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Symplectic varieties of complete intersection and contact geometry

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A normal complex algebraic variety X is a *symplectic variety* if there is a holomorphic symplectic 2-form ω on the regular part X_{reg} of X and ω extends to a (possibly degenerate) holomorphic 2-form on a resolution $f : \tilde{X} \to X$.

Example: Let $\mathfrak{g}$ be a semisimple complex Lie algebra and let G be its adjoint group. Let us consider the *adjoint quotient map* $\chi : \mathfrak{g} \to \mathfrak{g}//G$. If $\mathrm{rank}(\mathfrak{g}) = r$, then $\mathfrak{g}//G$ is isomorphic to the r-dimensional affine space $\cong \mathbf{C}^r$. The nilpotent variety N is, by definition, the set of all nilpotent elements of $\mathfrak{g}$ and we have $N = \chi^{-1}(0)$. The nilpotent variety decomposes into the disjoint union of (finite number of) nilpotent orbits. There is a unique nilpotent orbit O_{reg} that is open dense in N, which we call the *regular nilpotent orbit*. Then $N = \overline{O_{reg}}$. The regular nilpotent orbit O_{reg} coincides with the regular part of N and it admits a holomorphic symplectic form ω_{KK} so called the *Kostant-Kirillov* 2-form. Then (N, ω_{KK}) is a symplectic variety. Moreover $N \subset \mathfrak{g}$ is defined as a complete intersection of r homogeneous polynomials (with respect to the standard $\mathbf{C}^*$-action on $\mathfrak{g}$).

In this talk I characterize the nilpotent varieties of semisimple Lie algebras among affine symplectic varieties.

Let (X, ω) be a singular affine symplectic variety of dimension $2n$ embedded in an affine space $\mathbf{C}^{2n+r}$ as a complete intersection of r homogeneous polynomials. Assume that ω is also homogeneous, i.e. there is an integer l such that $t^*\omega = t^l \cdot \omega$ for $t \in \mathbf{C}^*$.

Main Theorem ([1]): *One has $(X, \omega) \cong (N, \omega_{KK})$, where N is the nilpotent variety of a semisimple Lie algebra $\mathfrak{g}$ together with the Kostant-Kirillov 2-form ω_{KK}.*

Before starting the proof, we give a few observations.

Observation 1. *$l = 1$, where $l := wt(\omega)$.*
Write X as $f_1 = ... = f_r = 0$ with homogeneous polynomials $f_i(z_1, ..., z_{2n+r})$. We put $a_i := \deg(f_i)$. As X is singular, we may assume that $a_i > 1$ for every i. The holomorphic volume form $\omega^n := \omega \wedge ... \wedge \omega$ can be written, by the adjunction formula (or the residue formula), as

$$\omega^n = c \cdot \mathrm{Res}(dz_1 \wedge ... \wedge z_{2n+r}/(f_1, ..., f_r))$$

with a nonzero constant c. Since X has only canonical singularities, we see that $wt(\omega) > 0$. Computing the weights of both sides, we get

$$0 < n \cdot wt(\omega) = 2n + r - \Sigma a_i < 2n.$$

The last inequality follows from the fact that $a_i > 1$ and so $\Sigma a_i > r$. Hence $wt(\omega) = 1$ and $\Sigma a_i = n + r$.

Observation 2. *X has a $\mathbf{C}^*$-equivariant crepant resolution $\pi : Y \to X$.*

