On Fano fourfolds with nef bundle $\Lambda^2\mathcal{T}_X$ and $\rho(X) \geqslant 2$

代数幾何学シンポジウム記録
2012年度 pp.125-125

Yasutake Kazunori (Kyushu University)

Abstract

In this poster, I explain about the structure of Fano fourfolds whose the second exterior power of tangent bundle $\Lambda^2\mathcal{T}_X$ is nef and Picard number $\rho(X)$ is at least 2.

Definition of nef vector bundle

X: smooth proj. var. / $\mathbb{C}$, $\mathcal{E}$:vector bundle on X,
$\pi : \mathbb{P}_X(\mathcal{E}) \to X$:projectivization, $\xi_{\mathcal{E}}$:tautol. div.

($\Diamond$) $\mathcal{E}$: **nef** (**ample**) $\overset{\text{def}}{\Leftrightarrow}$ $\xi_{\mathcal{E}}$: **nef** (**ample**)

Known results

One of a generalization of Mori's Theorem (Hartshorne conjecture), K.Cho and E.Sato gave a characterization of smooth quadric as a variety with ample bundle $\Lambda^2\mathcal{T}_X$.

Ample case, Cho-Sato [CS]

X sm. proj. variety. with ample vect. bundle $\Lambda^2\mathcal{T}_X$
$\Rightarrow$ $X \cong \mathbb{P}^n$ or Q_n : smooth quadric hypersurface

As a further generalization of this theorem, F.Campana and T.Peternell classified threefolds with nef bundle $\Lambda^2\mathcal{T}_X$.

Nef case in 3 dim., Campana-Peternell [CP1]

X sm. proj. threefold with nef vect. bundle $\Lambda^2\mathcal{T}_X$
$\Rightarrow$ Either $\mathcal{T}_X$: nef, the blowing up of $\mathbb{P}^3$ at a point.
or del Pezzo threefold of degree$\geqslant 2$ with $\rho(X) = 1$

Problem

Classify smooth projective fourfolds with nef vector bundle $\Lambda^2\mathcal{T}_X$.

Fourfolds with nef tangent bundle are already classified by F.Campana-T.Peternell, N.Mok and J.-M. Hwang. We review the classification in Fano case.

Fano fourfolds with $\mathcal{T}_X$ nef , [CP2], [M] [H]

X : smooth Fano fourfold with nef tangent bundle $\mathcal{T}_X$.
Then X is one of the following:
$\mathbb{P}^4$, Q_4, $\mathbb{P}^3 \times \mathbb{P}^1$, $Q_3 \times \mathbb{P}^1$, $\mathbb{P}^2 \times \mathbb{P}^2$,
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$, $\mathbb{P}_{\mathbb{P}^2}(\mathcal{T}_{\mathbb{P}^2}) \times \mathbb{P}^1$, $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$,
$\mathbb{P}_{\mathbb{P}^3}(\mathcal{N})$ with null correlation bundle $\mathcal{N}$.

General results

In the case where $\kappa(X) = 0$, we can classify in all dimension.

Kodaira dimension κ(X) = 0, Theorem1

X : smooth projective variety of $\kappa(X) = 0$.
Then the following conditions are equivalent:

1. $\Lambda^r\mathcal{T}_X$ is nef for $1 \leqslant r \leqslant n-1$;
2. $\mathcal{T}_X$ is nef;
3. There is an étale covering $\nu : A \to X$ from Abelian variety.

Proof. Nefness of $\Lambda^r\mathcal{T}_X$ implies that X has the flat tangent bundle. The theorem follows from the result of Yau $\square$

Next, we consider the case where X is Fano and obtained by the blowing up of a smooth variety along a smooth subvariety. This proposition plays an important role in my study.

Blowing up, Proposition2

X : blowing up of smooth variety Y of dimension n along smooth subvariety Z. If X is Fano and $\Lambda^2\mathcal{T}_X$ is nef
$\Rightarrow$ X is the blowing up of $\mathbb{P}^n$ at a point.

Main theorem

As a first step of classification of the case where $\kappa(X) = -\infty$, we consider Fano fourfolds with $\rho(X) \geqslant 2$.

Fano fourfolds with ρ(X) $\geqslant$ 2, Theorem3

X : smooth Fano fourfold with $\rho(X) \geqslant 2$ / $\mathbb{C}$.
If $\Lambda^2\mathcal{T}_X$ is nef and $\mathcal{T}_X$ is not nef
$\Rightarrow$ X is the blowing up of $\mathbb{P}^4$ at a point.

Proof. Using results about extremal contractions on smooth fourfolds $\square$

The proof of above theorem yields the following result.

Corollary

X : smooth Fano fourfold with $\rho(X) \geqslant 2$.
If $\Lambda^2\mathcal{T}_X$ is nef on every extremal rational curve in X
$\Rightarrow$ $\Lambda^2\mathcal{T}_X$ is nef.

References

[CP1] F. Campana and T. Peternell, *On the second exterior power of tangent bundles of threefolds*, Compos. Math. 83 no.3, (1992), 329–346.

[CP2] F. Campana and T. Peternell, *4-folds with numerically effective tangent bundles and second Betti numbers greater than one*, Manuscripta Math. 79 (1993), 225–238.

[CS] K. Cho and E. Sato, *Smooth projective varieties with the ample vector bundle $\Lambda^2\mathcal{T}_X$ in any characteristic*, J. Math. Kyoto Univ. 35 (1995), 1–33.

[H] J-M. Hwang, *Rigidity of rational homogeneous spaces*, Proc. of ICM. (2006), 613–626.

[M] N. Mok, *On Fano manifolds with nef tangent bundles admitting 1-dimensional varieties of minimal rational tangents*, Trans. of A.M.S. 357 no.7 (2002), 2639–2658.

[Y] K. Yasutake, *Fano fourfolds with Picard number greater than two whose the second exterior power of tangent bundles are numerically effective*, preprint.