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MULTIPLE FIBERS OF ELLIPTIC FIBRATIONS

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ABSTRACT. We review our studies on multiple fibers of elliptic fibrations [Mitc]. We study invariants of elliptic fibrations over a complete discrete valuation ring with an algebraically closed residue field. The invariants appear when we localize invariants appearing in the canonical bundle formula of elliptic surfaces. The study of the invariants is reduced to the case where the reduction of the special fiber is isomorphic to an elliptic curve, and the case is studied. Further, we develop the classification of elliptic fibrations with a multiple fiber of this type. As an application, we determine the combinations of multiple fibers appearing in elliptic surfaces of Kodaira dimension zero. The possible combinations were given by Bombieri and Mumford. As another application, we give methods of resolving multiple fibers via inseparable coverings. In particular, we complete the proof of Katsura and Ueno's resolution of multiple fibers. Our results also provide new methods of constructing unirational surfaces. In particular, we construct irrational elliptic Zariski surfaces of any geometric genus. This result gives a negative answer to Zariski's question in any characteristic.

1. INTRODUCTION

Multiple fibers of elliptic surfaces appear in Kodaira's classification of compact complex analytic surfaces [Kod63, §6]. It is important to study multiple fibers in the classification of elliptic surfaces because invariants associated to multiple fibers determine its Kodaira dimension. Any multiple fiber of multiplicity m of an elliptic fibration over a disk can be resolved by the base change via a totally-ramified cyclic covering of degree m and the normalization. The induced morphism between the total spaces is étale. Conversely, any elliptic fibration with a multiple fiber of multiplicity m can be obtained as the quotient of an equivariant group action of degree m on a smooth elliptic fibration. The quotient morphism between the total spaces is étale.

In the positive characteristic cases, multiple fibers of elliptic surfaces appear in Bombieri and Mumford's classification of algebraic surfaces [BM77]. A fiber of an elliptic fibration $f\colon X \to C$ over a closed point x on C is said to be *wildly ramified* if the point x is contained in the support of the torsion of $R^1 f_* \mathcal{O}_X$. Otherwise, the fiber is said to be *tamely ramified*. In contrast to the characteristic zero case, wildly ramified multiple fibers appear in the positive characteristic cases. Their examples and partial results on their resolution were obtained in [KU85] and [KU86]. We would like to study such multiple fibers in a comprehensive and systematic way. To this end, we study the case where the base space is local.

2. Notation and Terminology

Let R be a complete discrete valuation ring (CDVR) with an algebraically closed residue field k of characteristic p and a field of fractions K. Put $C := \operatorname{Spec} R$. An *elliptic fibration* over C is a proper flat scheme over C that is regular and whose generic fiber is a geometrically connected smooth curve of genus one. An elliptic fibration is said to be *minimal* if the special fiber does not contain an exceptional curve of the first kind. We study elliptic fibrations with a multiple fiber over C and apply the results to the classification of elliptic surfaces.

Let $f\colon X \to C$ be a minimal elliptic fibration. By X_k and X_K we denote the special fiber of f and the generic fiber of f, respectively. Let E be a minimal regular model of the Jacobian E_K of X_K. The model is unique up to isomorphism between the generic fibers. Let ${}_mT$ be the type of X_k where m is the multiplicity and T is the type (Kodaira's symbol) of the combination of the irreducible components of X_k/m. We define integers $u(T)$ and $v(T)$ by Table 1.

T	I_n	I_n^*	II	II*	III	III*	IV	IV*
$u(T)$	1	2	6	6	4	4	3	3
$v(T)$	0	0	4	0	2	0	1	0

TABLE 1. The definition of $u(T)$ and $v(T)$.

The study of X is reduced to the case $T = \mathrm{I}_n$. The case $T = \mathrm{I}_n$ $(n > 0)$ was studied in [LLR04, §8]. Thus, we mainly study the case $T = \mathrm{I}_0$. The results are summarized in the following sections. Let K'/K be a finite extension of degree d. Take the integral closure R' of R in K'. Put $C' := \operatorname{Spec} R'$ and $X_{K'} := X_K \times_K K'$. Let $f'\colon X' \to C'$ be the minimal regular model of $X_{K'}$ over C'. By ${}_{m'}T'$ we denote the type of the special fiber of f'. Let E' be the minimal regular model of the Jacobian of $X_{K'}$. We say that the multiple fiber of f *can be resolved* by K'/K if the equality $m' = 1$ holds.

