algebras , and as global sections of line bundles over moduli spaces (moduli-theoretic). These approaches are equivalent (see the survey [**16**]).

Recall that finite dimensional irreducible representations of $\mathfrak{g}$ are parameterized by the set of dominant integral weights P_+ considered a subset of $\mathfrak{h}^*$. To $\lambda \in P_+$, the corresponding irreducible representation V_λ contains a non-zero vector $v \in V_\lambda$ (the highest weight vector) such that (where Δ_+ is the set of positive roots):

$$Hv = \lambda(H)v, H \in \mathfrak{h}$$

$$X_\alpha v = 0, X_\alpha \in \mathfrak{g}_\alpha, \forall \alpha \in \Delta_+.$$

We will fix a level k in the sequel. Let P_k denote the set of dominant integral weights of level k. More precisely

$$P_k = \{\lambda \in P_+ \mid (\lambda, \theta) \leq k\}$$

where θ is the highest (longest positive) root, and (,) is a normalized Killing form ($(\theta, \theta) = 2$).

The affine Lie algebra $\hat{\mathfrak{g}}$ is defined to be

$$\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((\xi)) \oplus \mathbb{C}c$$

where c is an element in the center of $\hat{\mathfrak{g}}$ and the Lie algebra structure is defined by

$$[X \otimes f(\xi), Y \otimes g(\xi)] = [X, Y] \otimes f(\xi)g(\xi) + c(X, Y) \operatorname{Res}_{\xi=0}(gdf)$$

where $f, g \in \mathbb{C}((\xi))$ and $X, Y \in \mathfrak{g}$.

For each $\lambda \in P_k$ there is a unique "integrable" irreducible representation $\mathcal{H}_\lambda$ of $\hat{\mathfrak{g}}$ which contains V_λ and such that the central element c of $\hat{\mathfrak{g}}$ acts on it by multiplication by k. The representation $\mathcal{H}_\lambda$ when $\lambda = 0$ (still at level k) is called the vacuum representation at level k.

2.1. Conformal blocks. Fix a stable N-pointed curve of genus g with formal neighborhoods $\mathfrak{X} = (C; P_1, \ldots, P_N, \eta_1, \ldots, \eta_N)$ where $\eta_i : \hat{O}_{C,P_i} \tilde{\to} \mathbb{C}[[\xi_i]]$, $i = 1, \ldots, N$ as isomorphisms.

Let $\mathfrak{X}$ and be as above, and choose $\vec{\lambda} = (\lambda_1, \ldots, \lambda_N) \in P_k^N$.

Set

$$\mathcal{H}_{\vec{\lambda}} = \mathcal{H}_{\lambda_1} \otimes \ldots \otimes \mathcal{H}_{\lambda_N}.$$

DEFINITION 2.1. *Define the space of conformal blocks*

$$V_{\vec{\lambda}}^{\dagger}(\mathfrak{X}) = \mathrm{Hom}_{\mathbb{C}}(\mathcal{H}_{\vec{\lambda}}/\mathfrak{g}(\mathfrak{X})\mathcal{H}_{\vec{\lambda}}, \mathbb{C})$$

where $\mathfrak{g}(\mathfrak{X}) = \mathfrak{g} \otimes_{\mathbb{C}} \Gamma(C - \{P_1, \ldots, P_N\}, \mathcal{O})$ which acts on $\mathcal{H}_{\vec{\lambda}}$ by power series expansion around each P_i (see [**2**] and the references there in).

REMARK 2.2. *In the case of $g = 0$, conformal blocks naturally embed in the vector space dual of $V_{\lambda_1} \otimes \ldots \otimes V_{\lambda_N}$. The KZ connection is actually defined on this larger space (the connection operator is $\nabla_{\frac{\partial}{\partial z_i}} = \frac{\partial}{\partial z_i} + \frac{1}{\kappa} \sum_{j \neq i} \frac{\Omega_{ij}}{z_i - z_j}$, where Ω_{ij} is the Casimir acting on the ij factors and $\kappa = k + g^*$). In general the Hitchin/KZ connection has two definitions: one as heat operators (Hitchin), and the second uses the representation theory of Virasoro algebras.*

Following Dirac's bra-ket conventions, elements of $V_{\vec{\lambda}}^{\dagger}(\mathfrak{X})$ (or $\mathcal{H}_{\vec{\lambda}}^*$) are frequently denoted by $\langle \Psi |$ and those of $V_{\vec{\lambda}}(\mathfrak{X})$ (or of $\mathcal{H}_{\vec{\lambda}}$) by $| \Phi \rangle$ and the pairing by $\langle \Psi | \Phi \rangle$.