(Sketch of Proof): Take a resolution $f : W \to X$ and apply the MMP(Minimal Model Program) to the morphism f. Then we get a partial crepant resolution $\pi : Y \to X$ where Y may possibly have $\mathbf{Q}$-factorial terminal singularities. The $\mathbf{C}^*$-action on X extends to a $\mathbf{C}^*$-action in such a way that π is $\mathbf{C}^*$-equivariant. The symplectic form ω induces a Poisson structure on the regular locus X_{reg}. By the normality of X, this Poisson structure extends to a Poisson structure on X. The pull-back of ω by π is a symplectic form on Y_{reg} because π is crepant. Then it induces a Poisson structure on Y_{reg} and it extends to a Poisson structure on Y. Now let us consider a *Poisson deformation* $\mathcal{Y} \to \Delta$ of Y. Then π extends to a birational morphism $\tilde{\pi} : \mathcal{Y} \to \mathcal{X}$ over Δ for a Poisson deformation $\mathcal{X} \to \Delta$ of X. If the Poisson deformation $\mathcal{Y}/\Delta$ is very general, then $\tilde{\pi}_t : Y_t \to X_t$ is an isomorphism for $t \neq 0$. Since X has only complete intersection singularities, X_t also does. On the other hand, we have $\mathrm{Codim}_Y \mathrm{Sing}(Y) \geq 4$ because Y has only terminal singularities. This implies that $\mathrm{Codim}_{Y_t} \mathrm{Sing}(Y_t) \geq 4$; and hence $\mathrm{Codim}_{X_t} \mathrm{Sing}(X_t) \geq 4$. Notice that X_t is a symplectic variety and such a symplectic variety X_t must be smooth by a proposition of Beauville. As $Y_t \cong X_t$, we have seen that Y_t is smooth. Finally, by the $\mathbf{Q}$-factoriality of Y, we see that any Poisson deformation $\mathcal{Y} \to \Delta$ is a locally trivial flat deformation of Y. Therefore Y must be smooth and π is a crepant resolution.

We put $\mathbf{P}(X) := X - \{0\}/\mathbf{C}^*$ and $Z := Y - \pi^{-1}(0)/\mathbf{C}^*$. Then π induces a map $\bar{\pi} : Z \to \mathbf{P}(X)$. By using the fact that $wt(z_i) = 1$ for all i, we have:

Observation 3. *$\bar{\pi} : Z \to \mathbf{P}(X)$ is a crepant resolution.*

Notice that Z is a projective manifold of dimension $2n - 1$. An important fact is that Z has a *contact structure*.

Let W be a complex manifold of odd dimension $2n - 1$. A contact structure on W is an exact sequence of vector bundles

$$0 \to E \to \Theta_W \stackrel{\theta}{\to} M \to 0,$$

where M is a line bundle and the induced pairing $E \times E \to M$, $(x, y) \to \theta([x, y])$ is nondegenerate. Recall that a subbundle of Θ is called *integrable* if the bracket $[\ ,\ ]$ is closed in the subbundle. In this sense, E is a *highly non-integrable* subbundle of Θ. The line bundle M is called the *contact line bundle* and the twisted 1-form $\theta \in \Gamma(W, \Omega^1_W \otimes M)$ is called the *contact form*. If W admits a contact structure, then $-K_W \cong M^{\otimes n}$ and $(d\theta)^{n-1} \wedge \theta$ is a nondegenerate $2n - 1$-form on W.

Let us return to our situation. The $\mathbf{C}^*$-bundle $X - \{0\} \to \mathbf{P}(X)$ restricts to the $\mathbf{C}^*$-bundle $X_{reg} \to \mathbf{P}(X)_{reg}$. We put $L := O_{\mathbf{P}(X)}(1)|_{\mathbf{P}(X)_{reg}}$ and denote by $(L^{-1})^{\times}$ the $\mathbf{C}^*$-bundle on $\mathbf{P}(X)_{reg}$ obtained from the dual line bundle L^{-1} by removing the 0-section. Then $X_{reg} \to \mathbf{P}(X)_{reg}$ can be identified with $p : (L^{-1})^{\times} \to \mathbf{P}(X)_{reg}$. The $\mathbf{C}^*$-action on X_{reg} coincides with the natural $\mathbf{C}^*$-action on $(L^{-1})^{\times}$ as a $\mathbf{C}^*$-bundle. Let ζ be a vector field on $(L^{-1})^{\times}$ generating this $\mathbf{C}^*$-action. We regard ω as a symplectic 2-form on $(L^{-1})^{\times}$ by the identification of X_{reg} with $(L^{-1})^{\times}$. Then one can write

$$\omega(\zeta, \cdot) = p^*\theta$$

with a twisted 1-form $\theta \in \Gamma(\mathbf{P}(X)_{reg}, \Omega^1_{\mathbf{P}(X)_{reg}} \otimes L)$. We remark that p^*L has a natural trivialization on $(L^{-1})^\times$. This twisted 1-form θ determines a contact structure on $\mathbf{P}(X)_{reg}$.