3. Resolution

If $p \nmid u(T)$ and $d = u(T)$, then ${}_{m'}T' = {}_m\mathrm{I}_{n'}$. Assume that $T = \mathrm{I}_n$. Then there exists a separable extension K'/K of degree m such that ${}_{m'}T' = {}_1\mathrm{I}_{mn}$.

The scheme X' is isomorphic to the normalization of $X \times_C C'$ over C'. In particular, we obtain a morphism $\pi_X \colon X' \to X$.

Case 1: *$p \nmid m$ or $n > 0$.* The extension K'/K is cyclic and the morphism π_X is étale.

Case 2: *Otherwise*. The special fiber E_k of E is an elliptic curve (Theorem 6.6 in [LLR04]). The morphism π_X is not necessarily étale. Assume that K contains all the m-th roots of unity (e.g., R is equi-characteristic) and E_k is ordinary. Then we may suppose that K'/K is a cyclic extension. Without the first assumption, we may only suppose that K'/K is an abelian extension. Without the second assumption (i.e., E_k is supersingular), we may not suppose that K'/K is a Galois extension. In the equi-characteristic case, there exists an example of a minimal elliptic fibration with a multiple fiber of type ${}_p\mathrm{I}_0$ that cannot be resolved by any Galois extension of degree p nor the (unique) purely inseparable extension of degree p. Moreover, the morphism π_X is non-étale for any non-trivial finite extension K'/K.

The proof of the above results follow from the calculation of the invariants of X (§5) and the Galois cohomology group $H^1(K, E_K)$, and the following theorem proved by Liu, Lorenzini, and Raynaud:

Theorem 3.1 (Theorem 6.6 in [LLR04], Corollary 6.7 in [LLR04], and Corollary 7.4 in [LLR04]). *Let E_K be an elliptic curve over C_K. Take an element $\eta \in H^1(K, E_K)$ of order m. Let X_K be the smooth curve over C_K corresponding to η. Let E and X be the minimal regular models of E_K and X_K over C, respectively. Let T be the type of E_k. Then X_k is of type ${}_mT$. If $T \neq I_n$, then m is a power of p.*

Further, the following statement holds: Take a positive integer m'. If $T \neq I_n$, then we assume that m' is a power of p. Then there exists an element of $H^1(K, E_K)$ of order m'.

Finally, we complete the proof of Katsura and Ueno's resolution of multiple fibers [KU85, §6, §7]: when R is equi-characteristic and $T = \mathrm{I}_n$, the multiple fiber can be resolved by a finite succession of finite base changes and the normalizations each of which induces an étale covering or a purely inseparable covering of the total space. More precisely, the following holds:

Theorem 3.2. *Suppose that R is equi-characteristic. Assume that X_k is of type ${}_m\mathrm{I}_n$. Then there exist a finite cyclic extension K'/K (resp. K''/K') and a finite purely inseparable extension K''/K' (resp. K'/K) satisfying the following conditions: Let $\phi \colon X' \to X$ (resp. $\phi \colon X'' \to X'$) be the morphism between the minimal regular models induced by the base change and the normalization. Then ϕ is étale and X'' does not admit a multiple fiber.*

In the proof of Katsura and Ueno, they studied the Frobenius action on the cohomology group $H^1(Y, \mathscr{O}_Y)$ of an elliptic surface Y. In stead of the global case and the Frobenius action, we study the local case and the Galois cohomology group $H^1(K,E_K)$. We may also give more precise statements in some special cases. These results are useful in giving examples of unirational elliptic surfaces. The notion of *strange multiple fiber* was introduced as an obstruction to the proof of Katsura and Ueno's resolution. To avoid this obstruction and treat more general cases, first, we resolve a multiple fiber by a Galois base change and the normalization. The induced covering of the total space can be uniquely divided into two parts: the étale part and the non-étale part. It is relatively easy to study the étale part. Next, we study the non-étale part by explicit calculations of Galois cohomology groups. In particular, we obtain Katsura and Ueno's resolution. When $p \nmid u(T)$, we characterize the multiple fibers that can be resolved by a finite purely inseparable extension:

Theorem 3.3. *Suppose that R is equi-characteristic. Let ${}_mT$ be the type of X_k. Assume that $m > 1$, $p \nmid u(T)$, and $d = u(T)$. Then the multiple fiber of f can be resolved by a finite purely inseparable extension if and only if m is a power of p and one of the following conditions is satisfied:*

(1) $T = \mathrm{I}_0$ *and the special fiber of f is tamely ramified;*
(2) $X'_{k,\mathrm{red}}$ *is isomorphic to a supersingular elliptic curve over k.*

Further, we point out that the main results in [Kaw00] and [Kaw06] are incorrect. The results are based on partial results on the resolution of multiple fibers of elliptic surfaces satisfying the following condition: The reduction of each closed fiber is isomorphic to a supersingular elliptic curve. We give an alternative resolution of multiple fibers of elliptic fibrations of this type without any condition:

Theorem 3.4. *Suppose that R is equi-characteristic. Assume that X_k is of type ${}_m\mathrm{I}_0$ and E_K is a supersingular elliptic curve. For a non-negative integer n, by K_n/K we denote the (unique) purely inseparable extension of degree p^n in K^{alg}. Let R_n be the valuation ring of K_n. Put $C_n := \operatorname{Spec} R_n$. Let X_n/C_n be the minimal regular model of $X \times_C C_n$. Then X_n is canonically isomorphic to the normalization of $X \times_C C_n$ over C_n. Let m_n be the multiplicity of $X_{n,k}$. If $p \mid m_n$, then exactly one of the following equalities holds:*

(1) $(m_{n+1}, m_{n+2}) = (m_n/p, m_{n+1})$;
(2) $(m_{n+1}, m_{n+2}) = (m_n, m_{n+1}/p)$.

4. Construction

Any minimal elliptic fibration over C with a multiple fiber of type ${}_m\mathrm{I}_0$ can be constructed as the GIT quotient of a finite equivariant group action on a smooth elliptic fibration. When $p \nmid u(T)$ and $d = u(T)$, any minimal elliptic fibration over C with a multiple fiber of other type ${}_mT$ can be constructed from a minimal elliptic fibration over C' with a multiple fiber of type ${}_m\mathrm{I}_n$ by quotient and birational transformation. We may give this construction explicitly.

5. INVARIANTS

Let ω_f be the relative dualizing sheaf for f. Put $V := X_k/m$. We study the following invariants (l,a):

(1) The length l of the torsion of the R-module $\Gamma(C, R^1 f_* \mathscr{O}_X)$.
(2) The integer a appearing in the isomorphism

$$\omega_f \cong f^* f_* \omega_f \otimes \mathscr{O}_X(aV)$$

induced by the canonical injective $\mathscr{O}_X$-module homomorphism

$$f^* f_* \omega_f \longrightarrow \omega_f.$$

The inequalities $0 \leq a < m$ hold. If $m = 1$, then $(l,a) = (0,0)$. The following statements hold (Proposition 1 in [Mitb]):

(1) The divisor V_f contains an irreducible component whose multiplicity is equal to m (by the classification of the special fibers). In particular, the equality $m = 1$ holds if and only if the equality $X(K) \neq \emptyset$ holds.
(2) The special fiber of f is tamely ramified if and only if the order of the line bundle $\mathscr{O}_X(V_f)|_{V_f}$ on V_f in the Picard group of V_f is equal to m_f.
(3) If the special fiber of f is tamely ramified, then $a_f = m_f - 1$.
(4) If the special fiber of f is wildly ramified, then $p \mid m_f$.

