代数幾何学シンポジウム記録
2011年度 pp.91-113

SURFACES WITH $p_g = 0$: CONSTRUCTIONS AND MODULI SPACES, BURNIAT SURFACES AND DEFORMATIONS OF AUTOMORPHISMS

F. CATANESE

INTRODUCTION

The following is a slightly extended version of the talk, with the same title, which I gave at the Kinosaki Symposium on Algebraic Geometry in October 2011, and dealing with the classification of complex projective surfaces of general type (here the reader may find a few more references).

As mentioned in the talk, there exist two ways to do "classification theory":

- one is similar to the activity of collecting beautiful and/or interesting objects at your home,
- the other is like planning on the onset to build a large museum, starting by collecting financial support and experts who are supposed to work there: in other words, giving priority to the organizational and social aspects of your enterprise.

But even if you choose method one, your home might become a museum after your death, so both methods could converge to the same goal in the end; the main difference is therefore psychological, and the choice reflects mainly personal taste.

Also, at first glance the first method seems to be simpler than the second: still it might face you with non trivial technical problems (even if you only collect wine labels, it is not easy to peel them off the bottle, especially for the French wines...).

Our beloved objects are here the surfaces of general type, and interesting patterns emerge while collecting examples and studying them.

So, let S be a smooth complex projective *minimal* surface of general type. This means that S does not contain any rational curve of self intersection (-1) or, equivalently, that the canonical divisor K_S of S is nef and big ($K_S^2 > 0$). Then it is well known that

$$K_S^2 \geq 1,\ \chi(S) := 1 - q(S) + p_g(S) \geq 1.$$

Date: January 21, 2012.

The present article follows the expository thread of the talk given at the Kinosaki Algebraic Geometry Symposium, October 27, 2011. It reports on research which took place in the realm of the DFG Forschergruppe 790 "Classification of algebraic surfaces and compact complex manifolds".

Recall that the *geometric genus* of S:

$$p_g(S) := h^0(S, \Omega_S^2) := \dim H^0(S, \Omega_S^2) = \dim H^0(S, \mathcal{O}_S(K_S)),$$

and the *irregularity* of S:

$$q(S) := h^1(S, \mathcal{O}_S) := \dim H^1(S, \mathcal{O}_S) = h^0(S, \Omega_S^1) := \dim H^0(S, \Omega_S^1),$$

are *birational invariants* of S, as well as K_S^2, since

$$K_S^2 = P_2(S) - \chi(S), P_2(S) := \dim H^0(S, \mathcal{O}_S(2K_S)).$$

Recall (see [Bom73]) that the canonical model of S is $X := Proj(\mathcal{R}(S))$, where $\mathcal{R}(S) := \oplus_m H^0(S, \mathcal{O}_S(mK_S))$ is the canonical ring of S. X is a normal surface with Rational Double Points as singularities, and with K_X ample. Moreover there is a birational morphism p: $S \to X$ contracting exactly the finitely many (-2)-curves (the irreducible curves C such that $K_S \cdot C = 0$, which satisfy $C^2 = -2$, hence $C \cong \mathbb{P}^1$).

We have a coarse moduli space for the canonical models X of surfaces S of general type with fixed χ and K^2 ([Gies77]).

Theorem 0.1. *For each pair of natural numbers (x, y) we have the Gieseker moduli space $\mathfrak{M}^{can}_{(x,y)}$, which is a quasi projective scheme, and whose points correspond to the isomorphism classes of minimal surfaces S of general type with $\chi(S) = x$ and $K_S^2 = y$.*

It is a coarse moduli space for the canonical models X of minimal surfaces S of general type with $\chi(S) = x$ and $K_S^2 = y$.

Concerning the range attained by the pair of numerical invariants above, an upper bound for K_S^2 is given by the Bogomolov-Miyaoka-Yau inequality:

Theorem 0.2 ([Miy77b], [Yau77], [Yau78], [Miy82]). *Let S be a smooth surface of general type. Then*

$$K_S^2 \leq 9\chi(S),$$

and equality holds if and only if the universal covering of S is the complex ball $\mathbb{B}_2 := \{(z, w) \in \mathbb{C}^2 | |z|^2 + |w|^2 < 1\}$.

0.1. **Surfaces wih very low invariants.** The above inequality is relevant when one is looking at the classification of surfaces of general type with 'very low' invariants, for instance with the minimal possible value $\chi(S) = 1$ for $\chi(S)$.

In this case classification means therefore to "understand" the nine moduli spaces $\mathfrak{M}^{can}_{(1,n)}$ for $1 \leq n \leq 9$, in particular to describe their connected and irreducible components and their respective dimensions. Observe that

$$\chi(S) = 1 \iff p_g(S) = q(S).$$

Remark 0.3. If we assume that S is irregular (i.e., $q(S) > 0$), then by a result of Debarre ([Deb82]) it follows that $K_S^2 \geq 2p_g(S)$.

Therefore in our case $p_g(S) \leq 4$. Moreover it was shown by Beauville ([Deb82]) that $p_g(S) = q(S) = 4$ if and only if S is the product of two curves of genus 2.

Also surfaces with $p_g = q = 3$ were described in [CCML98]: there are only two families, namely, the symmetric squares of curves of genus 3 (these have $K_S^2 = 6$), or the quotients of a product of curves $C_2 \times C_3'$, where C_2 has genus 2 and C_3' has genus 3, by an involution of product type, acting as the hyperelliptic involution on C_2 and freely on C_3' (these surfaces have $K_S^2 = 8$).

In [CCML98] a partial classification was shown (it was shown for instance that $6 \leq K_S^2 \leq 9$), and the classification was then finished independently by [HaPa02], and[Pi02].

There has been lately a revival of interest for the surfaces in the second family, from the topological side (see [Akb12]).

The case $p_g = q = 2$ is already harder: there is a substantial literature (see [BCP06b]), but they are still far from being classified in spite of work by several authors: Zucconi, Ciliberto and Mendes Lopes, Hacon and Chen, Polizzi and Penegini .

One knows that $4 \leq K_S^2 \leq 9$ by the cited inequalities, but it is unknown whether they do exist for $K_S^2 = 9$ (there have been repeated but failing attempts by Yeung to show that these ball quotients cannot occur), or whether the surfaces with $p_g = q = 2$ and $K_S^2 = 4$ are all double covers of a principally polarized Abelian surface, and with branch curve a divisor $\Delta \in |2\Theta|$ (this was proven by Manetti in [Man03] under the assumption that K_S is ample). There are also no examples known with $p_g = q = 2, K_S^2 = 7$.

Surfaces with $p_g = q = 1$ have $2 \leq K_S^2 \leq 9$ and have been classified for $K_S^2 = 2, 3$ ($K_S^2 = 2$ in [Cat81], $K_S^2 = 3$ in [CC91], [CC93], cf. also [CatPig06] for an exact determination of the connected components of the moduli space). Existence of the case of ball quotients ($p_g = q = 1, K_S^2 = 9$) has been announced by Cartwright and Steger. If this is correct, there are no gaps and surfaces with $p_g = q = 1$ do exist for each value of $K_S^2 = 2, \ldots, 9$.

As the reader may know or surmise, the case $p_g = q = 0$ is the most difficult, and also the one for which there are more examples (see [BCP10]). Surfaces with $p_g = q = 0$ exist for all values of $K_S^2 = 1, \ldots, 9$, but only the case $K_S^2 = 9$ has been classified by Cartwright and Steger, giving a precision to the fundamental work of Prasad-Yeung ([CaSt10], [PY07]); there are 50 fundamental groups, and 100 isolated points of the moduli space, corresponding to 50 pairs of complex conjugate non isomorphic surfaces ([KK02]).

Since the 1970's there was a big revival of interest (see [Dolga81] for an early survey) in the construction of these surfaces and in a possible attempt to classification, and the Bloch conjecture and differential topological questions raised by Donaldson Theory were a further

reason for raising further interest about surfaces of general type with $p_g = 0$. There has been recent important progress in the last 5 years (see [BCP10]) but there is no hope at the moment to even conjecturally finish the classification. E.g., for $K_S^2 = 7$ there is only one family of surfaces with $p_g = 0$ known, constructed by Inoue (cf. [Ino94]), while for $K_S^2 = 8$ the only known examples have the bidisk as universal cover (the reducible case has been classified in [BCG08], and a missing case was then added by [Fra11]).

At this point it seems appropriate to stop reporting on classification results and to concentrate on 'philosophical' issues. Consider the following provocative question of D. Mumford, posed at the Montreal Conference in 1980:

Can a computer classify all surfaces of general type with $p_g = 0$?