2.2. Relations to non-abelian theta functions. Let us consider a (simply connected) semisimple lie group G with Lie algebra $\mathfrak{g}$, and no parabolic structures. The starting point is an observation (made rigorous by many authors, see the survey [**16**]) that since principal bundles on C minus a point p are trivial, the moduli stack $M_G(X)$ of principal G-bundles is a double quotient (where z is a formal coordinate at p)

$$G(\mathcal{O}(X - p)) \backslash G(\mathbb{C}((z)))/G(\mathbb{C}[[z]])$$

The moduli-stack has Picard group $= \mathbb{Z}$ if G is simple. Let $\mathcal{L}$ be the positive generator of the Picard group. We can go about trying to calculate the space of generalized theta functions $H^0(M_G(X)(r), \mathcal{L}^k)$

as follows: Let $\mathcal{Q} = G(\mathbb{C}((z)))/G(\mathbb{C}[[z]])$ and $\mathcal{L}'$ the pull back of $\mathcal{L}^k$ under the natural map from $\mathcal{Q}$ to the double quotient. The space $H^0(M_G(X)(r), \mathcal{L}^k)$ is the subspace of sections of $H^0(\mathcal{Q}, \mathcal{L}')$ invariant under the action of $G(\mathbb{C}[[z]])$. However the bundle $\mathcal{L}'$ is linearized not for the action of $G(\mathbb{C}((z)))$, but rather, for a central extension of it, and the space $H^0(\mathcal{Q}, \mathcal{L}')$ is the dual V_k^* of an irreducible representation of this central extension (V_k is the same as $\mathcal{H}_\lambda$ defined above for $\lambda = 0$). Passing to Lie algebras, one obtains that $H^0(M_G(X), \mathcal{L}^k)$ is isomorphic to the space of conformal blocks associated to the pointed curve (C, p, z) with the vacuum representation (at level k) attached at p. There is a generalization of this picture to the case of "insertions" (i.e. parabolic structures).

2.3. Propagation of vacua. Add a new point P_{N+1} together with the vacuum representation V_0 of level k, at P_{N+1}. Also fix a formal neighborhood at P_{N+1}. We therefore have a new pointed curve $\mathfrak{X}'$, and an extended $\vec{\lambda}' = (\lambda_1, \dots, \lambda_N, \lambda_{N+1} = 0)$. There is an isomorphism ("propagation of vacua")

$$V^\dagger_{\vec{\lambda}'}(\mathfrak{X}') \tilde{\to} V^\dagger_{\vec{\lambda}}(\mathfrak{X}),\ \langle\widehat{\Psi}| \mapsto \langle\Psi|.$$

2.4. Correlation functions. Suppose $\mathfrak{X} \in \mathfrak{M}_{g,N}$. Let $\langle\Psi| \in V^\dagger_{\vec{\lambda}}(\mathfrak{X})$, $Q_1, \dots, Q_M \in C - \{P_1, \dots, P_N\}$, $|\Phi\rangle \in \mathcal{H}_{\vec{\lambda}}$, $Q_1, \dots, Q_M \in C - \{P_1, \dots, P_N\}$, $Q_i \neq Q_j, i < j$ and corresponding elements $X_1, \dots, X_M \in \mathfrak{g}$. There is a very important differential called a correlation function

$$\Omega = \langle\Psi|X_1(Q_1)X_2(Q_2)\dots X_M(Q_M)|\Phi\rangle \in \bigotimes_{i=1}^{M} \Omega^1_{C,Q_i}.$$

Here Ω^1_C is the vector bundle of holomorphic one-forms on C. One way to define Ω is via propagation by vacua: add points $Q_1, \dots Q_M$ with formal coordinates $\psi_1, \dots, \psi_M$ and consider the elements $X_a(-1)|0\rangle$ in the vacuum representation at those points where $X_a(n) = X_a \otimes \xi^n$. Then

$$\Omega = \langle\widehat{\Psi}|X_1(-1)|0\rangle \otimes X_2(-1)|0\rangle \dots X_M(-1)|0\rangle \otimes |\Phi\rangle d\psi_1 \dots d\psi_M.$$

The differential form Ω is independent of the chosen coordinates and can be thought of a suitable multiderivative of a theta function.

