Example of Weierstrass semigroups of double covering type

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1 Numerical semigroup

We call a subsemigroup H of the monoid $\mathbb{N}_0$ consisting of non-negative integers a **numerical semigroup** if $\mathbb{N}_0 \backslash H$ is a finite set. The genus $g(H)$ of a numerical semigroup H is defined by the cardinality of the set $\mathbb{N}_0 \backslash H$.

Example 1. $H = \langle 3, 4\rangle$ or $\langle 3, 5, 7\rangle \Longrightarrow g(H) = 3$.

2 Weierstrass semigroup

We work over the complex number field $\mathbb{C}$. A curve means a smooth projective curve. For a curve C and a point P on C, we call a non-negative integer $n \in \mathbb{N}_0$ a **gap** if there is no meromorphic function which is holomorphic on $C\backslash\{P\}$ and has a pole of order n at P.

Fact 1. For the set $G(P)$ consisting of gaps at a point P on a curve C, the set $\mathbb{N}_0\backslash G(P)$ forms a numerical semigroup.

We call the numerical semigroup $\mathbb{N}_0\backslash G(P)$ a **Weierstrass semigroup** and denote it by $H(P)$. For instance, both of two semigroups in Example 1 are Weierstrass semigroups of a plane pointed curve of degree 4.

3 Plane curve case

Theorem 3.1 (E. Kang, S. J. Kim). Let C be a plane curve of degree d, and let $P \in C$. then, we have the following results.

(i) If $I_P(C \cap T_P(C)) = d$, then $H(P) = \langle d, d-1\rangle$.
(ii) If $I_P(C \cap T_P(C)) = d-1$, then

$$H(P) = \langle \{d-1+r(d-2)\}_{0\le r\le d-2}\rangle,$$

where $T_R(C)$ is the tangent line at R on a curve C, and $I_Q(C_1\cap C_2)$ is the intersection multiplicity at an intersection point Q of two curves C_1 and C_2.

It is well known that, in the case where C is a smooth plane curve of degree d, if the intersection multiplicity at P of C and the tangent line $T_P(C)$ at P on C is equal to d, $d-1$, or $d-2$, then the Weierstrass semigroup $H(P)$ of (C, P) is uniquely determined. Moreover, if $d \le 7$, then $H(P)$ is completely determined by Komeda and Kim.

4 Weierstrass semigroups of double covering of curves

For a numerical semigroup H, we set $d_2(H) = \{\frac{h}{2} \mid h \in H \text{ is even}\}$.

Fact 2. $\pi : \tilde{C} \to C$ is a double covering of curves with a ramification point $\tilde{P} \Longrightarrow d_2(H(\tilde{P})) = H(\pi(\tilde{P}))$.

We call a numerical semigroup H the **double covering type** if there is a double cover of curves $\pi : \tilde{C} \to C$ with $H = H(\tilde{P})$ as in Fact 2.

Question. For a pointed curve (C, P), what is a condition for a numerical semigroup $\tilde{H}$ with $d_2(\tilde{H}) = H(P)$ to be the double covering type ?

In general, it is difficult to consider this problem. However, if the genus of a pointed curve (C, P) is sufficiently small, then the following result is known.

Theorem 4.2 (Komeda). Let $\tilde{H}$ be a numerical semigroup of genus ≥ 9 with $g(d_2(\tilde{H})) = 3$. Then $\tilde{H}$ is the double covering type.

5 Main results

Theorem 5.1 Let X be an algebraic K3 surface which is given by a double cover $\pi : X \to \mathbb{P}^2$. Let C be a smooth projective curve on X with $\pi^{-1}\pi(C) = C$ which is not the ramification divisor of π, and let P be a ramification point of $\pi|_C : C \to \pi(C)$. Assume that the curve $\pi(C)$ is a plane curve of degree $d \ge 4$. Then, we have the following results.

(i) If $I_{\pi(P)}(T_{\pi(P)}(\pi(C)) \cap \pi(C)) = d$, then

$$H(P) = 2H(\pi(P)) + (6d-1)\mathbb{N}_0.$$

(ii) Assume that $I_{\pi(P)}(T_{\pi(P)}(\pi(C)) \cap \pi(C)) = d-1$ and let

$$T_{\pi(P)}(\pi(C))|_{\pi(C)} = (d-1)\pi(P) + Q.$$

If $I_Q(T_Q(\pi(C)) \cap \pi(C)) = d$, then

$$\begin{aligned} H(P) = 2H(\pi(P)) &+ (8d-9)\mathbb{N}_0 + (10d-13)\mathbb{N}_0 \\ &+ \cdots + (8d-9+2r(d-2))\mathbb{N}_0 \\ &+ \cdots + (2(d-1)^2+5)\mathbb{N}_0. \end{aligned}$$

In theorem 5.1 (ii), in the case where $d = 4$ and $I_Q(T_Q(\pi(C)) \cap \pi(C)) \le 4$, the Weierstrass semigroup $H(P)$ is classified as follows.

n	$H(P)$
23	$2H(\pi(P)) + 23\mathbb{N}_0$
27	$2H(\pi(P)) + 27\mathbb{N}_0 + 31\mathbb{N}_0 + 35\mathbb{N}_0$, $2H(\pi(P)) + 27\mathbb{N}_0 + 29\mathbb{N}_0$
29	$2H(\pi(P)) + 29\mathbb{N}_0 + 31\mathbb{N}_0 + 33\mathbb{N}_0$

References

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