代数幾何学シンポジウム記録
2011年度 pp.65-69

QUARTIC K3 SURFACES AND CREMONA TRANSFORMATIONS - A QUESTION OF MARAT GIZATULLIN.

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ABSTRACT. We prove that there is a smooth quartic K3 surface automorphism that is not derived from the Cremona transformation of the ambient three-dimensional projective space. This gives a negative answer to a question of Professor Marat Gizatullin.

1. INTRODUCTION

Throughout this note, we work over the complex number field $\mathbf{C}$.

In his lecture "Quartic surfaces and Cremona transformations" in the workshop on Arithmetic and Geometry of K3 surfaces and Calabi-Yau threefolds held at the Fields Institute (August 16-25, 2011), Professor Igor Dolgachev discussed the following question with several beautiful examples supporting it:

Question 1.1. Let $S \subset \mathbf{P}^3$ be a smooth quartic K3 surface. Is any biregular automorphism g of S (as abstract variety) derived from a Cremona transformation of the ambient space $\mathbf{P}^3$? More precisely, is there a birational automorphism $\tilde{g}$ of $\mathbf{P}^3$ such that $\tilde{g}_*(S) = S$ and $g = \tilde{g}|S$? Here $\tilde{g}_*(S)$ is the proper transform of S and $\tilde{g}|S$ is the, necessarily biregular, birational automorphism of S induced then by $\tilde{g}$.

Later, Professor Igor Dolgachev pointed out to me that, to his best knowledge, Professor Marat Gizatullin is the first who asked this question. The aim of this short note is to give a negative answer to the question:

Theorem 1.2. *(1) There exists a smooth quartic K3 surface $S \subset \mathbf{P}^3$ of Picard number 2 such that* $\mathrm{Pic}\,(S) = \mathbf{Z}h_1 \oplus \mathbf{Z}h_2$ *with intersection form:*

$$((h_i.h_j)) = \begin{pmatrix} 4 & 20 \\ 20 & 4 \end{pmatrix} .$$

(2) Let S be as above. Then $\mathrm{Aut}\,(S)$ *has an element g such that it is of infinite order and $g^*(h) \neq h$. Here* $\mathrm{Aut}\,(S)$ *is the group of biregular automorphisms of S as abstract variety and $h \in \mathrm{Pic}\,(S)$ is the hyperplane section class.*

(3) Let S and g be as above. Then there is no element $\tilde{g}$ of $\mathrm{Bir}\,(\mathbf{P}^3)$ *such that $\tilde{g}_*(S) = S$ and $g = \tilde{g}|S$. Here* $\mathrm{Bir}\,(\mathbf{P}^3)$ *is the Cremona group of $\mathbf{P}^3$, i.e., the group of birational autmorphisms of $\mathbf{P}^3$.*

1991 *Mathematics Subject Classification.* 14J28, 14E07, 14J50.

supported by JSPS Gran-in-Aid (B) No 22340009, JSPS Grant-in-Aid (S), No 22224001, and by KIAS Scholar Program.

Our proof is based on a result of Takahashi concerning log Sarkisov program ([Ta]), which we quote as Theorem (3.1), and standard argument concerning K3 surfaces.

Remark 1.3. (1) Let $C \subset \mathbf{P}^2$ be a smooth cubic curve, i.e., a smooth curve of genus 1. It is classical that any element of $\mathrm{Aut}\,(C)$ is derived from a Cremona transformation of the ambient space $\mathbf{P}^2$. In fact, this follows from the fact that any smooth cubic curve is written in the Weierstrass form after linear change of the coordinate and the explicit form of the group law in terms of the coordinate.

(2) Let n be an integer such that $n \geq 3$ and $Y \subset \mathbf{P}^{n+1}$ be a smooth hypersurface of degree $n+2$. Then Y is an n-dimensional Calabi-Yau manifold. It is well-known that $\mathrm{Bir}\,(Y) = \mathrm{Aut}\,(Y)$, it is a finite group and any element of $\mathrm{Aut}\,(Y)$ is derived from a biregular automorphism of the ambient space $\mathbf{P}^{n+1}$. In fact, the statement follows from $K_Y = 0$ in $\mathrm{Pic}\,(Y)$ (adjunction formula), $H^0(T_Y) = 0$ (by $T_Y \simeq \Omega_Y^{n-1}$ together with Hodge symmetry), and $\mathrm{Pic}\,(Y) = \mathbf{Z}h$, where h is the hyperplane class (Lefschetz hyperplane section theorem). We note that $K_Y = 0$ implies that any birational automorphism of Y is isomorphic in codimension one, so that for any birational automorphism g of Y, we have a well-defined group isomorphism g^* on $\mathrm{Pic}\,(Y)$. Then $g^*h = h$. This implies that g is biregular and it is derived from an element of $\mathrm{Aut}\,(\mathbf{P}^{n+1})$.