The meaning of the question is clear, and confirmed by recent progress: these surfaces are so many, that it takes more than man's power to 'conquer' their classification. And it is indeed true that a computer algebra program is necessary to construct systematically certain surfaces, as it was carried out in [BCGP09], [BP11a] for product-quotient surfaces with $p_g = 0$. More generally, it is conceivable that computer programs, may be quantum computers, may describe all the possible canonical rings of such surfaces in some not so distant future.

There remains however a major difficulty: these rings will belong to different families, for instance according to the several possibilities for the degrees of a minimal system of generators, and of relations and higher syzygies.

But, how to find out how these locally closed subsets will fit in together inside the moduli space?

This difficulty is witnessed already by the work of Horikawa ([Hor78]) in the much simpler instance of surfaces with $p_g(S) = 4, K_S^2 = 6$. Horikawa, looking at the canonical map, was able to divide these surfaces in 11 families, and began then to analyse the problem of incidence among these locally closed strata of the moduli space, the question being: is stratum $\mathcal{A}$ in the closure of stratum $\mathcal{B}$? This is a typical hard problem in the theory of surfaces, and Horikawa showed that the corresponding subset of the moduli space has 4 irreducible components, and at most 3 connected components: leaving open the question whether there are 1, 2, or 3 connected components. Asnwering one of these questions turned out to be quite difficult, namely in [BCP06a] it was shown that the number of connected components is at most 2, but it is still open the question whether the number is 1 or 2.

One may sometimes be in a lucky situation, where it is possible to describe completely a connected component of the moduli space.

This may happen in several ways, for instance because there is only one mode of presentation for the canonical ring, or because this phenomenon happens for some finite unramified covering $\hat{S}$ of S (see the next section).

Or, topology may dictate the existence of certain holomorphic maps to Abelian varieties or products of curves, and this geometric feature allows to determine a connected component of the moduli space.

On the other hand, if we are not in a lucky situation, or if there is no good topological reason which determines a connected component, it is very hard to show that an irreducible component is indeed a connected component. One has to study deformations of a given family of surfaces (determining an open set $\mathfrak{U}$ in the moduli space), then one-parameter limits of the deformed objects (degenerations of the surfaces corresponding to points in the open set $\mathfrak{U}$, i.e., determine the closure of the open set $\mathfrak{U}$) and then the deformations of these limits (try to see whether the closure $\overline{\mathfrak{U}}$ is also an open set).

Together with Ingrid Bauer, also motivated by the problem of sorting out the surfaces constructed in [BCGP09], we took as a benchmark the problem of determining completely the connected components of the moduli spaces containing the so called Burniat surfaces (some surfaces with $p_g(S) = 0$ constructed in 1966 by Pol Burniat, see [Bu66]). The problem is now solved for $K_S^2 = 2, 4, 5, 6$, but there is a single remaining final step missing in the case $K_S^2 = 3$ of tertiary Burniat surfaces.

The paper is organized as follows:

In the first section we shall briefly recall some by now classical 'lucky' case where some connected component of the moduli space of surfaces with $p_g = 0$ can be determined. This part should also be seen as a 'warm up' for the sequel.

In section 2 we define the Burniat surfaces and in section 3 we state the main classification theorem concerning them. In section 4 we treat primary Burniat surfaces, which have a large fundamental group, and we illustrate via this case the principle "topology can determine connected components of the moduli space", a phenomenon which has been explored in various other cases.

In section 5 we introduce extended Burniat surfaces, which are deformations of nodal Burniat surfaces (they yield a concrete example of an open set $\mathfrak{U}$ as previously mentioned).

Finally, in section 6 we describe a pathological behaviour of the moduli space, which is related to the degeneration of extended Burniat surfaces to Burniat surfaces; namely, the fact that continuous families of canonical models yield, at the level of minimal models, families of branch loci which vary discontinuously. The explanation goes through the remarkable phenomenon that, even if the automorphism group of the minimal model is the same as the automorphism group of the canonical model, the same does not hold for families; so that, if G

is the group of automorphisms of the general surface, then $\mathrm{Def}(S,G)$ is not proper onto $\mathrm{Def}(X,G)$; and, for tertiary Burniat surfaces, while $\mathrm{Def}(S)$ and $\mathrm{Def}(X,G)$ surject onto $\mathrm{Def}(X)$, $\mathrm{Def}(S,G)$ just maps to a nowhere dense set.

For the convenience of the reader we have drawn pictures of the line configurations in the plane corresponding to the branch divisors of the Burniat surfaces with $K_S^2 = 2, \dots 6$. They can be found in figure 1.2 attached below.

1. Lucky cases

Here are two classical examples of surfaces of general type where everything runs smoothly (see [Miy76], [Miy77a],[Rei78], [Dolga81]).

1.1. Classical Godeaux surfaces.

(1) Here $G := \mathbb{Z}/5$, it acts on a 4-dimensional vector space V via the 4 non trivial characters, hence also on $\mathbb{P}^3 := \mathbb{P}(V^\vee)$ and $X = \hat{X}/G$ is the quotient of an invariant 5-ic surface $\hat{X}$ (with RDP's as singularities) on which G acts freely.

Hence, if S is the minimal resolution of X, $\pi_1(S) \cong G = \mathbb{Z}/5$.

(2) It turns out that the representation V is isomorphic to the representation $H^0(\hat{S}, \mathcal{O}_{\hat{S}}(K_{\hat{S}}))$, therefore S has $p_g(S) = 0$, and $K_S^2 = 1$. Since $G^\vee \cong \mathbb{Z}/5$ is also the torsion part of $H^2(S,\mathbb{Z})$, to $\chi \in G^\vee$ corresponds a divisor class M_χ, and

$$V \cong H^0(\hat{S}, \mathcal{O}_{\hat{S}}(K_{\hat{S}})) \cong [\oplus_{\chi \in G^\vee, \chi \neq 0} H^0(S, \mathcal{O}_S(K_S + M_\chi))]$$

(3) Conversely, if S has $p_g(S) = 0$, and $K_S^2 = 1$, and torsion $T \cong G^\vee \cong \mathbb{Z}/5$, then the subspaces $H^0(S, \mathcal{O}_S(K_S + M_\chi))$ have dimension 1, and they yield a basis for the vector space $H^0(\hat{S}, \mathcal{O}_{\hat{S}}(K_{\hat{S}}))$.

Let $x_\chi \in H^0(S, \mathcal{O}_S(K_S + M_\chi))$ be a non zero element: then there cannot be any relation of the form $x_\chi x_{\chi'} = x_\psi x_{\psi'}$, because the associated divisors $div(x_\chi)$ are irreducible on the canonical model X of S. From this one concludes that the canonical map of $\hat{S}$ cannot have a quadric as image, hence it induces an isomorphism of the canonical model $\hat{X}$ with a quintic surface in $\mathbb{P}(V^\vee) = \mathbb{P}^3$.

1.2. Standard Campedelli surfaces.

(1) These are, by definition, the Campedelli surfaces with torsion group $T \cong (\mathbb{Z}/2)^3$.

(2) Here $\hat{S}$ is the natural unramified covering associated to the torsion group (again here equal to the full first homology group $H_1(S,\mathbb{Z})$ since $q(S) = 0$), and $S = \hat{S}/G$.

(3) The best description of $\hat{X}$ is as the maximal abelian covering of exponent 2 of the plane $\mathbb{P}^2$ branched on 7 lines D_i, one for each $g_i^\vee \in G^{\vee *} := G^\vee \setminus \{0\}$. X is smooth if the 7 lines are in linear general position.

(4) The Galois group of $\hat{X} \to \mathbb{P}^2$ is the group $G' \cong (\mathbb{Z}/2)^6$

$$G' := \oplus_{g_i^\vee \in G^{\vee *}} (\mathbb{Z}/2) g_i^\vee / (\mathbb{Z}/2)(\sum_i g_i^\vee).$$

There is a natural surjection $G' \to G^\vee$, with kernel canonically isomorphic to G, since to each element $g \in G$ corresponds the sum of the elements lying in its annihilator $g^\perp = Ann(g)$ in $G^\vee$.

In this way we see that, being $X = \hat{X}/G$, $X \to \mathbb{P}^2$ is ramified on the 7 lines, and it has the property that the local monodromy around the line D_i is the element $g_i^\vee \in G^\vee$ in the Galois group. Instead $\hat{X} \to X$ is unramified, with Galois group G.

(5) Indeed $\phi_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-2)^7$, and X is contained in the rank 7 vector bundle $\oplus_{g_i \in G^*} L_i$ whose sheaf of sections is isomorphic to $\mathcal{O}_{\mathbb{P}^2}(2)^7$.