3. In genus 0

We will henceforth consider the case $C = \mathbb{P}^1$, with a chosen ∞ and a coordinate z on $\mathbb{A}^1 = \mathbb{P}^1 - \{\infty\}$. Consider distinct points $P_1, \dots, P_N \in \mathbb{A}^1 \subset \mathbb{P}^1$ with z-coordinates $z_1, \dots, z_N$ respectively. The standard coordinate z endows each P_i with a formal coordinate. Let $\mathfrak{X} = \mathfrak{X}(\vec{z})$ be the resulting N-pointed curve with formal coordinates.

For every positive root δ, make a choice of a non-zero element f_δ in $\mathfrak{g}_{-\delta}$. Suppose $\lambda_1, \dots, \lambda_N \in P_k$, such that $\mu = \sum_{i=1}^N \lambda_i$ is in the root lattice. Write $\mu = \sum n_p \alpha_p$, where α_p are the simple positive roots, where $n_p \in \mathbb{Z}_{\geq 0}$.

Let $|\vec{\lambda}\rangle = |\lambda_1\rangle \otimes \dots \otimes |\lambda_N\rangle$ be the product of the corresponding highest weight vectors. Now consider and fix a map $\beta : [M] = \{1, \dots, M\} \to R$ (R is the set of simple positive roots), so that $\mu = \sum_{a=1}^M \beta(a)$ (where M is $M = \sum n_p$).

Introduce variables $t_1, \dots, t_M$ considered points on $\mathbb{P}^1 - \{\infty, P_1, \dots, P_N\}$ Consider, for every $\langle\Psi| \in V_{\vec{\lambda}}^\dagger(\mathfrak{X})$, the correlation function

$$\Omega = \Omega_\beta(\langle\Psi|) = \langle\Psi| f_{\beta(1)}(t_1) f_{\beta(2)}(t_2) \dots f_{\beta(M)}(t_M) |\vec{\lambda}\rangle.$$

REMARK 3.1. *There is an explicit formula (see* [**2**]*) for Ω in terms of $\langle\Psi| \in (V_{\lambda_1} \otimes \dots \otimes V_{\lambda_N})^*$ (a Schechtman-Varchenko form).*

The above correlation function can be considered as a suitable "normal" multiderivative of the corresponding theta function on a Harder-Narasimhan stratum (the theta function vanishes on the stratum).

Let $\kappa = k + g^*$ where g^* is the dual Coxeter number of $\mathfrak{g}$. The following "master function" was discovered by Schechtman-Varchenko [**15**]:
(3.1)

$$\mathcal{R} = \prod_{1 \leq i < j \leq N} (z_i - z_j)^{\frac{-(\lambda_i, \lambda_j)}{\kappa}} \prod_{a=1}^{M} \prod_{j=1}^{N} (t_a - z_j)^{\frac{(\lambda_j, \beta(a))}{\kappa}} \prod_{1 \leq a < b \leq M} (t_a - t_b)^{\frac{-(\beta(a), \beta(b))}{\kappa}}$$

3.1. The extension theorem. Suppose $M = \sum n_p$ (and hence $\beta : [M] \to R \subseteq \Delta_+$). Let

$$X_{\vec{z}} = \{(t_1, \dots, t_M) \in \mathbb{A}^M : t_a \neq t_b, t_a \neq z_i, i \in [N], a \neq b \in [M]\}.$$

Consider an unramified (possibly disconnected) cover of $X_{\vec{z}}$ given by $Y_{\vec{z}} = \{(t_1, \dots, t_M, y) \mid y^\kappa = \mathcal{R}^\kappa\}$, (where assume for simplicity that the exponents in $\mathcal{R}^\kappa$ are integers).

Now fix $\langle\Psi| \in V_{\vec{\lambda}}^\dagger(\mathfrak{X}(\vec{z}))$ and set $\Omega = \Omega_\beta(\langle\Psi|)$. The following extension result was shown recently in [**2**]:

THEOREM 3.2. (1) *The multi-valued meromorphic form $\mathcal{R}\Omega$ on $X_{\vec{z}}$ is square integrable.*
(2) *The differential form $\mathcal{R}\Omega$ extends to an everywhere regular, single valued, differential form of the top order on any smooth and projective compactification $\overline{Y}_{\vec{z}} \supset Y_{\vec{z}}$.*

The above theorem was proved earlier by Ramadas [**14**], for $\mathfrak{g} = \mathfrak{sl}_2$. Ramadas' strategy [**14**] is to prove this kind of theorem by showing that the logarithmic degree of $\mathcal{R}\Omega$ along any "abnormal stratum" for $\mathcal{R}\Omega$ is positive (see [**2**] and references therein). These abnormal strata are of three kinds:

(S1) A certain subset of the $t's$ come together (to an arbitrary moving point). That is $t_1 = t_2 = \cdots = t_L$ after renumbering (possibly changing β).
(S2) A certain subset of the $t's$ come together to one of the z's. That is $t_1 = t_2 = \cdots = t_L = z_1$ after renumbering (possibly changing β).
(S3) A certain subset of the $t's$ come together to ∞. That is $t_1 = t_2 = \cdots = t_L = \infty$ after renumbering (possibly changing β).

