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Compactifying $\operatorname{Spec}\mathbb{Z}$

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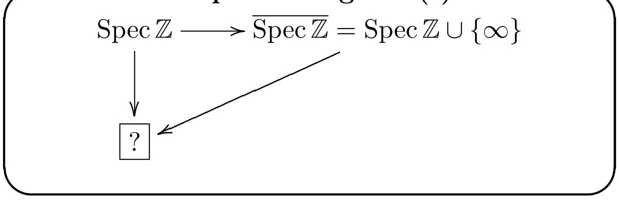
1 Motivation

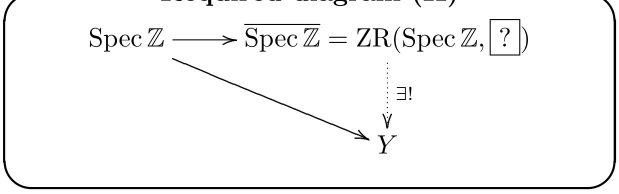
We want a category in which we can compactify $\operatorname{Spec}\mathbb{Z}$:

Required diagram (I)



We want $\overline{\operatorname{Spec}\mathbb{Z}}$ to be the *Zariski-Riemann space* $\mathrm{ZR}(\operatorname{Spec}\mathbb{Z}, \boxed{?})$ of $\operatorname{Spec}\mathbb{Z}$, namely to have the universal property as follows: $\operatorname{Spec}\mathbb{Z} \to \boxed{?}$ is proper, and if $\operatorname{Spec}\mathbb{Z} \to Y$ is a morphism over $\boxed{?}$ and $Y \to \boxed{?}$ is proper, then there should exist a unique morphism $\overline{\operatorname{Spec}\mathbb{Z}} \to Y$:

Required diagram (II)



However, these diagrams cannot be obtained in the category (**Sch**) of schemes:

(1) $\operatorname{Spec}\mathbb{Z}$ is the initial object in (**Sch**), hence we don't have any object $\boxed{?}$ under $\operatorname{Spec}\mathbb{Z}$.

(2) The infinite place ∞ is not a prime ideal.

Several attempts have been made in the past (Haran, Durov, $\cdots$), but have not reached the characterization of $\overline{\operatorname{Spec}\mathbb{Z}}$ by the universal property.

Therefore, we must consider an algebraic type which generalizes that of commutative rings; this is what we call 'convexoid rings', as is defined below.

2 Definitions

An algebraic type τ is *commutative*, if any m-ary operator φ and n-ary operator ψ commutes:

$$\varphi(\psi(x_{11},\cdots,x_{1n}),\cdots,\psi(x_{m1},\cdots,x_{mn})) = \psi(\varphi(x_{11},\cdots,x_{m1}),\cdots,\varphi(x_{1n},\cdots,x_{mn}))$$

This enables us to give a τ-algebra structure on Hom sets, and hence to define tensor products $\otimes$.

A *convexoid* is a quadruple $(M, \boxplus, -, 0)$ which satisfies:

(1) M is a set, $0 \in M$, $\boxplus$ (resp. $-$) is a binary (resp. unary) operator on M,

(2) The algebraic structure on M is commutative in the above sense, and

(3) $a \boxplus b = b \boxplus a$, $(-a) \boxplus a = 0$.

A *convexoid ring* is a commutative monoid object with respect to $\otimes$ in the category of convexoids.

Example 2.0.1. $\mathrm{D}\mathbb{Q} = \{x \in \mathbb{Q} \mid |x| \leq 1\}$ is a convexoid ring, by setting $a \boxplus b = (a+b)/2$. Note that $\boxplus$ is not associative.

We can define 'convexoid schemes', just as in the way of usual schemes; however, this is NOT sufficient for our purpose.

If R is a convexoid ring, then $\gamma_R = 1 \boxplus 0$ is the *fundamental constant.*

A ring is a localization of a convexoid

$R[\gamma_R^{-1}]$ becomes a ring, by setting $a + b = \gamma_R^{-1}(a \boxplus b)$.

A morphism $A \to B$ of convexoid rings is an *equivalence*, if $\gamma_A B = \gamma_B B$ and $A[\gamma_A^{-1}] \to B[\gamma_B^{-1}]$ is a ring isomorphism.

A *convexoid scheme* is a (multiplicative) monoid-valued space X which is locally isomorphic to the spectrum of some convexoid ring, and transition maps are equivalences.

3 Results

Let R_0 be the initial object in the category of convexoid rings. This is a submonoid of the polynomial ring $\mathbb{Z}[\gamma]$, and hence equipped with a canonical grading. We can define a convexoid scheme $\operatorname{Proj} R_0$.

Valuation convexoid rings can be defined just like valuation rings. A *proper morphism* of convexoid schemes is defined by the valuative criterion.

Main theorem(T-) [1]

$$\overline{\operatorname{Spec}\mathbb{Z}} = \mathrm{ZR}(\operatorname{Spec}\mathbb{Z}, \operatorname{Proj} R_0).$$

This is realized in the pro-category of convexoid schemes.

Remark 3.0.2.
- The infinity place canonically appears, without using the terminology of norms. This can be shown by proving a generalization of Ostrowski's theorem.
- The above theorem can be generalized to any ring of algebraic integers $\mathcal{O}_K$.
- A locally free sheaf on $\overline{\operatorname{Spec}\mathcal{O}_K}$ is, by definition a projective $\mathcal{O}_K$-module equipped with a norm satisfying some finiteness property.

References

[1] Takagi, S.: *Compactifying* $\operatorname{Spec}\mathbb{Z}$, arXiv: math.AG/1203.4914