X maps to the fibre product of the 7 double coverings

$$y_{g_i}^2 = \Pi_{g_j^\vee \notin Ann(g_i)} \delta_j,$$

where $D_j = div(\delta_j)$, and is indeed defined in the above rank 7 vector bundle by the following equations (see [Par91], and also [Cat08], page 146)

$$y_{g_i} \cdot y_{g_j} = y_{g_i + g_j} \Pi_{g_j^\vee \notin (Ann(g_i) \cup Ann(g_j))} \delta_j,$$

(6) Again we have a 7 dimensional vector space V corresponding to the 7 non trivial characters of G,

$$V \cong H^0(\hat{S}, \mathcal{O}_{\hat{S}}(K_{\hat{S}})) \cong [\oplus_{\chi \in G^\vee, \chi \neq 0} H^0(S, \mathcal{O}_S(K_S + M_\chi))].$$

Each summand has dimension 1 and a generator x_χ corresponds just to an equation δ_j for a line D_j, using the established notation $\{g_j^\vee\}$ for $G^{\vee *}$.

The bicanonical map of S is the Galois covering of $\mathbb{P}^2$ with group $G^\vee$, and $\hat{X}$ is embedded in $\mathbb{P}^6 := \mathbb{P}(V^\vee)$ as the complete intersection of 4 quadrics. The 4 quadrics are sums of squares, and are easily obtained because the 7 elements x_χ^2 belong to the 3 dimensional vector space $W \cong H^0(S, \mathcal{O}_S(2K_S)) = p^* H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(2))$.

(7) Conversely, given a Campedelli surface S with torsion group $G \cong (\mathbb{Z}/2)^3$, one considers the natural unramified Galois covering $\hat{S} \to S$ with group G, and shows that $H^0(S, \mathcal{O}_S(K_S + M_\chi))$ has dimension 1 for each character. Hence one has a 7 dimensional vector space V corresponding to the 7 non trivial

characters of G,

$$V \cong H^0(\hat{S}, \mathcal{O}_{\hat{S}}(K_{\hat{S}})) \cong [\oplus_{\chi \in G^\vee, \chi \neq 0} H^0(S, \mathcal{O}_S(K_S + M_\chi))].$$

Again, one can show that $\hat{X}$ is canonically embedded in $\mathbb{P}^6 := \mathbb{P}(V^\vee)$ as the complete intersection of 4 quadrics, and 3 of the 7 elements x_χ^2 are linearly independent and yield the bicanonical map of S. Since the 4 quadrics are sums of squares it follows that $\hat{X}$ is also invariant by the bigger group $G' \cong (\mathbb{Z}/2)^6$, and then the bicanonical map of S factors through the projection onto the canonical model X and the Galois cover $X \to \mathbb{P}^2$ with Galois group $G^\vee \cong G'/G$.

2. What is a ... Burniat surface?

Burniat surfaces are surfaces of general type with geometric genus $p_g(S) = 0$, and they were constructed by Pol Burniat in 1966 in [Bu66], where the method of singular bidouble covers was introduced in order to attack the geography problem for surfaces of general type.
The birational structure of Burniat surfaces is rather simple to explain: let $P_1, P_2, P_3 \in \mathbb{P}^2$ be three non collinear points (which we assume to be the points $(1:0:0)$, $(0:1:0)$ and $(0:0:1)$), and let $D_i = \{\Delta_i = 0\}$, for $i \in \mathbb{Z}/3\mathbb{Z}$, be the union of three distinct lines through P_i, including the line $D_{i,1}$ which is the side of the triangle joining the point P_i with P_{i+1}.
We furthermore assume that $D = D_1 \cup D_2 \cup D_3$ consists of nine different lines.

Definition 2.1. A Burniat surface S is the minimal model for the function field

$$\mathbb{C}(\sqrt{\frac{\Delta_1}{\Delta_2}}, \sqrt{\frac{\Delta_1}{\Delta_3}}).$$

Proposition 2.2. *Let S be a Burniat surface, and denote by m the number of points, different from P_1, P_2, P_3, where the curve D has multiplicity at least 3 (hence indeed equal to 3). Then $0 \leq m \leq 4$, and the invariants of the smooth projective surface S are:*

$$p_g(S) = q(S) = 0, \ K_S^2 = 6 - m.$$

The heart of the calculation, based on the theory of bidouble covers, as explained in [Cat99], is that the singularities where the three curves have multiplicities $(3,1,0)$ lower K^2 and the difference $p_g - q$ both by 1, while the singularities where the three curves have multiplicities $(1,1,1)$ lower K^2 by 1 and leave $p_g - q$ unchanged (in fact, for a bidouble cover branched on 3 smooth cubics, one has $K_S^2 = 9, p_g = 3$).

Example 2.3. (Singularities of Galois Coverings). Take three general lines l_1, l_2, l_3 through a point $P \in S$. Choosing appropriate local (in the analytic topology) coordinates we can assume $P = 0 \in \mathbb{C}^2$, $l_1 =$

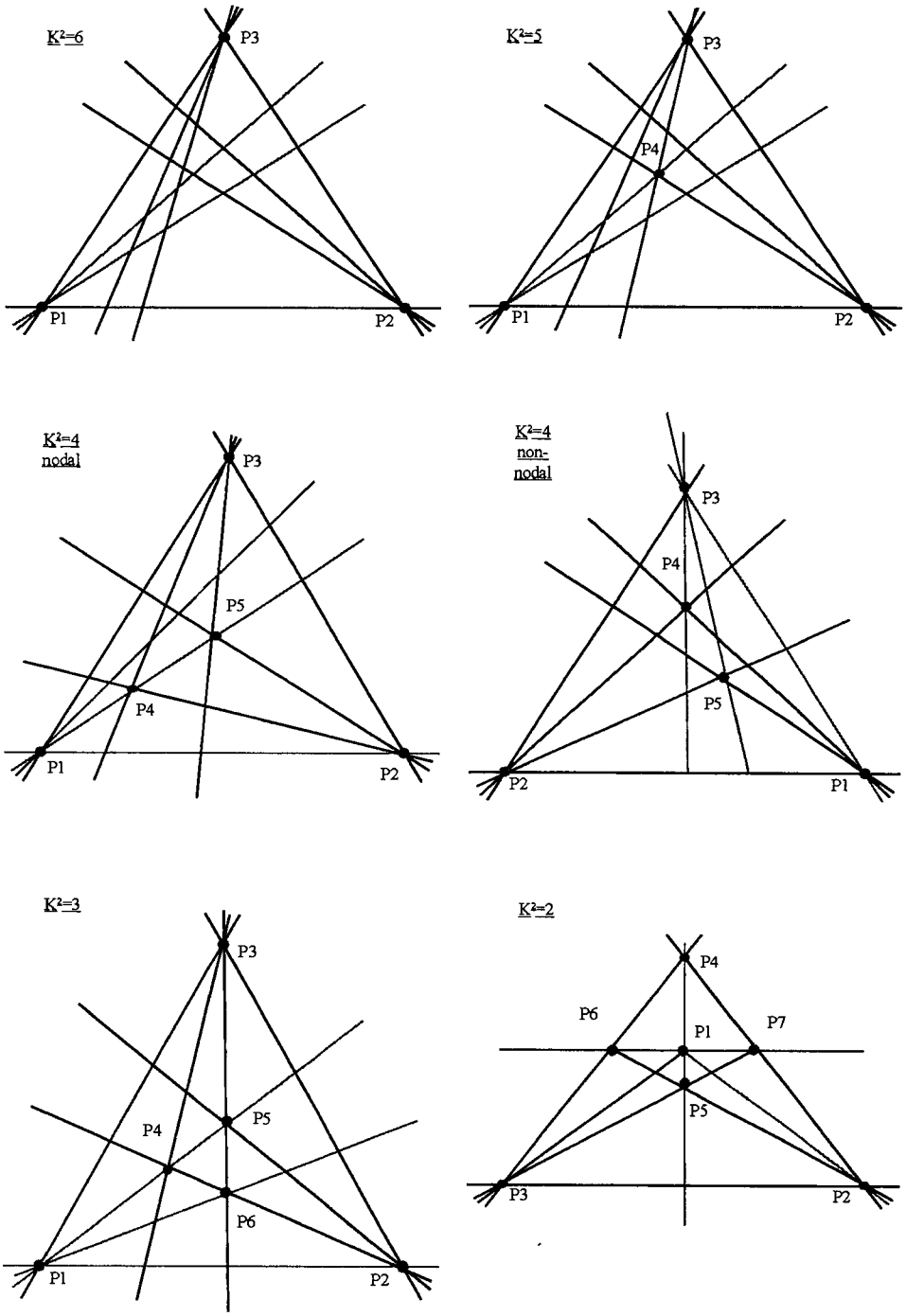


FIGURE 1. Configurations of lines

$\{x = 0\}$, $l_2 = \{y = 0\}$ and $l_3 = \{x - y = 0\}$. Taking the maximal Abelian cover of exponent 2 branched in l_1, l_2, l_3 we get:

$$u^2 = x, \quad v^2 = y, \quad w^2 = x - y \quad \Longleftrightarrow \quad u^2 - v^2 = w^2,$$

i.e. we get an ordinary double point

$$Y := \{(u, v, w) \in \mathbb{C}^3 : u^2 - v^2 = w^2\} \subset \mathbb{C}^3,$$

(an A_1-singularity) over 0.