The invariants (l,a) appear when we localize invariants appearing in the canonical bundle formula of elliptic surfaces (§7). The invariants determine the Kodaira dimensions of the elliptic surfaces (Proposition 7.1). By the same method, we define the invariants (l',a') for X'/C'. We study a relationship between (l,a) and (l',a'). The study of (l,a) can be reduced to the case $T = \mathrm{I}_n$ in any characteristic p except in some small characteristics depending on T:

Theorem 5.1. *Assume that $p \nmid u(T)$ and $d = u(T)$. Then the equality*

$$u(T)(ml + a) = m'l' + a' + v(T)(m-1)$$

holds.

By $d_{C'/C}$ we denote the valuation of the different of C'/C. The case $T = \mathrm{I}_n$ $(n > 0)$ was studied in [LLR04, §8]:

Theorem 5.2 (Proposition 8.11 (b) in [LLR04]). *Assume that $T = \mathrm{I}_n$ $(n > 0)$. Suppose that $d = m$ and $m' = 1$. Then the equality*

$$ml + a = d_{C'/C}$$

holds.

Assume that $T = \mathrm{I}_0$. By $d_{X'/X}$ we denote the valuation of the different of $\pi_X \colon X' \to X$ (§3) along the special fibers. The remaining case $T = \mathrm{I}_0$ is studied:

Theorem 5.3. *Assume that $T = \mathrm{I}_0$. Put $d' := dm'/m$. Then the equality*

$$d'(ml + a) = m'l' + a' + m'd_{C'/C} - d_{X'/X}$$

holds.

The above three theorems follow from a unified theorem including the case $p \mid u(T)$. In the proof, we study the relationship between the homomorphisms

$$\tau_X \colon H^1(X, \mathscr{O}_X) \longrightarrow H^1(E, \mathscr{O}_E)$$

and

$$\tau_{X'} \colon H^1(X', \mathscr{O}_{X'}) \longrightarrow H^1(E', \mathscr{O}_{E'})$$

constructed in Theorem 3.8 in [LLR04]. The canonical isomorphism

$$H^1(X_K, \mathscr{O}_{X_K}) \otimes_K K' \cong H^1(X_{K'}, \mathscr{O}_{X_{K'}})$$

allows us to compare the images of

$$H^1(X, \mathscr{O}_X) \quad \text{and} \quad H^1(X', \mathscr{O}_{X'})$$

in $H^1(X_{K'}, \mathscr{O}_{X_{K'}})$. The difference between the images can be described by the ramifications of $\pi_C \colon C' \to C$ and $\pi_X \colon X' \to X$ if π_X exists. To treat the general case, we study fibrations whose total spaces are not necessarily regular. Using Theorem 3.8 in [LLR04] and the Grothendieck duality, we obtain the unified theorem. Our proof of the second theorem is based on the unified theorem and the fact that any minimal Weierstrass model is normal and its singularities are rational. In particular, the proof is different from that in [LLR04]. Finally, we calculate $d_{X'/X}$ in the third theorem from a cocycle that represents the element of $H^1(K, E_K)$ corresponding to X_K in order to obtain the desired invariants (l, a).

6. Classification of Multiple Fibers of Type ${}_m\mathrm{I}_0$

We count the number M of isomorphism classes of elliptic fibrations with a special fiber of type ${}_m\mathrm{I}_0$, a fixed Jacobian E_K of the generic fibers, and a fixed resolving Galois extension K'/K of degree m. When $p \nmid m$, only tamely ramified multiple fibers appear. There exists a one-to-one correspondence between the isomorphism classes and the elements of $E(k)$ of order m. In particular, the number M is non-zero and finite. When E_k is an ordinary elliptic curve and $p \mid m$, both of tamely and wildly ramified multiple fibers appear. Their numbers are determined by the Galois group of K'/K. The total number M is non-zero and finite. When E_k is a supersingular elliptic curve and $m = p$, only wildly ramified multiple fibers appear. The number M is infinite except in one case where the conductor of K'/K is equal to two. In this exceptional case, the number M is equal to zero.