REMARK 3.3. *The logarithmic degree $d^S(\mathcal{R}\Omega)$ along a stratum S is the following. Blow up $(\mathbb{P}^1)^M$ along S and let E be the exceptional divisor. Then, $d^S(\mathcal{R}\Omega) - 1$ is the order of vanishing of $\mathcal{R}\Omega$ along E. Note that "order of vanishing" is an additive function and the order of vanishing of $\mathcal{R}$ is $\frac{1}{C\kappa}$ times the order of vanishing of the single valued function $\mathcal{R}^{C\kappa}$ (for a sufficiently divisible integer C).*

More precisely, we prove the following theorem.

THEOREM 3.4. *The logarithmic degree of $\mathcal{R}\Omega$ along each of the strata $(S1)$, $(S2)$ and $(S3)$ is positive.*

The author while following Ramadas' overall strategy, replaced the used of quot schemes by Kac-Moody algebras. The following were crucial

- Ω is a correlation function in the language of [**18**]. It is a log form in the sense of Hodge theory, which has a pole along $t_a = t_b$ only if $\beta(a) + \beta(b)$ is a root, and the residue is again a correlation function.
- Correlation functions have explicit power series expansions as collections of points come together - For example on the stratum $t_1 = t_2 = \cdots = t_L = z_1$, we have a formula (on suitable angular sectors)

$$\Omega = \sum_{b_1,\ldots,b_L} \omega_{\vec{b}} u_1^{-b_1-1} \ldots u_L^{-b_L-1}$$

where $\omega_{\vec{b}}$ equals

$$\langle \Psi | f_{\beta(L+1)}(t_{L+1}) \ldots f_{\beta(M)}(t_M) | \rho_1(f_{\beta(1)}(\xi_1^{b_1})) \ldots \rho_1(f_{\beta(L)}(\xi_1^{b_L})) | \vec{\lambda} \rangle d\vec{u}$$

($d\vec{u} = du_1 \dots du_L$, $u_i = t_i - z_1$ and ρ_1 acts on the the first factor of $|\vec{\lambda}\rangle$.)

- The logarithmic order of Ω are controlled by its power-series coefficients. The author showed in [**2**] that the power series coefficients have to vanish as "too many points come together" - this vanishing pattern is shown to adequately "compensate" the (zero and) pole producing patterns of the master function $\mathcal{R}$. For example
$$f_{\beta(1)}(\xi_1^{b_1}) \cdots f_{\beta(L)}(\xi_1^{b_L})|\lambda_1\rangle = 0$$
if
$$\sum b_a > \frac{2(\lambda_1, \gamma) - (\gamma, \gamma)}{2k} \tag{3.2}$$
where $\gamma = \sum_{i=1}^{L} \beta(i)$. Note that k appears in the denominator of (3.2) and not κ.
- Vanishing statements of power series coefficients are an essential consequence of the integrability assumption on the representations of $\hat{\mathfrak{g}}$ (see Theorem 12.5, part (d) of [**9**]). In geometric terms these vanishings occur because of the compactness of the moduli of semi-stable bundles. The finite dimensional analogue is the following: the vanishing $f^{k+1}v = 0$ where v is the highest weight of an irreducible finite dimensional $\mathrm{SL}_2(\mathbb{C})$ representation of dimension k (with $f \in \mathfrak{sl}_2$ as usual).

3.2. Gawedzki's proposal in genus 0. Theorem 3.2 leads to a unitary metric and an answer to Question 1.1 in genus 0. Before we do this, it is instructive to step back and take a look at the general Gawedzki proposal (influenced by Ramadas' work):

(1) Conformal blocks can be viewed as sections of line bundles on moduli of parabolic bundles on the curve. In this setting, according to Ramadas, one should first find derivatives of theta functions on Harder-Narasimhan strata, which can be taken as a suggestion to look at correlation functions. Roughly speaking, one modifies the corresponding G-bundles (in the Harder-Narasimhan stratum, actually the part where the extension data are trivial) around a finite set of additional points, and then takes a suitable mixed partial derivative of the (non-abelian) theta function ($\langle\Psi|$) in the direction of these changes to obtain a correlation function $\Omega(\langle\Psi|)$. The next step is multiplication by a Schechtman-Varchenko master function $\mathcal{R}$, available only in genus 0 and perhaps in genus 1 (in genus 1, in terms of classical theta functions [**5**]), but in principle one

should be able to guess the form of the Schechtman-Varchenko function by prescribing its local behavior.