Y is invariant under the involution

$$\sigma \colon \mathbb{C}^3 \to \mathbb{C}^3, \quad (u, v, w) \mapsto (-u, -v, -w),$$

and we get a factorization of the $(\mathbb{Z}/2\mathbb{Z})^3$- Galois covering $(Y, 0) \to (\mathbb{C}^2, 0)$ as

$$(Y, 0) \to (Y/\sigma, 0) \to (\mathbb{C}^2, 0).$$

Note that $(Y/\sigma, 0)$ is a $\frac{1}{4}(1, 1)$ - singularity, which is not Gorenstein, but $\frac{1}{2}$-Gorenstein. Acquiring such a singularity leaves the geometric genus p_g and the irregularity q invariant, but lowers K_S^2 by 1.

Indeed, the minimal resolution of such a singularity has an exceptional curve $E \cong \mathbb{P}^1$ with $E^2 = -4$, hence the canonical divisor on the resolution S is the pull back of the canonical divisor of Y diminished by $\frac{1}{2}E$. Because $(K_S + E) \cdot E = -2 \Rightarrow K_S \cdot E = +2$.

One may understand the biregular structure of a Burniat surface S through the blow up W of the plane at the points $P_1, P_2, P_3, \ldots P_m$ of D of multiplicity at least three.

W is a weak Del Pezzo surface of degree $6 - m$ (i.e., a surface with nef and big anticanonical divisor).

Proposition 2.4. *The Burniat surface S is a finite bidouble cover (a finite Galois cover with group $(\mathbb{Z}/2\mathbb{Z})^2$) of the weak Del Pezzo surface W. Moreover the bicanonical divisor $2K_S$ is the pull back of the anticanonical divisor $-K_W$. The bicanonical map of S is the composition of the bidouble cover $S \to W$ with the anticanonical quasi-embedding of W, as a surface of degree $K_S^2 = K_W^2$ in a projective space of dimension $K_S^2 = K_W^2$.*

3. The main classification theorem

Fixing the number $K_S^2 = 6 - m$, one sees immediately that the Burniat surfaces are parametrized by a rational family of dimension $K_S^2 - 2$, and that this family is irreducible except in the case $K_S^2 = 4$.

Definition 3.1. The family of Burniat surfaces with $K_S^2 = 4$ of nodal type is the family where the points P_4, P_5 are collinear with one of the other three points P_1, P_2, P_3, say P_1.
The family of Burniat surfaces with $K_S^2 = 4$ of non-nodal type is the family where the points P_4, P_5 are never collinear with one of the other three points.

Our main classification result of Burniat surfaces is summarized in the following table, giving information concerning the families of Burniat surfaces, and where $\mathbb{H}_8$ denotes the quaternion group of order 8. More information will be given in the subsequent theorems.

K^2	dim	is conn. comp.?	name	π_1
6	4	yes	primary	$1 \to \mathbb{Z}^6 \to \pi_1 \Rightarrow (\mathbb{Z}/2\mathbb{Z})^3$
5	3	yes	secondary	$\mathbb{H}_8 \oplus (\mathbb{Z}/2\mathbb{Z})^3$
4 non nodal	2	yes	secondary	$\mathbb{H}_8 \oplus (\mathbb{Z}/2\mathbb{Z})^2$
4 nodal	2	no: $\subset$ 3-dim. irr. conn. component $\supset$ $\supset$ extended Burniats	secondary nodal	$\mathbb{H}_8 \oplus (\mathbb{Z}/2\mathbb{Z})^2$
3	1	no: $\subset$ 4-dim. irr. component $\supset$ $\supset$ extended Burniats	tertiary	$\mathbb{H}_8 \oplus \mathbb{Z}/2\mathbb{Z}$
2	0	no: $\in$ conn. component of standard Campedelli	quaternary	$(\mathbb{Z}/2\mathbb{Z})^3$

Theorem 3.2. **(see** [BC11b] **and** [BC10])

1) The three respective subsets of the moduli spaces of minimal surfaces of general type $\mathfrak{M}^{can}_{1,K^2}$ corresponding to Burniat surfaces with $K^2 = 6$, resp. with $K^2 = 5$, resp. Burniat surfaces with $K^2 = 4$ of non nodal type, are irreducible connected components, normal, rational of respective dimensions 4,3,2.
Moreover, the base of the Kuranishi family of such surfaces S is smooth.

Observe that the above result for $K^2 = 6$ was first proven by Mendes Lopes and Pardini in [MLP01]. We showed in [BC11b] the stronger theorem

Theorem 3.3 (Primary Burniat surfaces theorem). *Any surface homotopy equivalent to a Burniat surface with $K^2 = 6$ is a Burniat surface with $K^2 = 6$.*

Theorem 3.4 (Secondary nodal Burniat surfaces theorem). **(see** [BC10] **and** [BC11])

Secondary nodal Burniat surfaces, together with secondary extended nodal Burniat surfaces form an irreducible connected component of the moduli space.

For $K^2 = 2$ another realization of the Burniat surface is (as shown by Kulikov in [Ku04]) as a special element of the family of Campedelli surfaces with torsion $(\mathbb{Z}/2\mathbb{Z})^3$, considered in the previous section.

We saw that they are Galois covers of the plane with group $(\mathbb{Z}/2\mathbb{Z})^3$, branched on seven lines. For the Burniat surface we have the special configuration of a complete quadrilateral together with its three diagonals.

4. The homotopy equivalence method

This section is devoted to the idea of the proof of theorem 3.3. There is a general philosophy behind the method of proof which applies to many more cases.

A Burniat surface S with $K_S^2 = 6$ is called a *primary Burniat surface.* Recall that by proposition 2.4 S is a finite bidouble cover of a Del Pezzo surface $Y = \hat{\mathbb{P}}^2(P_1, P_2, P_3)$ of degree 6, which can be seen as

$$Y := \{((y_1, y_1'), (y_2, y_2'), (y_3, y_3')) \in (\mathbb{P}^1)^3 : y_1 y_2 y_3 = y_1' y_2' y_3'\}.$$

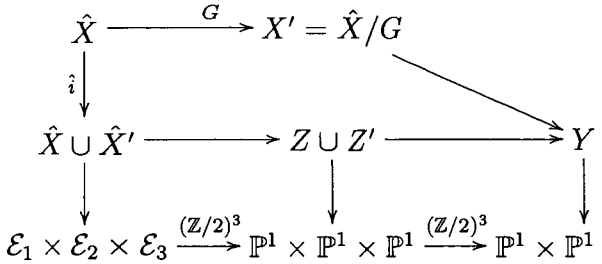
We take the $(\mathbb{Z}/2\mathbb{Z})^3$-covering of

$$\pi\colon \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1,$$

given by

$$(v_i)^2 = y_i, \quad (v_i')^2 = y_i', \quad i \in \{1, 2, 3\}.$$

Then $\pi^{-1}(Y)$ splits as the union of two Del Pezzo surfaces of degree 6, $Z \cup Z'$, where $Z := \{v_1 v_2 v_3 = v_1' v_2' v_3'\}$, and $Z' := \{v_1 v_2 v_3 = -v_1' v_2' v_3'\}$. What we have done is the following: we have taken the square root of the two points in each $\mathbb{P}^1$ corresponding to two of the four lines passing through each P_j. Now we take the square root of the other two lines through each of the three points P_j and obtain a $(\mathbb{Z}/2\mathbb{Z})^3$-covering $\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3 \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, where each $\mathcal{E}_j$ is therefore an elliptic curve. We get the following diagram:



X' is the normal $(\mathbb{Z}/2\mathbb{Z})^2$-covering of Y whose resolution is a Burniat surface.

We have the following:

Facts 4.1.