7. Elliptic Surfaces

Let C be a proper smooth curve over an algebraically closed field k. An *elliptic fibration* over C is a proper flat scheme over C that is regular and whose generic fiber is a geometrically connected smooth curve of genus one. An elliptic fibration over C is said to be *relatively minimal* if any closed fiber does not contain an exceptional curve of the first kind. A proper smooth surface X is called a (*relatively minimal*) *elliptic surface* over C if X admits a (relatively minimal) elliptic fibration over C. Let $f\colon X \to C$ be a relatively minimal elliptic fibration. For each closed point t on C, we write the fiber $f^{-1}(t)$ over t as $m_t D_t$ where m_t is the multiplicity of $f^{-1}(t)$ and D_t is a divisor on X. For each closed point t on C, let l_t be the length of the torsion of the $\mathscr{O}_{C,t}$-module $(R^1 f_* \mathscr{O}_X)_t$. Put

$$l(f) := \sum_{t \in C} l_t.$$

Then the canonical bundle formula (Theorem 2 in [BM77]) gives an isomorphism

$$\mathscr{K}_X \cong f^* \mathscr{L} \otimes_{\mathscr{O}_X} \mathscr{O}_X(D)$$

where the line bundle $\mathscr{L}$ on C and the divisor D on X satisfy the equalities

$$\deg \mathscr{L} = \chi(\mathscr{O}_X) + 2g(C) - 2 + l(f)$$

and

$$D = \sum_{t \in C} a_t D_t.$$

Here, each coefficient a_t is an integer satisfying the inequalities $0 \le a_t < m_t$. We write the combination of the multiple fibers as

$$(a_1/m_1^*, \ldots, a_r/m_r^*, a_{r+1}/m_{r+1}, \ldots, a_s/m_s).$$

Here, the symbol $*$ means that the fiber is wildly ramified. The canonical bundle formula gives the following proposition:

Proposition 7.1. *Put*

$$\delta := \chi(\mathscr{O}_X) + 2g(C) - 2 + l(f) + \sum_{i=1}^{s} \frac{a_i}{m_i}.$$

Then the equalities

$$\kappa(X) = \begin{cases} -\infty, & \delta < 0, \\ 0, & \delta = 0, \\ 1, & \delta > 0 \end{cases}$$

hold.

8. Examples of Elliptic Surfaces

First, we construct previously unknown types of elliptic surfaces with a strange multiple fiber [KU85]. Next, we give examples of all five types of elliptic Enriques surfaces in characteristic two [Kat82, §1]. Finally, we construct irrational elliptic Zariski surfaces of any geometric genus p_g. In particular, when $p_g = 0$, the examples give a negative answer to Zariski's question [BBL94] in any characteristic. To determine the types of surfaces in the second and third examples, we use our result on resolution of multiple fibers by purely inseparable extensions. In our construction, we globalize elliptic fibrations over an equi-characterisitic CDVR by the following proposition:

Proposition 8.1 (Corollary 5.4.6 in [CD89] and Remark 5.4.2 in [CD89]). *For each closed point t on C, let C_t be the completion of the localization of C at t, M_t the function field of C_t, and M the function field of C. Let E be the relatively minimal regular model over C of an elliptic curve E_M over M. Assume that E/C is non-trivial, i.e., there does not exist an elliptic curve E_0 over k such that E/C is given by the second projection $E_0 \times_k C \longrightarrow C$. We denote the generic fiber of $E \times_C C_t/C_t$ by E_{M_t}. Then the global-to-local map*

$$H^1(M, E_M) \longrightarrow \bigoplus_{t \in C} H^1(M_t, E_{M_t})$$

is surjective.