(2) This should lead to a "generalized Schechtman-Varchenko form": A form on a cover of C^M (C is the curve), with local coefficients (of classical thetas with finite monodromy).

(3) There will then be the difficult task of showing that the assignment of a conformal block to an element in a cohomology group: $\langle\Psi| \to \mathcal{R}\Omega(\langle\Psi|)$ is flat for connections (as the marked curve varies) - which is available in genus 0 thanks to [**15**].

We will now describe the genus 0 resolution of this proposal. Instead of the moduli of N points on $\mathbb{P}^1$, we will work over the configuration space $\mathcal{C}$ of N points in $\mathbb{A}^1$ (which will then lead to a projective metric over the moduli space, see [**2**]). The varieties Λ_x from the introduction are $\overline{Y}_{\vec{z}}$, which are smooth projective compactifications of the canonical $Y_{\vec{z}}$. The most important local system in this picture is the image $H_{\vec{z}}$ of the cohomology of $\overline{Y}_{\vec{z}}$ in that of $Y_{\vec{z}}$ (which is independent of the choice of the smooth compactification $\overline{Y}_{\vec{z}}$, by mixed Hodge theory):

$$H_{\vec{z}} = \mathrm{Im} : H^M(\overline{Y}_{\vec{z}}, \mathbb{C}) \to H^M(Y_{\vec{z}}, \mathbb{C})$$

(an isomorphism on $(M,0)$ parts)

One obtains an inclusion (from Theorem 3.2):

$$V^{\dagger}_{\vec{\lambda}}(\mathfrak{X}(\vec{z})) \hookrightarrow H^{M,0}(\overline{Y}_{\vec{z}}, \mathbb{C}) \tag{3.3}$$

taking to $\langle\Psi|$ to $\mathcal{R}\Omega_\beta(\langle\Psi|)$, and where the last group is isomorphic to the $(M,0)$ part of $H_{\vec{z}}$.

It follows from the work of Schechtman and Varchenko that the above inclusions preserve connections (more precisely, they prove that $V^{\dagger}_{\vec{\lambda}}(\mathfrak{X}(\vec{z})) \to H^M(Y_{\vec{z}}, \mathbb{C})$ preserves connections, by some arguments in Hodge theory this implies that $V^{\dagger}_{\vec{\lambda}}(\mathfrak{X}(\vec{z})) \to H^M(\overline{Y}_{\vec{z}}, \mathbb{C})$ preserves connections as well).

Note that the $H^{M,0}$ part of a variation of Hodge structure is, in general, not preserved by the Gauss-Manin connection, the failure is measured by Griffiths' transversality. However in our situation there is a local subsystem of $H^M(\overline{Y}_{\vec{z}}, \mathbb{C})$ which lives in $H^{M,0}$. It is meaningful to study the entire Hodge structure $H_{\vec{z}}$, not just its $(M,0)$ part.

The Hodge metric on $H^{M,0}(\overline{Y}_{\vec{z}}, \mathbb{C})$ restricts to a unitary (and non-degenerate) metric on the space of conformal blocks; it is preserved by the KZ-connection. Given a conformal block $\langle\Psi|$, let $\pi(\langle\Psi|) = \mathcal{R}\Omega(\langle\Psi|)$ be the corresponding Schechtman-Varchenko form. Then, a

(KZ-invariant) unitary metric (upto a constant) is given by the convergent integral (as conjectured in [4]):

$$|\langle\Psi|\|^2 = (\sqrt{-1})^M \int_{(\mathbb{P}^1)^M} \pi(\langle\Psi|) \wedge \overline{\pi(\langle\Psi|)}$$

An obvious question is to characterize the image of (3.3). Here are two other questions:

QUESTION 3.5.
Consider a more general hyperplane arrangement (and a parameter space for topologically equivalent hyperplane arrangements, and weights). Can one single out a part of the corresponding variation of cohomology groups ("those that extend to compactifications"?): For example if in the master function (3.1) *there are general (rational) exponents.*

QUESTION 3.6.
(In the spirit of a question of Nori to the author) Is there a "modular interpretation" for the entire Hodge structure $H_{\vec{z}}$? Perhaps as sections of suitable line bundles over moduli of Higgs bundles (with growth conditions at infinity)? It will be interesting to see if this chain of thought leads to Hitchin type connections on other spaces of global sections.

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