(1) $\hat{X} \to X'$ *is étale (with group* $G = (\mathbb{Z}/2\mathbb{Z})^2$*)* $\iff$ S *is a primary Burniat surface.*

(2) $\hat{X} \subset \mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3$ *is a hypersurface of multidegrees* $(2, 2, 2)$.

(3) $X' = \hat{X}/G$ *is the quotient of a free action except for some* A_1*-singularities with stabilizer* $\mathbb{Z}/2\mathbb{Z}$*, yielding* $\frac{1}{4}(1, 1)$ *- points on* X'.

4.1. **Idea of the proof of theorem 3.3.** Assume that S is homotopically equivalent to a primary Burniat surface. Then S has the same fundamental group as a primary Burniat surface. Hence there is an étale $(\mathbb{Z}/2\mathbb{Z})^3$-covering $\hat{X} \to X$ with $q(\hat{X}) = 3$.

Step 1. One shows that the Albanese variety $\mathrm{Alb}(\hat{X})$ is the product of three elliptic curves $\mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3$. In fact, for each $i \in \{1, 2, 3\}$ there is an intermediate cover

$$\hat{X} \to X_i \to X'$$

with $q(X_i) = 1$. By the universal property of the Albanese variety we get a morphism

$$\lambda\colon \mathrm{Alb}(\hat{X}) \to \mathrm{Alb}(X_1) \times \mathrm{Alb}(X_2) \times \mathrm{Alb}(X_3) \cong \mathcal{E}_1' \times \mathcal{E}_2' \times \mathcal{E}_3'.$$

Looking at the fundamental group $\pi_1(\hat{X})$, one sees that the isogeny λ is of product type, whence the claim follows.

Step 2. Consider now the Albanese map of $\hat{X}$:

$$f\colon \hat{X} \to f(\hat{X}) =: \hat{Y} \subset \mathcal{E}_1 \times \mathcal{E}_2 \times \mathcal{E}_3.$$

Then the class of $\hat{Y}$ is the same as for the Albanese image of the corresponding étale covering of a primary Burniat surface (since we have a map to a $K(\pi, 1)$ space with fundamental group π equal to $\pi_1(\hat{X})$ hence the class of the image is invariant by homotopy equivalence).

Moreover, since this class is $2F_1+2F_2+2F_3$, we see that the Albanese map of $\hat{X}$ is birational. Finally, an argument using adjunction shows that $\hat{X} \cong \hat{Y}$.

Step 3. X (the canonical model of S) is $\hat{X}/G$, and it is a bidouble cover of a Del Pezzo surface of degree 6 as required.

For more details we refer to [BC11b].

Remark 4.2. The same method proves similar theorems in the following cases:

(1) *Keum-Naie surfaces* with $K^2 = 4$ (cf. [BC11a]).

For Keum-Naie surfaces S the key idea is to find a representation $X = \hat{X}/G$, where $G = (\mathbb{Z}/2\mathbb{Z})^2$, acting on $\mathcal{E}_1 \times \mathcal{E}_2$ ($\mathcal{E}_1$, $\mathcal{E}_2$ being again elliptic curves). $\hat{X}$ is a double cover of $\mathcal{E}_1 \times \mathcal{E}_2$ branched on a G-invariant divisor Δ of bidegree $(4, 4)$.

Then G acts freely on $\hat{X}$ for a suitable twist of the action.

(2) *Inoue surfaces* with $K^2 = 7$ (cf. [BC12]).

Inoue surfaces are of the form $X = \hat{X}/(\mathbb{Z}/2\mathbb{Z})^4$, where $\hat{X}$ is a $(\mathbb{Z}/2\mathbb{Z})^4$-invariant divisor of multidegree $(2, 2, 4)$ in $\mathcal{E}_1 \times \mathcal{E}_2 \times D$, where $\mathcal{E}_1$, $\mathcal{E}_2$ are again elliptic curves, while D is a curve of genus 5 which is a maximal abelian cover of $\mathbb{P}^1$ of exponent 2 branched on 5 points.

(3) *Kulikov surfaces* (cf. [ChCo10]).

These surfaces are $(\mathbb{Z}/3\mathbb{Z})^2$-coverings of the plane branched on the sides of a triangle and on the three medians (the lines connecting the three vertices with the barycenter).

The above cases also show another common feature:

$\hat{X}$ has A_1-singularities $\iff$ G acts no longer freely.

This implies that then $X = \hat{X}/G$ acquires ν singularities of type $\frac{1}{4}(1,1)$, and K_S^2 (S being a minimal model) drops by ν. Furthermore $\pi_1(S) = \pi_1(X)$ becomes finite.

Therefore for these families the *investigation of the connected components* of the moduli spaces has to be done by

(1) showing openness of the subset of the moduli space induced by such a family using local deformation theory;
(2) investigating the closure via 1-parameter limits.

Remark 4.3. All the Burniat surfaces X we consider are $G = (\mathbb{Z}/2\mathbb{Z})^2$-covers of a normal Del Pezzo surface Z of degree K_X^2.

For nodal Burniat surfaces with $K_S^2 = 4$ and Burniat surfaces with $K_S^2 = 3$, we need to introduce a larger family, including the so-called *extended Burniat surfaces*. They will be introduced in the next section in the more symmetric case of tertiary Burniat surfaces.

5. Extended Burniat surfaces

We recall the following definitions from [BC11]. Let $P_1, P_2, P_3 \in \mathbb{P}^2$ be three non collinear points, and let $P_4, \dots, P_{3+m}$, $m = 2, 3$, be further (distinct) points not lying on the sides of the triangle with vertices P_1, P_2, P_3.

Assume moreover that, for $m = 2$, the points P_1, P_4, P_5 are collinear, while, for $m = 3$, we shall moreover assume that also P_2, P_4, P_6 and P_3, P_5, P_6 are collinear (in particular, no four points are collinear).

Let's denote by $\tilde{Y} := \hat{\mathbb{P}}^2(P_1, P_2, \dots, P_{3+m})$ the weak Del Pezzo surface of degree $6 - m$, obtained by blowing up $\mathbb{P}^2$ in the $3 + m$ points $P_1, P_2, \dots, P_{3+m}$.

Saying that $\tilde{Y}$ is a weak Del Pezzo surface means that the anticanonical divisor $-K_{\tilde{Y}}$ is nef and big; in our case it is not ample, because of the existence of (-2)-curves, i.e. curves $N_i \cong \mathbb{P}^1$, with $N_i \cdot K_{\tilde{Y}} = 0$.

Contracting the (-2)-curves N_i we obtain a normal singular Del Pezzo surface Y' with $-K_{Y'}$ very ample.

In order to simplify the formulae, let us treat the case $m = 3$, denoting $P_3' := P_4, P_2' := P_5, P_1' := P_6$.

Then we have that P_i, P_{i+1}', P_{i+2}' are collinear (here $i \in \mathbb{Z}/3\mathbb{Z}$).

We denote by L the divisor on $\tilde{Y}$ which is the total transform of a general line in $\mathbb{P}^2$, by E_i the exceptional curve lying over P_i, by E_i' the exceptional curve lying over P_i', and by $D_{i,1}$ the unique effective divisor

in $|L - E_i - E_{i+1}|$, i.e., the proper transform of the side of the triangle joining the points P_i, P_{i+1}.

For $m = 3$ we have (-2)-curves N_1, N_2, N_3 such that

$$\{N_i\} = |L - E_i - E'_{i+1} - E'_{i+2}|,$$

Therefore the anticanonical image of $\tilde{Y}$ is a normal surface $Y' \subset \mathbb{P}^{6-m}$ of degree $6-m$, whose singularities are one node ν_1 (an A_1 singularity) in the case $m = 2$, and three nodes ν_1, ν_2, ν_3 in the case $m = 3$ (the (-2)-curve N_i is the total transform of the point ν_i).

Definition 5.1. *1) The* Burniat branch divisors for $m = 3$ *are defined to be the divisors* D_1, D_2, D_3 *such that:*

$$\{D_i\} = |L - E_i - E_{i+1}| + N_i + |L - E_i - E'_i| + E_{i-1},$$

2) The strictly extended Burniat branch divisor classes for $m = 3$ *are defined as follows:*

$$\Delta_i \equiv D_i - N_i + N_{i-1} + N_{i+1},$$

3) The strictly extended Burniat branch divisors for $m = 3$ *are defined taking an irreducible curve*

$$\Gamma_i \in |2L - E_i - E'_i - E_{i+1} - E'_{i+1}|$$

and replacing in Δ_i

$$N_{i-1} + |L - E_i - E_{i+1}| + E_{i-1}$$

by Γ_i*, so that*

$$\Delta_i = \Gamma_i + N_{i+1} + |L - E_i - E'_i|.$$

For each Burniat divisor, we have the option to replace it (or not) by a strictly extended Burniat divisor. By taking the corresponding bidouble cover, we obtain an extended Burniat surface.