9. Elliptic Surfaces of Kodaira Dimension Zero

We determine the combinations of multiple fibers appearing in relatively minimal elliptic surfaces $f\colon X \longrightarrow C$ of Kodaira dimension zero (Table 2). The possible combinations were given in the complex analytic case (Proposition 3.23 in [FM94]), in the algebraic case [BM77, p. 33], and in the rigid

χ	p_g	g	l	(a_i/m_i)	n	p	N	j	examples or non-existence
0	0	0	0	$(1/2,1/2,1/2,$	2	$\neq 2$	H	C	[H] a, [Mitb]
				$1/2)$		2	Q	O	[Q] a, [Mitb]
				$(2/3,2/3,2/3)$	3	$\neq 3$	H	C	[H] b, [Mitb]
						3	Q	O	[Q] a, [Kat95, §5.2], [Mitb]
				$(1/2,3/4,3/4)$	4	$\neq 2$	H	C	[H] c, [Mitb]
						2	Q	O	[Q] d, [Mitb]
				$(1/2,2/3,5/6)$	6	$\neq 2,3$	H	C	[H] d, [Mitb]
						2	Q	O	[Q] b, [Mitb]
						3	Q	O	[Q] b, [Mitb]
0	0	0	1	$(0/2^{*},1/2,1/2)$	2	2	Q	O	[Q] c, [Mitb]
				$(1/2^{*},1/2)$	2	2			not exist
				$(1/3^{*},2/3)$	3	3	Q	O	[Q] c, [Kat95, §5.2], [Mitb]
				$(1/4^{*},3/4)$	4	2	Q	O	[Q] e
				$(2/4^{*},1/2)$	2	2			not exist
				$(2/6^{*},2/3)$	3	2	H	O	[H] d, [KU85, §8.4]
							Q	S	[Q] g
				$(3/6^{*},1/2)$	2	3	H	O	[H] d, [KU85, §8.4]
							Q	S	[Q] e
0	1	0	2	$(0/p^{u*})$					
				$u=1$	1	2	H	O	[H] a, [KU85, §8.1]
							Q	S	[Q] f ($\lambda=0$), [Q] h
				$u=1$	1	3	H	O	[H] b, [KU85, §8.1]
							Q	S	[Q] d, [Q] f
				$u=2$	1	2	H	O	[H] c, [KU85, §8.2]
				the other cases	1	>0			not exist
				$(0/p^{u*},0/p^{v*})$					
				$u=1, v=1$	1	2	H	O	[H] a
							Q	S	[Q] f ($\lambda\neq 0$)
				the other cases	1	>0			not exist
0	1	1	0	none	1	any	A	C	the product of elliptic curves
						2	H	C	[H] a, [H] c,
									[KU85, §8.1, §8.2]
						3	H	S	[H] b, [KU85, §8.1]
0	0	1	0	none	2	$\neq 2$	H	C	[H] a, [Mitb]
					3	$\neq 3$	H	C	[H] b, [Mitb]
					4	$\neq 2$	H	C	[H] c, [Mitb]
					6	$\neq 2,3$	H	C	[H] d, [Mitb]
1	0	0	0	$(1/2,1/2)$	2	any	E		§8
1	1	0	1	$(0/2^{*})$	1	2	N		§8
2	1	0	0	none	1	any	K		$K3$ Kummer surfaces

TABLE 2. The invariants of relatively minimal elliptic surfaces $f\colon X \to C$ of Kodaira dimension zero. See §9 for the notation. The columns N and j are valid when X is an algebraic surface. We use the following abbreviation: [H]=[BM77, §3, p. 37], [Q]=[BM76, §2, p. 214].

analytic case [Mita]. Examples or non-existence theorems are given in each possible case.

Let us explain the notation in Table 2. By χ (resp. p_g, g, l, n, p) we denote $\chi(\mathscr{O}_X)$ (resp. the geometric genus of X, the genus of C, the length of the torsion of the R-module $\Gamma(C, R^1 f_* \mathscr{O}_X)$, the order of the canonical bundle $\mathscr{K}_X$ of X in the Picard group of X, the characteristic of the base field). By (a_i/m_i) we denote the combination of multiple fibers. Assume that X is an algebraic surface. Then the column N gives the name of X ([BM77] and [BM76]): K: $K3$ surface, E: classical Enriques surface, N: non-classical Enriques surface, A: abelian surface (the product of two elliptic curves), H: hyperelliptic surface, Q: quasi-hyperelliptic surface. The column j gives the j-invariant of the Jacobian fibration associated to f in the following way: C: any element of k, O: any constant ordinary j-invariant in k, S: any supersingular j-invariant in k.

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