Acknowledgement. I would like to express my thank to Professor Igor Dolgachev for his kind explanation on this question with history, the organizers of the workshop "Arithmetic and Geometry of K3 surfaces and Calabi-Yau threefolds" held at the Fields Institute (August 16-25, 2011) for invitation with full financial support. I also would like to express my thank to the organizers of Kinosaki symposium of algebraic geometry (2011) for giving me an opportunity to talk there.

2. Proof of Theorem (1.2) (1), (2)

In this section, we shall prove Theorem (1.2)(1)(2) by dividing into several steps. The last lemma (Lemma (2.6)) will be also used in the proof of Theorem (1.2) (3).

Lemma 2.1. *There is a projective K3 surface such that* $\mathrm{Pic}\,(S) = \mathbf{Z}h_1 \oplus \mathbf{Z}h_2$ *with*

$$((h_i.h_j)) = \begin{pmatrix} 4 & 20 \\ 20 & 4 \end{pmatrix} .$$

Proof. Note that the abstract lattice given by the symmetric matrix above is even lattice of rank 2 with signature $(1,1)$. Hence the result follows from [Mo], Corollary (2.9), which is based on the surjectivity of the period map for K3 surfaces (see eg. [BHPV], Page 338, Theorem 14.1) and Nikulin's theory ([Ni]) of integral bilinear form. □

From now, S is a K3 surface in Lemma (2.1).

Note that the cycle map $c_1 : \mathrm{Pic}\,(S) \to \mathrm{NS}\,(S)$ is an isomorphism for a K3 surface. So, we identify these two spaces. $\mathrm{NS}\,(S)_{\mathbf{R}}$ is $\mathrm{NS}\,(S) \otimes_{\mathbf{Z}} \mathbf{R}$. A positive cone $P(S)$ of S is the connected component of the set

$$\{x \in \mathrm{NS}\,(S)_{\mathbf{R}} \,|\, (x^2)_S > 0\} ,$$

containing the ample classes. Ample cone $\mathrm{Amp}\,(S) \subset \mathrm{NS}\,(S)_{\mathbf{R}}$ of S is the open convex cone generated by the ample classes.

Lemma 2.2. NS (S) *represents neither* 0 *nor* -2. *In particular,* S *has no smooth rational curve and no smooth elliptic curve and* $(C^2)_S > 0$ *for all non-zero effective curve* C *in* S.

Proof. We have $((xh_1 + yh_2)^2)_S = 4(x^2 + 10xy + y^2)$. Hence there is no $(x, y) \in \mathbf{Z}^2$ such that $((xh_1 + yh_2)^2)_S \in \{-2, 0\}$. □

Lemma 2.3. *After replacing* h_1 *by* $-h_1$, *the line bundle* h_1 *is very ample. In particular* $\Phi_{|h_1|} : S \to \mathbf{P}^3$ *is an isomorphism onto a smooth quartic surface.*

Proof. h_1 is non-divisible in Pic (S) by the construction. It follows from Lemma (2.2) and $(h_1^2)_S = 4 > 0$ that one of $\pm h_1$ is ample with no fixed component. By replacing h_1 by $-h_1$, we may assume that it is h_1. Then, by [SD], Theorem 6.1, h_1 is a very ample line bundle with the last assertion. □

By Lemma (2.3), *we may and will assume that* $S \subset \mathbf{P}^3$ *and denote this inclusion by* ι, *and a general hyperplane section by* h. *That is,* $h = H \cap S$ *for a general hyperplane* $H \subset \mathbf{P}^3$, *from now on.* Note that $h = h_1$ in Pic (S).

Lemma 2.4. Amp $(S) = P(S)$ *and it is*

$$\{xh_1 + yh_2 \in \mathrm{NS}\,(S)_{\mathbf{R}} \,|\, x > -(5 + 2\sqrt{6})y, x > -(5 - 2\sqrt{6})y, x + 5y > 0\} \ .$$

In particular, it is an irrational cone.

Proof. By Lemma (2.2), (2.3), Amp $(S) = P(S)$ and $xh_1 + yh_2 \in \mathrm{NS}\,(S)_{\mathbf{R}}$ is in $P(S)$ if and only if

$$((xh_1 + yh_2)^2)_S > 0 \ , \ ((xh_1 + yh_2).h_1)_S > 0 \ .$$

By the explicit form of $((h_i.h_j)_S)$, the result follows from these two inequalities. □

Lemma 2.5. *There is an automorphism* g *of* S *such that* g *is of infinite order and* $g^*(h) \neq h$ *in* Pic (S).

Proof. Let $^* : \mathrm{Aut}\,(S) \to \mathrm{O}(\mathrm{NS}\,(S)))$ be the natural representation. Let $G < \mathrm{O}(\mathrm{NS}\,(S))$ be the image.