Remark 5.2. 1) Observe that $(D_1 + D_2 + D_3) \in |-3K_{\tilde{Y}}|$ is a reduced normal crossing divisor.

2) Similarly, $(\Delta_1 + \Delta_2 + \Delta_3) \in |-3K_{\tilde{Y}} + \sum N_i|$ is a reduced normal crossing divisor.

3) On the normal Del Pezzo surface Y', for $m = 3$,

Δ_j yields a conic and one line, D_j yields three lines.

In particular, if the conic corresponding to Δ_j specializes to contain the line corresponding to E_{j-1}, we obtain D_2 subtracting the divisor $N_{j-1} + N_{j+1}$ and adding the divisor N_j.

We can now consider (cf. [Cat84], [Cat99]) the associated bidouble covers $S \to \tilde{Y}$ with branch divisors the Burniat divisors, respectively the extended Burniat divisors.

Definition 5.3. *A* tertiary nodal Burniat surface *is obtained, for* $m = 3$*, as a bidouble cover* $S \to \tilde{Y}$ *with branch divisors the three Burniat divisors.*

S is then a minimal surface of general type with $p_g(S) = q(S) = 0$, $K_S^2 = 6 - m$ (cf. [BC10]*).*

If we let some of the three branch divisors be extended Burniat divisors, then we obtain a non minimal surface S' whose minimal model S is called a tertiary extended Burniat surface.

In [BC11] it is shown that these (discontinuous) degenerations of the branch divisors from extended to nodal Burniat surfaces produce a flat family of G-covers $X_t \to Y'$ of the canonical models over a normal Del Pezzo surface of degree $6 - m$ (4, respectively 3).

More precisely, we have the following two auxiliary results:

Proposition 5.4. *There exists a family, with connected base*

$$B \subset \{(C_1, \Gamma_2) | C_1 \in |L - E_1|, \Gamma_2 \in |2L - E_2 - E_3 - E_4 - E_5|\}$$

where C_1 is irreducible and either Γ_2 is irreducible, or splits as $N_1 + E_1 + |L - E_2 - E_3|$), parametrizing a flat family of canonical models, including exactly all the nodal Burniat surfaces and the extended Burniat surfaces with $K_X^2 = 4$.

Proposition 5.5. *There exists a family, with connected base*

$$T \subset \{(\Gamma_1, \Gamma_2, \Gamma_3)\}$$

where $\Gamma_1, \Gamma_2, \Gamma_3$ are as in Definition 5.1, parametrizing a flat family of canonical models, including exactly all the nodal Burniat surfaces and the extended Burniat surfaces with $K_X^2 = 3$.

Remark 5.6. 1) In the nodal Burniat case the surface S does not have an ample canonical divisor K_S, due to the existence of (-2)-curves, which are exactly the inverse images of the (-2)-curves $N_i \subset \tilde{Y}$.

For this reason we call the above Burniat surfaces of *nodal type*. We denote their canonical model by X, and observe that X is a finite bidouble cover of the normal Del Pezzo surface Y'.

For $m = 2$ X has precisely one node (an A_1-singularity, corresponding to the contraction of the (-2)-curve) as singularity. While, for $m = 3$, X has exactly three nodes as singularities.

2) In the extended Burniat case S' is not minimal. In the strictly extended Burniat case the inverse image of each N_i splits as the union of two disjoint (-1)-curves. In this latter case S has ample canonical divisor, hence $S = X$.

3) In all cases, the morphism $X \to Y'$ is exactly the bicanonical map of X (see [BC10]. [BC11]).

4) Nodal Burniat surfaces are parametrized by a family with smooth base of dimension 2 for $m = 2$, of dimension 1 for $m = 3$.

Strictly extended Burniat surfaces are parametrized by a family with smooth base of dimension 3 for $m = 2$, of dimension 4 for $m = 3$.

The key feature is that, both for nodal Burniat surfaces, and for extended Burniat surfaces, the canonical model X is a finite bidouble

cover of a singular Del Pezzo surface Y', which has one node in the case $m = 2$, and three nodes for $m = 3$ (in this case Y' is a cubic surface in $\mathbb{P}^3$).

In this case the direct image $p_*(\mathcal{O}_X)$ splits as a direct sum of four reflexive character sheaves of generic rank 1.
For $K^2 = 3$ it is shown in [BC11] that a small deformation of a Burniat surface or of an extended Burniat surface is a Galois covering with group $(\mathbb{Z}/2\mathbb{Z})^2$ of a cubic surface with three singular points, and with branch locus equal to three plane sections. Hence one sees that the locus of Burniat and extended Burniat surfaces is open. Moreover in loc.cit. it is shown that the closure of the subset corresponding to extended Burniat surfaces with $K_S^2 = 3$ contains G-covers of cubic surfaces with a D_4-singularity, and G-covers of the four nodal cubic.

Yifan Chen shows in his Bayreuth Ph.D. thesis that there are no further degenerations.

Summarizing, we have the following theorem

Theorem 5.7 (Bauer,Catanese, Chen). *The irreducible component $\mathcal{N}$ of the moduli space containing the Burniat surfaces with $K_S^2 = 3$ consists exactly of*

(1) *Burniat surfaces,*
(2) *extended Burniat surfaces,*
(3) *G-covers of a normal cubic with a D_4-singularity,*
(4) *G-covers of the four nodal cubic, which are étale exactly over one of the four nodes.*

Moreover, 1),2),3) are contained in $\mathcal{N} \setminus \partial\mathcal{N}$.

The key technique used in the above theorem (developed in [BC11]) is the one of blowing up and down logarithmic sheaves in order to calculate the tangent cohomology. It would take too long to explain this technique in detail here.

There remains the challenging

Question 5.8. *Is $\mathcal{N}$ a connected component of the moduli space?*

Another approach was proposed to construct a family of surfaces including the tertiary Burniat surfaces, in [NePi11]; the deformation theoretic aspects were not addressed in [NePi11] and it could be interesting to do it.

Using the techniques developed for the above results Yi-fan Chen has been able to prove a conjecture of Mendes-Lopes and Pardini (cf. [MLP04]):

Theorem 5.9 (Y. Chen). *The six dimensional family constructed by Mendes -Lopes and Pardini in* [MLP04], *containing the Keum-Naie surfaces with $K_S^2 = 3$ as a proper algebraic subset, is indeed an irreducible component of the moduli space of surfaces of general type.*

The new idea which made the long sought for proof of the above result possible is the representation of a special surface in the above family as some $(\mathbb{Z}/2\mathbb{Z})^2$-cover of a four nodal cubic and the use of the methods mentioned above (to calculate spaces of sections of logarithmic differential forms on blow ups). Chen shows that the tangent dimension of Kuranishi space is bounded from above by 6, and then the Kuranishi inequality gives that the dimension is exactly 6.

6. Deformation of automorphisms.

Here is what we have learnt from extended Burniat surfaces.

In this section S will be the minimal model of a nodal Burniat surface with $K_S^2 = 4$ or $K_S^2 = 3$, and X its canonical model. Observe that for $K^2 = 4$, X has one ordinary node, while for $K^2 = 3$, X has three ordinary nodes.

A very surprising and new phenomenon occurs for these surfaces, confirming Vakil's 'Murphy's law' philosophy ([Va06]).

To explain what happens for the moduli spaces of extended and nodal Burniat surfaces, let us recall again an old result due to Burns and Wahl (cf. [BW74]).

Let S be a minimal surface of general type and let X be its canonical model. Denote by $\mathrm{Def}(S)$, resp. $\mathrm{Def}(X)$, the base of the Kuranishi family of S, resp. of X.

Their result explains the relation between $\mathrm{Def}(S)$ and $\mathrm{Def}(X)$.

Theorem 6.1 (Burns - Wahl). *Assume that K_S is not ample and let $p : S \to X$ be the canonical morphism.*

Denote by $\mathcal{L}_X$ the space of local deformations of the singularities of X and by $\mathcal{L}_S$ the space of deformations of a neighbourhood of the exceptional curves of p. Then $\mathrm{Def}(S)$ is realized as the fibre product associated to the Cartesian diagram

$$\begin{array}{ccc} \mathrm{Def}(S) & \longrightarrow & \mathcal{L}_S \cong \mathbb{C}^\nu, \\ \downarrow & & \downarrow \lambda \\ \mathrm{Def}(X) & \longrightarrow & \mathcal{L}_X \cong \mathbb{C}^\nu, \end{array}$$

where ν is the number of rational (-2)-curves in S, and λ is a Galois covering with Galois group $W := \oplus_{i=1}^r W_i$, the direct sum of the Weyl groups W_i of the singular points of X.

