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Non-normal very ample integral polytopes

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1. Definitions

$\mathcal{P} \subset \mathbb{R}^N$: an integral polytope of dimension n
$\mathcal{A}_\mathcal{P} = \{(\alpha, 1) \in \mathbb{R}^{N+1} : \alpha \in \mathcal{P}\} \cap \mathbb{Z}^{N+1}$

- We say that $\mathcal{P}$ is **normal** if $\mathcal{P}$ satisfies

$$\mathbb{Z}_{\geq 0}\mathcal{A}_\mathcal{P} = \mathbb{R}_{\geq 0}\mathcal{A}_\mathcal{P} \cap \mathbb{Z}\mathcal{A}_\mathcal{P}.$$

- We say that $\mathcal{P}$ is **very ample** if $\mathcal{P}$ satisfies

$$|(\mathbb{R}_{\geq 0}\mathcal{A}_\mathcal{P} \cap \mathbb{Z}\mathcal{A}_\mathcal{P}) \setminus \mathbb{Z}_{\geq 0}\mathcal{A}_\mathcal{P}| < \infty.$$

the elements of $(\mathbb{R}_{\geq 0}\mathcal{A}_\mathcal{P} \cap \mathbb{Z}\mathcal{A}_\mathcal{P}) \setminus \mathbb{Z}_{\geq 0}\mathcal{A}_\mathcal{P}$: the **holes** of $\mathcal{P}$
Note : **"normal $\Longrightarrow$ very ample"**

For simplicity, we assume

$$N = n \quad \text{and} \quad \mathbb{Z}\mathcal{A}_\mathcal{P} = \mathbb{Z}^{n+1}.$$

Then

$\mathcal{P}$ is normal (resp. very ample) $\Longleftrightarrow$

$\mathcal{P}$ satisfies $(*)$ for **all** $m \geq 1$ (resp. for **sufficiently large** m) :

$$\forall \alpha \in m\mathcal{P} \cap \mathbb{Z}^n,$$
$$(*) \quad \exists \alpha_1, \ldots, \exists \alpha_m \in \mathcal{P} \cap \mathbb{Z}^n \text{ s.t.}$$
$$\alpha = \alpha_1 + \cdots + \alpha_m.$$

♣ $\mathcal{P}$ is called **k-normal** if $\mathcal{P}$ satisfies $(*)$ for all $m \geq k$.

2. Backgrounds

From **Commutative Algebra**

K : field
$\mathbb{Z}_{\geq 0}\mathcal{A}_\mathcal{P}, \mathbb{R}_{\geq 0}\mathcal{A}_\mathcal{P} \cap \mathbb{Z}^{n+1}$: affine semigroups

- $K[\mathcal{P}] = K[\mathbb{Z}_{\geq 0}\mathcal{A}_\mathcal{P}]$, $\overline{K[\mathcal{P}]} = K[\mathbb{R}_{\geq 0}\mathcal{A}_\mathcal{P} \cap \mathbb{Z}^{n+1}]$: affine semigroup graded K-algebras

$\Longrightarrow$ $\overline{K[\mathcal{P}]}$ is the ***normalization*** of $K[\mathcal{P}]$.

- $K[\mathcal{P}]$ and $\overline{K[\mathcal{P}]}$ are graded by grading $\deg X^\alpha = i$, where $\alpha = (\alpha_1, \ldots, \alpha_n, i) \in \mathbb{R}_{\geq 0}\mathcal{A}_\mathcal{P} \cap \mathbb{Z}^{n+1}$.
- $\mathcal{P}$ is normal (resp. very ample) $\Longleftrightarrow$ $K[\mathcal{P}]$ (resp. m^k) is generated by degree one elements (resp. for $k \gg 0$, where $m = K[\mathcal{P}]_+$).

From **Algebraic Geometry**

X : toric variety of dim n $/K$, L : ample line bundle on X
$\mathcal{P} \subset \mathbb{R}^n$: the corresponding integral polytope to (X, L)
Then

$$H^0(X, L) \cong \bigoplus_{v \in \mathcal{P} \cap \mathbb{Z}^n} K\mathbf{e}_v$$
$$R(X, L) := \bigoplus_{i \geq 0} H^0(X, L^{\otimes i}) \cong \overline{K[\mathcal{P}]}$$

- $\mathcal{P}$ is normal $\Longleftrightarrow$ L is *normally generated* (in the sense of Mumford)
- $\mathcal{P}$ is very ample $\Longleftrightarrow$ L is very ample

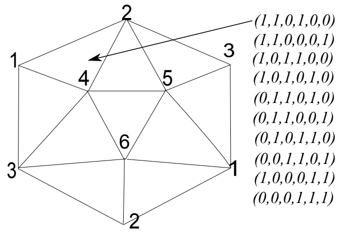
3. Examples

Our main interest is the existence of

non-normal very ample integral polytopes !!

EXAMPLE (Bruns–Gubeladze '02)

dim=5



∘ non-normal : $(1,1,1,1,1,1) =$
$1/2((1,1,0,1,0,0) + (0,1,1,0,1,0) + (0,0,1,1,0,1) + (1,0,0,0,1,1))$
∘ very ample ; $(1,1,1,1,1,1) + (1,1,0,1,0,0) =$
$(1,0,1,1,0,0) + (0,1,0,1,1,0) + (1,1,0,0,0,1)$ and so on...
∘ the number of holes = 1

EXAMPLE (Bruns–Gubeladze '09)

$$\text{conv}\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 & 3 & 4 & 1 & 2 \end{pmatrix}, \qquad \dim = 3,$$

∘ $(1,1,4) = \frac{1}{2}((0,0,1) + (1,0,2) + (0,1,3) + (1,1,2))$
∘ $(1,1,4) + (0,0,0) = (0,0,1) + (0,0,1) + (1,1,2)$ ......
∘ the number of holes = 1

REMARK (Ogata '11)

In [3], infinitely many non-normal very ample integral polytopes of dim 3 are constructed.

4. Main Result

How about higher dimensions ?

Theorem
Let n and h be integers with $n \geq 3$ and $h \geq 1$. Then there exists a non-normal very ample integral polytope of dimension n having exactly h holes.

REMARK (Katthän '12)
$\mathcal{P}$: non-normal very ample integral polytope
$\Longrightarrow$ $K[\mathcal{P}]$ is **NEVER** Cohen–Macaulay !!

Reference

[1] W. Bruns and J. Gubeladze, "Polytopes, rings and K-theory", Springer–Verlag, Heidelberg, 2009.

[2] D. Cox, J. Little and H. Schenck, "Toric varieties", American Mathematical Society, 2011.

[3] S. Ogata, Very ample but not normal lattice polytopes, to appear in *Beiträge Algebra Geom.*, (2011)