G naturally acts on Amp (S). By Sterk [St], Lemma (2.3), (2.4), this action has a finite rational polyhedral fundamental domain, say Δ. We may choose Δ so that $h \in \Delta$ (it can be in one of the two boundary rays). Since Amp (S) is irrational by Lemma (2.4), it follows that $|G| = \infty$. By the Burnside property of the linear group [Bu], it follows that G has then an element of infinite order, say τ^* with $\tau \in \mathrm{Aut}\,(S)$. We have $(\tau^*)^2(h) \neq h$ (even if h belongs to one of the two boundary rays of Δ). This is because $\dim \mathrm{NS}\,(S) = 2$ and τ^* is of infinite order. Hence $g = \tau^2$ satisfies the requirement. □

Let g be as in Lemma (2.5). Then the pair $(S \subset \mathbf{P}^3, g)$ satisfies all the requirement of Theorem (1.2)(1), (2).

Lemma 2.6. *Let* $(S \subset \mathbf{P}^3, g)$ *be as in Theorem (1.2)(1),(2). Let* $C \subset S$ *be a non-zero effective curve of degree* < 16, *i.e.,*

$$(C, h)_S = (C.H)_{\mathbf{P}^3} < 16 \ .$$

Then $C = S \cap T$ *for some hypersurface* T *in* $\mathbf{P}^3$.

Proof. Recall that $h = h_1$ in $\mathrm{Pic}\,(S)$. There are $m, n \in \mathbf{Z}$ such that $C = mh_1 + nh_2$ in $\mathrm{Pic}\,(S)$. Then

$$(C.h)_S = 4(m + 5n) > 0\ ,\ \ (C^2)_S = 4(n^2 + 10mn + m^2) > 0\ .$$

Here the last inequality follows from Lemma (2.2). Thus, if $(C, h)_S < 16$, then $m + 5n$ is either 1, 2 or 3 by $m, n \in \mathbf{Z}$. Hence we have either one of:

$$m = 1 - 5n\ ,\ ,m = 2 - 5n\ ,\ \ m = 3 - 5n\,.$$

Substituting into $n^2 + 10mn + m^2 > 0$, we obtain one of either

$$1 - 24n^2 > 0\ ,\ \ 4 - 24n^2 > 0\ ,\ \ 9 - 24n^2 > 0\ .$$

Since $n \in \mathbf{Z}$, it follows that $n = 0$ in each case. Therefore, in $\mathrm{Pic}\,(S)$, we have $C = mh$ for some $m \in \mathbf{Z}$. Since $H^1(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(\ell)) = 0$ for all $\ell \in \mathbf{Z}$, the natural restriction map

$$\iota^* : H^0(\mathbf{P}^3, \mathcal{O}_{\mathbf{P}^3}(m)) \to H^0(S, \mathcal{O}_S(m))$$

is surjective for all $m \in \mathbf{Z}$. This implies the result. □

3. Proof of Theorem (1.2) (3)

In his paper [Ta], Theorem 2.3 and Remark 2.4, N. Takahashi proved the following remarkable theorem as a nice application of log Sarkisov program (For terminologies, we refer to [KMM]):

Theorem 3.1. *Let X be a Fano manifold of dimension ≥ 3 with Picard number 1, $S \in |-K_X|$ be a smooth hypersurface. Let $\Phi : X - - \to X'$ be a birational map to a $\mathbf{Q}$-factorial termial variety X' with Picard number 1, which is not an isomorphism, and $S' := \Phi_* S$. Then:*

(1) If $\mathrm{Pic}\,(X) \to \mathrm{Pic}\,(S)$ is surjectve, then $K_{X'} + S'$ is ample.

(2) Let $X = \mathbf{P}^3$ and H be a hyperplane of $\mathbf{P}^3$. Note then that S is a smooth quartic K3 surface. Assume that any irreducible reduced curve $C \subset S$ such that $(C.H)_{\mathbf{P}^3} < 16$ is of the form $C = S \cap T$ for some hypersurface $T \subset \mathbf{P}^3$. Then $K_{X'} + S'$ is ample.

Applying Theorem (3.1)(2), we shall complete the proof of Theorem (1.2)(3). Let $(S \subset \mathbf{P}^3, g)$ be the pair constructed in Theorem (1.2)(1)(2). We argue by contradiction, i.e., *Assuming to the contrary that there would be a birational map $\tilde{g} : \mathbf{P}^3 - - \to \mathbf{P}^3$ such that $\tilde{g}_*(S) = S$ and that $g = \tilde{g}|S$, we shall derive a contradiction.*

We shall devide into two cases:

(i) $\tilde{g}$ is an isomorphism, (ii) $\tilde{g}$ is not an isomorphism.

Case (i). Since $\tilde{g}$ is an isomorphism, $\tilde{g}^* H = H$ in $\mathrm{Pic}\,(\mathbf{P}^3)$. However, then $g^* h = h$ in $\mathrm{Pic}\,(S)$, a contradiction to Theorem (1.2)(2).

Case (ii). By the case assumption and Lemma (2.6), our S would satisfy all the conditions of Theorem (3.1)(2). Recall also that $\tilde{g}_* S = S$. However, then, by Theorem (3.1)(2), $K_{\mathbf{P}^3} + S$ would be ample, a contradiction to $K_{\mathbf{P}^3} + S = 0$ in $\mathrm{Pic}\,(\mathbf{P}^3)$.

This completes the proof.

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