An immediate consequence is the following

Corollary 6.2. (Burns - Wahl) *1) $\psi : \mathrm{Def}(S) \to \mathrm{Def}(X)$ is a finite morphism, in particular, ψ is surjective.*
2) If the derivative of $\mathrm{Def}(X) \to \mathcal{L}_X$ is not surjective (i.e., the singularities of X cannot be independently smoothened by the first order infinitesimal deformations of X), then $\mathrm{Def}(S)$ is singular.

Assume now that we have $1 \neq G \leq \mathrm{Aut}(S) = \mathrm{Aut}(X)$.

Then we can consider the space of G-invariant local deformations of S, $\mathrm{Def}(S,G)$, resp. $\mathrm{Def}(X,G)$ of X, and we have a natural map $\mathrm{Def}(S,G) \to \mathrm{Def}(X,G)$.

We indeed show here that, unlike the case for the corresponding morphism of local deformation spaces, this map needs not to be surjective; and, as far as we know, the following result gives the first global example of such a phenomenon.

Theorem 6.3. *The deformations of nodal Burniat surfaces with $K_S^2 = 4,3$ to extended Burniat surfaces with $K_S^2 = 4,3$ yield examples where $\mathrm{Def}(S,(\mathbb{Z}/2\mathbb{Z})^2) \to \mathrm{Def}(X,(\mathbb{Z}/2\mathbb{Z})^2)$ is not surjective.*

Moreover, $\mathrm{Def}(S,(\mathbb{Z}/2\mathbb{Z})^2) \subsetneq \mathrm{Def}(S)$, whereas for the canonical model we have: $\mathrm{Def}(X,(\mathbb{Z}/2\mathbb{Z})^2) = \mathrm{Def}(X)$.

The moduli space of pairs, of an extended (or nodal) Burniat surface with $K_S^2 = 4,3$ and a $(\mathbb{Z}/2\mathbb{Z})^2$-action, is disconnected; but its image in the moduli space is a connected open set.

The reason for this phenomenon can already be seen locally around the node.

Let G be the group $G \cong (\mathbb{Z}/2)^2$ acting on $\mathbb{C}^3$ as follows:

$$G = \{1, \sigma_1, \sigma_2, \sigma_3 = \sigma_1 + \sigma_2\}$$

acts by $\sigma_1(u,v,w) = (u,v,-w)$, $\sigma_2(u,v,w) = (-u,-v,w)$.

The invariants for the action of G on $\mathbb{C}^3 \times \mathbb{C}$ are:

$$x := u^2, y := v^2, z := uv, s := w^2, t.$$

Observe that the hypersurfaces $X_t = \{(u,v,w) | w^2 = uv + t\}$ are G-invariant, and the quotient X_t/G is the fixed hypersurface

$$Y_t \cong Y_0 = \{(x,y,z) | z^2 = xy\},$$

which has a nodal singularity at the point $x = y = z = 0$.

In fact, G acts on the family $\mathcal{X}_t = \{w^2 = uv + t\}$, which admits a simultaneous resolution only after the base change $\tau^2 = t$: and then we have two small resolutions

$$\mathcal{S} := \{((u,v,w,\tau),\xi) \in \mathcal{X} \times \mathbb{P}^1 | \frac{w-\tau}{u} = \frac{v}{w+\tau} = \xi\},$$

$$\mathcal{S}' := \{((u,v,w,\tau),\eta) \in \mathcal{X} \times \mathbb{P}^1 | \frac{w+\tau}{u} = \frac{v}{w-\tau} = \eta\}.$$

Then it is easy to see that G has several liftings to $\mathcal{S}$, but

- either G acts only as a group of *birational not biregular* automorphisms on $\mathcal{S}$ and leaves τ fixed,
- or, G acts as a group of *biregular* automorphisms on $\mathcal{S}$ but does not leave τ fixed.

Looking at the local picture in detail, one sees how the above family yields a discontinuous variation of the three branch divisors on the blow up $\tilde{Y}_0$ of Y_0 at its singular point.

Acknowledgement. I would like to thank Ingrid Bauer for helping to finish the article without too much delay, the responsibility of misprints (I hope not too many) remaining mine. I would like to thank prof. Katsura and the organizers of the Conference, prof. Iritani, prof. Kawaguchi, prof. Nakaoka, for their kind hospitality.

References

[Akb12] Akbulut, S. *The Catanese-Ciliberto-Mendes Lopes surface*, Journal of Gökova Geometry Topology Volume 5 (2011), 1–17.

[BCGP09] Bauer, I., Catanese, F., Grunewald, F., Pignatelli, R. *Quotients of a product of curves by a finite group and their fundamental groups.* arXiv:0809.3420, to appear in Amer. J. Math..

[BC11b] Bauer, I., Catanese, F. *Burniat surfaces I: fundamental groups and moduli of primary Burniat surfaces* Classification of algebraic varieties, 49-76, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011.

[BC10] Bauer, I., Catanese, F. *Burniat surfaces II: secondary Burniat surfaces form three connected components of the moduli space* Invent. Math. 180 (2010), 559-588.

[BC11a] Bauer, I., Catanese, F., *The moduli space of Keum-Naie surfaces* Groups Geom. Dyn. 5 (2011), no. 2, 231–250.

[BC11] Bauer, I., Catanese, F. *Burniat surfaces III: deformations of automorphisms and extended Burniat surfaces*, arXiv:1012.3770, 40 + 4 pages.

[BC12] Bauer, I., Catanese, F. *Inoue manifolds: Inoue surfaces with $K^2 = 7$, $p_g = 0$ form a connected component of the moduli space.*

[BCG08] Bauer, I., Catanese, F., Grunewald, F., *The classification of surfaces with $p_g = q = 0$ isogenous to a product of curves.* , Pure Appl. Math. Q. **4** (2008), no. 2, part 1, 547–586.

[BCP06a] Bauer, I., Catanese, F., Pignatelli, R.: *The moduli space of surfaces with $p_g = 4$ and $K^2 = 6$* Math. Ann. 336 (2006), no. 2, 421–438.

[BCP06b] Bauer, I., Catanese, F., Pignatelli, R., *Complex surfaces of general type: Some recent progress.* Catanese, Fabrizio (ed.) et al., Global aspects of complex geometry. Berlin: Springer. 1–58 (2006).

[BCP10] Bauer, I., Catanese, F., Pignatelli, R., *Surfaces of general type with geometric genus zero: a survey.* Ebeling, Wolfgang (ed.) et al., Complex and differential geometry. Conference held at Leibniz Universität Hannover, Germany, September 14–18, 2009. Proceedings. Berlin: Springer. Springer Proceedings in Mathematics, 8 (2011), 1–48 .

[BP11a] Bauer, I, Pignatelli, R., *The classification of minimal product-quotient surfaces with $p_g = 0$.* arXiv:1006.3209 to appear in *Math. of Comp.*

[Bom73] Bombieri, E., *Canonical models of surfaces of general type.* Inst. Hautes Études Sci. Publ. Math. No. 42 (1973), 171–219.

[Bu66] Burniat, P. *Sur les surfaces de genre $P_{12} > 1$.* Ann. Mat. Pura Appl. (4) 71 (1966) 1–24.

[BW74] Burns, D. M., Jr., Wahl, J. *Local contributions to global deformations of surfaces.* Invent. Math. 26 (1974), 67–88

[CaSt10] Cartwright, D. I.; Steger, T., *Enumeration of the 50 fake projective planes.* C. R. Math. Acad. Sci. Paris 348 (2010), no. 1-2, 11–13.

[Cat81] Catanese, F. *On a class of surfaces of general type.* Algebraic surfaces, 267–284, C.I.M.E. Summer School 1977, 76, Liguori Editore, Napoli (1981), reedited by Springer, Heidelberg, (2010).

[Cat84] Catanese, F. *On the moduli spaces of surfaces of general type.* J. Differential Geom. 19 (1984), no. 2, 483–515.

[Cat87] Catanese, F. *Automorphisms of rational double points and moduli spaces of surfaces of general type.* Compositio Math. 61 (1987), no. 1, 81–102.

[Cat88] Catanese, F. *Moduli of algebraic surfaces.* Theory of moduli (Montecatini Terme, 1985), 1–83, Lecture Notes in Math., 1337, Springer, Berlin, (1988).

[Cat89] Catanese, F. *Everywhere nonreduced moduli spaces.* Invent. Math. 98 (1989), no. 2, 293–310.

