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Moduli Spaces of Tautological Sheaves on degree two Hilbert Schemes

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Introduction

Since the proof of the Calabi conjecture by Yau in '77 the classification of compact Ricci-flat Kähler manifolds has made big progress. By the theorems of de Rham, Berger and Beauville every such manifold is – up to finite covering – isomorphic to a product of three kinds of basic building blocks: complex tori, so-called strict Calabi-Yau manifolds and hyperkähler manifolds. Whereas for strict Calabi-Yaus there is a vast range of examples, the situation in the hyperkähler case is much different:

Hyperkähler/IHS manifolds

Definition. A compact complex manifold X is called *hyperkähler manifold* or *irreducible holomorphic symplectic manifold (IHS)* if the following conditions are satisfied:

- $\pi_1(X) = 0$ and
- $H^2(X) = \mathbb{C} \cdot \omega$, where ω is a nowhere degenerate closed holomorphic two-form (also called holomorphic symplectic form).

There is a quite short list of known examples:

- Moduli spaces of sheaves on K3 surfaces. *This includes K3 surfaces and Hilbert schemes of points on these surfaces.*
- Two sporadic examples by O'Grady. *They as well are constructed from moduli spaces of sheaves on K3 surfaces.*
- *Generalized Kummer Varieties.*

Since the list of examples is this short one big aim of mathematicians working with holomorphic symplectic varieties is to construct new examples. As can be seen above, the starting point for most of the examples are moduli spaces of sheaves on a K3 surface. Such a surface is an IHS itself. So we end up with the following central question:

Questions

Let X be a projective IHS manifold. Let M be a moduli space of sheaves on X.
Does M admit a symplectic structure?
Or even:
Does M admit a symplectic resolution being again an IHS manifold?

Almost nothing is known in this direction and, as it seems, it is very difficult to answer this question in this generality. Therefore we pick for our IHS manifold one of the most basic examples, the Hilbert square $X^{[2]}$ of a projective K3 surface X. Then let us formulate a short list of seemingly more achievable aims:

Aims

- Find examples of stable sheaves on $X^{[2]}$.
- Find conditions such that the moduli spaces of these sheaves are smooth.
- Construct symplectic structures on these moduli spaces.

Tautological Sheaves 1 - Stability

The theory of moduli spaces of sheaves on K3 surfaces is very well understood. It is based on the fundamental paper on the symplectic structure and smoothness of these spaces by Mukai (cf. [Muk]). The central result is the following:

Theorem (Mukai). *Let X be a projective K3 surface. Then the moduli space of stable sheaves with fixed numerical data is a smooth quasi-projective variety admitting a symplectic structure.*

If one also considers semistable sheaves then the moduli spaces become compact but often singular. There is a complete classification of the possible cases and of possible symplectic resolutions due to the work of Yoshioka, Lehn, O'Grady, Zowislok and many others.
The idea now is to use these results and to transfer them to the Hilbert square. There is a big class of sheaves on the Hilbert square of a surface which is quite well understood: the so-called tautological sheaves.

Tautological Sheaves

Let X be a projective K3 surface and denote by $X^{[2]}$ its Hilbert square. It consists of zero dimensional length two subschemes $\xi \subset X$. On the product $X \times X^{[2]}$ there exists a universal family $\Xi := \{(x,\xi) | x \in \xi\}$. Let $p\colon X \times X^{[2]} \to X$ and $q\colon X \times X^{[2]} \to X^{[2]}$ denote the first and second projection and let $\mathcal{F}$ be a sheaf on X. We set $\mathcal{F}^{[2]} := q_*(\mathcal{O}_\Xi \otimes p^*\mathcal{F})$. Since $q|_\Xi\colon \Xi \to X^{[2]}$ is two-to-one this is a sheaf on $X^{[2]}$ and we will call it *the tautological sheaf associated with $\mathcal{F}$*. Note that if $\mathcal{F}$ is a vector bundle then $\mathcal{F}^{[2]}$ is again a vector bundle where the rank of $\mathcal{F}^{[2]}$ is twice the rank of $\mathcal{F}$.

These tautological objects where studied by several people with very different interests like Ellingsrud, Göttsche and Lehn, Boissière and Nieper-Wikirchen, Danila, Scala, and Krug. The most useful results for us concern the cohomology and extension groups of tautological sheaves.

Theorem (Scala ([Sca]), Krug ([Kru])). *Let $\mathcal{F}$ and $\mathcal{G}$ be sheaves on a projective surface X. Then we have the following formulas for the cohomology and the extension groups of the associated tautological sheaves:*

$$\mathrm{H}^*(X^{[2]}, \mathcal{F}^{[2]}) \cong \mathrm{H}^*(X,\mathcal{F}) \otimes \mathrm{H}^*(X, \mathcal{O}_X)$$
and
$$\mathrm{Ext}^*_{X^{[2]}}(\mathcal{F}^{[2]}, \mathcal{G}^{[2]}) \cong \mathrm{Ext}^*_X(\mathcal{F},\mathcal{G}) \otimes \mathrm{H}^*(X,\mathcal{O}_X) \oplus \mathrm{H}^*(X,\mathcal{F}^\vee) \otimes \mathrm{H}^*(X,\mathcal{G}).$$

On a K3 surface a stable sheaf $\mathcal{F}$ always satisfies $h^2(\mathcal{F}) \cdot h^0(\mathcal{F}) = 0$ since otherwise the structure sheaf would be a destabilizing quotient or subbundle. Thus we get the following consequence from the theorem:

Corollary. *Let $\mathcal{F}$ be a stable sheaf on a projective K3 surface. Then the associated tautological sheaf is simple.*

Now this is a first hint that the tautological sheaf associated with a stable sheaf could be again stable. The first result in this direction was proven by Schlickewei (cf. [Schl]). I could prove an extensive generalisation of it (cf. [Wan]):

Stability of Tautological Sheaves

Theorem. *Let (X,H) be a polarized K3 surface and let $\mathcal{F}$ be either a torsion free rank one sheaf or a μ_H-stable rank two vector bundle on X. Assume $c_1(\mathcal{F}) \neq 0$. Then $\mathcal{F}^{[2]}$ is a rank two (rank four resp.) μ_{H_N}-stable sheaf where H_N is a carefully chosen polarization on $X^{[2]}$ depending on H.*

Example. We explicitely excluded the case $\mathcal{F} = \mathcal{O}_X$. In fact, for the associated rank two tautological bundle $\mathcal{O}_X^{[2]}$ we can find a destabilizing subbundle, namely the trivial bundle:

$$\mathcal{O}_{X^{[2]}} \hookrightarrow \mathcal{O}_X^{[2]}.$$

Tautological Sheaves 2 - Deformations

So we have found a big number of examples of stable sheaves on the Hilbert square. We can therefore consider the moduli space of these sheaves. The most important question then would be whether these spaces are smooth manifolds or singular spaces. In order to answer this question we have to study the deformation theory of tautological sheaves. Therefore we fix some stable sheaf $\mathcal{F}$ on X such that $\mathcal{F}^{[2]}$ is again stable. We may assume that $h^2(\mathcal{F}) = 0$. Again, the tangent space of the moduli space at the point corresponding to $\mathcal{F}^{[2]}$ is naturally isomorphic to the group of infinitessimal deformations of $\mathcal{F}^{[2]}$ which is given by the first extension group. Looking at Krug's formula () we see

$$\mathrm{Ext}^1_{X^{[2]}}(\mathcal{F}^{[2]}, \mathcal{F}^{[2]}) \cong \mathrm{Ext}^1_X(\mathcal{F}^{[2]}, \mathcal{F}^{[2]}) \bigoplus \mathrm{H}^1(X,\mathcal{F})^\vee \otimes \mathrm{H}^0(X,\mathcal{F}). \quad (1)$$

The first summand is just given by the infinitessimal deformations which come from the sheaf $\mathcal{F}$ on the surface and will therefore be called *surface-deformations*. Let us denote the second summand by the *additional deformations*. One can immediately deduce:

Surface-Deformations of Tautological Sheaves

Proposition. *The deformations of $\mathcal{F}^{[2]}$ coming from the surface are unobstructed. We therefore have an embedding of the moduli space of stable sheaves on X into the corresponding moduli space of stable sheaves on the Hilbert square.*

Corollary. *If in addition $h^1(X,\mathcal{F}) = 0$ we have an isomorphism of the moduli space of stable sheaves on X with a connected component of the corresponding moduli space of stable sheaves on $X^{[2]}$.*

Example. We can find an explicit example of a sheaf where the additional deformations are not unobstructed, i.e. the moduli space is singular in the point corresponding to this sheaf. The idea is the following: First of all find a sheaf $\mathcal{F}$ on X satisfying $h^0(\mathcal{F}) \cdot h^1(\mathcal{F}) \neq 0$ and $\dim \mathrm{Ext}^1_X(\mathcal{F},\mathcal{F}) \neq 0$. Now, if $\mathcal{F}$ has a deformation $e \in \mathrm{Ext}^1_X(\mathcal{F},\mathcal{F})$ such that some global section $\phi \in \mathrm{H}^0(X,\mathcal{F})$ does not deform with the sheaf then the dimension of the tangent space $\mathrm{Ext}^1_{X^{[2]}}(\mathcal{F}^{[2]},\mathcal{F}^{[2]})$ has to drop when we move in the direction of the deformation $(e, \psi \otimes \phi)$ (where ψ is some nontrivial element in $\mathrm{H}^1(X,\mathcal{F})^\vee$) and this deformation has a nontrivial obstruction.
So how to find such a sheaf? The easiest example is a tensor product of an ideal sheaf and a line bundle. Let X be a elliptically fibred K3 surface with fibre class E and section C. Following Donagi and Morrison (cf. [DM]) the linear system of $L := C + kE$ for $k \geq 2$ has C as a base component. Furthermore let $p \in C$ be a point, denote by $\mathcal{I}_p$ its ideal sheaf and set $\mathcal{F} := \mathcal{I}_p \otimes L$. Since p is a base point of L we have $\mathrm{H}^0(\mathcal{F}) \cong \mathrm{H}^0(L)$. Now the deformations of $\mathcal{F}$ correspond to deformations of the point p inside X. Thus if we deform $\mathcal{F}$ in a direction not tangent to C, p will not be longer a base point and the dimension of the space of global sections will drop.

Outlook

The most urgent task will be to study the behaviour of the additional deformations in broader generality. Is the defect described in the example above the only case where singularities may occur?
Suppose one can find examples where the additional deformations are unobstructed. What can be said about the whole moduli space in these cases. Can one find an explicit description of the deformed tautological sheaves?
So far, we did not address the question for symplectic structures on the moduli spaces. As described in [Bot] and in Chapter 10 of [HL] the Atiyah class of a sheaf may play a central role in the construction of symplectic forms. Hence a good description of the Atiyah class of tautological bundles would be desirable. I could accomplish partial results in this direction.

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