[Cat99] Catanese, F. *Singular bidouble covers and the construction of interesting algebraic surfaces.* Algebraic geometry: Hirzebruch 70 (Warsaw, 1998), 97–120, Contemp. Math., 241, Amer. Math. Soc., Providence, RI, (1999).

[Cat00] Catanese, F., *Fibred Surfaces, varieties isogeneous to a product and related moduli spaces.* Amer. J. Math. **122** (2000), no. 1, 1–44.

[Cat08] Catanese, F. *Differentiable and deformation type of algebraic surfaces, real and symplectic structures.* Symplectic 4-manifolds and algebraic surfaces, 55–167, Lecture Notes in Math., 1938, Springer, Berlin, (2008).

[CC91] Catanese, F., Ciliberto, C.: *Surfaces with $p_g = q = 1$.* In: Catanese F. (ed) et al. Problems in the theory of surfaces and their classification (Cortona, 1988), 49–79. Sympos. Math., XXXII. Academic Press, London (1991)

[CC93] Catanese, F., Ciliberto, C.: *Symmetric products of elliptic curves and surfaces of general type with $p_g = q = 1$.* J. Algebraic Geom. 2, no. 3, 389–411 (1993).

[CCML98] Catanese, F., Ciliberto, C., Mendes Lopes, M. *On the classification of irregular surfaces of general type with nonbirational bicanonical map.* Trans. Amer. Math. Soc. 350 (1998), no. 1, 275–308.

[CatPig06] Catanese, F., Pignatelli, R.: *Fibrations of low genus. I.* Ann. Sci. École Norm. Sup. (4) 39 (2006), no. 6, 1011–1049.

[ChCo10] Chan, M., Coughlan, S. *Kulikov surfaces form a connected component of the moduli space.* arXiv:1011.5574, 22 pages.

[ChenHac06] Chen, J. A., Hacon, C. D. *A surface of general type with $p_g = q = 2$ and $K_X^2 = 5$.* Pacific J. Math. 223 (2006), no. 2, 219–228.

[CilML02] Ciliberto, C., Mendes Lopes, M. *On surfaces with $p_g = q = 2$ and non-birational bicanonical maps.* Adv. Geom. 2 (2002), no. 3, 281–300.

[Deb82] Debarre, O., *Inégalités numériques pour les surfaces de type général.* With an appendix by A. Beauville. Bull. Soc. Math. France 110 (1982), no. 3, 319–346.

[Dolga81] Dolgachev, I. *Algebraic surfaces with $q = p_g = 0$.* Algebraic surfaces, 97-215, C.I.M.E. Summer School 1977, 76, Liguori Editore, Napoli (1981), reedited by Springer, Heidelberg, (2010).

[EnrMS] Enriques, F., *Memorie scelte di geometria, vol. I, II, III.* Zanichelli, Bologna, 1956, 541 pp., 1959, 527 pp., 1966, 456 pp.

[Fra11] Frapporti, D., *Mixed surfaces, new surfaces of general type with $p_g = 0$ and their fundamental group.* arXiv:1105.1259

[Gies77] Gieseker, D. *Global moduli for surfaces of general type.* Invent. Math. 43 (1977), no. 3, 233–282.

[HaPa02] Hacon, C.D., Pardini, R.: *Surfaces with $p_g = q = 3$.* Trans. Amer. Math. Soc. 354, no. 7, 2631–2638 (2002).

[Hor78] Horikawa, E.: *Algebraic surfaces of general type with small c_1^2 III.* Invent. Math. 47, no. 3, 209–248 (1978)

[Ino94] Inoue, M., *Some new surfaces of general type.* Tokyo J. Math. 17 (1994), no. 2, 295–319.

[KK02] Kulikov, V. S., Kharlamov, V. M., *On real structures on rigid surfaces.* Izv. Ross. Akad. Nauk Ser. Mat. 66 (2002), no. 1, 133–152; translation in Izv. Math. 66 (2002), no. 1, 133–150 .

[Kim05] Kimura, S., *Chow groups are finite dimensional, in some sense.* Math. Ann. **331** (2005), no. 1, 173–201.

[Ku04] Kulikov, V., S. *Old examples and a new example of surfaces of general type with $p_g = 0$.* (Russian) Izv. Ross. Akad. Nauk Ser. Mat. 68 (2004), no. 5, 123–170; translation in Izv. Math. 68 no. 5, (2004), 965–1008.

[Man03] Manetti, M.: *Surfaces of Albanese general type and the Severi conjecture.* Math. Nachr. 261/262, 105–122 (2003).

[MLP01] Mendes Lopes, M., Pardini, R. *A connected component of the moduli space of surfaces with $p_g = 0$.* Topology 40 no. 5,(2001), 977–991.

[MLP04] Mendes Lopes, M., Pardini, R. *A new family of surfaces with $p_g = 0$ and $K^2 = 3$.* Ann. Sci. École Norm. Sup. (4) 37 (2004), no. 4, 507–531.

[MP10] Mistretta, E., Polizzi, F., *Standard isotrivial fibrations with $p_g = q = 1$ II,* J. Pure Appl. Algebra **214** (2010), 344–369.

[Miy76] Miyaoka, Y. *Tricanonical maps of numerical Godeaux surfaces.* Invent. Math. **34** (1976), no. 2, 99–111.

[Miy77a] Miyaoka, Y., *On numerical Campedelli surfaces.* Complex analysis and algebraic geometry, 113–118. Iwanami Shoten, Tokyo, 1977.

[Miy77b] Miyaoka, Y. *On the Chern numbers of surfaces of general type.* Invent. Math. **42** (1977), 225–237.

[Miy82] Miyaoka, Y. *Algebraic surfaces with positive indices.* Classification of algebraic and analytic manifolds (Katata, 1982), 281–301, Progr. Math. **39**, Birkhäuser Boston, Boston, MA, 1983.

[NePi11] Neves, J. , Pignatelli, R. *Unprojection and deformations of tertiary Burniat surfaces*, arXiv:1101.3160.

[Par91] Pardini, R. *Abelian covers of algebraic varieties.* J. Reine Angew. Math. 417 (1991), 191–213.

[Pe11] M. Penegini, *The classification of isotrivial fibred surfaces with $p_g = q = 2$,* Collectanea Math. (2011).

[Pe-Pol10] M. Penegini, F. Polizzi *On surfaces with $p_g = q = 2, K^2 = 5$ and Albanese map of degree 3*, arXiv:1011.4388.

[Pe-Pol11] M. Penegini, F. Polizzi *Surfaces with $p_g = q = 2, K^2 = 6$ and Albanese map of degree 2*, arXiv:1105.4983.

[Pet77] Peters, C. A. M. *On certain examples of surfaces with $p_g = 0$ due to Burniat.* Nagoya Math. J. 66 (1977), 109–119.

[Pi02] Pirola, G. P. *Surfaces with $p_g = q = 3$.* Manuscripta Math. 108 (2002), no. 2, 163–170.

[P10] Polizzi, F., *Numerical properties of isotrivial fibrations.* Geom. Dedicata 147 (2010), 323–355.

[P06] Polizzi, F., *Surfaces of general type with $p_g = q = 1$, $K^2 = 8$ and bicanonical map of degree 2.* Trans. Amer. Math. Soc. 358 (2006), no. 2, 759–798.

[PY07] Prasad, G., Yeung, S., *Fake projective planes* Invent. Math. **168** (2007), no. 2, 321–370, addendum Invent. Math. 182 (2010), no. 1, 213–227.

[Rei78] Reid, M., *Surfaces with $p_g = 0$, $K^2 = 1$.* J. Fac. Sci. Tokyo Univ. **25** (1978), 75–92

[Va06] Vakil, R. *Murphy's law in algebraic geometry: badly-behaved deformation spaces.* Invent. Math. 164 (2006), no. 3, 569–590.

[Yau77] Yau, S.T., *Calabi's conjecture and some new results in algebraic geometry.* Proc. Nat. Acad. Sci. U.S.A. **74** (1977), no. 5, 1798–1799.

[Yau78] Yau, S.T., *On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampére equation. I.* Comm. Pure Appl. Math. **31** (1978), no. 3, 339–411.

[Zuc03] Zucconi, F., *Surfaces with $p_g = q = 2$ and an irrational pencil.* Canad. J. Math. 55 (2003), no. 3, 649–672.

Author's Address:
Fabrizio Catanese
Lehrstuhl Mathematik VIII
Mathematisches Institut der Universität Bayreuth, NW II
Universitätsstr. 30; D-95440 Bayreuth, Germany