Denote by $i : \mathbf{P}(X)_{reg} \to \mathbf{P}(X)$ the inclusion map. Then $i_*\Omega^1_{\mathbf{P}(X)_{reg}} = \bar{\pi}_*\Omega^1_Z$. Since $\Gamma(\mathbf{P}(X), i_*\Omega^1_{\mathbf{P}(X)_{reg}} \otimes O_{\mathbf{P}(X)}(1)) = \Gamma(Z, \Omega^1_Z \otimes \bar{\pi}^*O_{\mathbf{P}(X)}(1))$, the twisted 1-form θ can be regarded as an element of $\Gamma(Z, \Omega^1_Z \otimes \bar{\pi}^*O_{\mathbf{P}(X)}(1))$.

Proposition 1. *Z has a contact structure with the contact line bundle $\bar{\pi}^*O_{\mathbf{P}(X)}(1)$.*

For a contact projective manifold Z, the following structure theorem was proved by Kebekus, Peternell, Sommese and Wisniewski.

Theorem 2: *Assume that Z is a contact projective manifold with $b_2(Z) > 1$ and K_Z not nef. Then Z is a projectivized cotangent bundle $\mathbf{P}(T^*M) := T^*M - (0 - section)/\mathbf{C}^*$ of a projective manifold M. Moreover, the contact line bundle is isomorphic to $O_{\mathbf{P}(T^*M)}(1)$.*

As $K_Z \cong \bar{\pi}^*O_{\mathbf{P}(X)}(-n)$, it is not nef. Moreover, since X is a symplectic variety of complete intersection, $\mathrm{Codim}_X \mathrm{Sing}(X) = 2$. Thus, if $n \geq 2$, then $\mathbf{P}(X)$ has singularities and $\bar{\pi}$ has an exceptional locus. This implies that $b_2(Z) > 1$. When $n = 1$, it is easily checked that X is isomorphic to an A_1-surface singularity $z_1^2 + z_2^2 + z_3^2 = 0$ in $\mathbf{C}^3$.

In the remainder we assume that $n \geq 2$. Then Z is a projectivized cotangent bundle $\mathbf{P}(T^*M)$ for some projective manifold by the theorem above. Which kind of manifold is M ?

First notice that $O_{\mathbf{P}(T^*M)}(1) = \bar{\pi}^*O_{\mathbf{P}(X)}(1)$. In particular, $O_{\mathbf{P}(T^*M)}(1)$ is nef. The following theorem was proved by Demailly, Peternell and Schneider.

Theorem 3. *Let M be a projective manifold with $O_{\mathbf{P}(T^*M)}(1)$ nef. Assume that $\chi(M, O_M) \neq 0$. Then M is a Fano manifold.*

Actually it is conjectured that M is a rational homogeneous space under the same assumption; but it is still open except when $\dim M = 2$ or 3. But, in our case, we can prove more:

Proposition 4. *Let M be a Fano manifold such that $|O_{\mathbf{P}(T^*M)}(1)|$ is free from base points. Then M is a rational homogeneous soace, that is, $M = G/P$ with a complex semisimple Lie group G and its parabolic subgroup P.*

(Sketch of Proof). By the assumption we see that the map

$$H^0(M, \Theta_M) \otimes O_M \to \Theta_M$$

is surjective. Let G be the neutral component of $\mathrm{Aut}(M)$. Then G is a linear algebraic group. By the surjectivity G acts transitively on M. Hence M can be written as G/P with a parabolic subgroup P. Let us prove that G is semisimple. Let $r(G)$ be the radical of G. Then $r(G)$ is contained in P. But this implies that $r(G)$ acts trivially on G/P. By the definition of G, G acts effectively on M. This implies that $r(G) = 1$.

By the proposition $Z = \mathbf{P}(T^*(G/P))$. Let us consider the moment map $\mu : T^*(G/P) \to \mathfrak{g}^*$. Since $\mathfrak{g}$ is semisimple, $\mathfrak{g}^* \cong \mathfrak{g}$. Then $\mathrm{Im}(\mu)$ coincides with a nilpotent orbit closure $\bar{O}$ of $\mathfrak{g}$. One can take the projectivization $\bar{\mu} : \mathbf{P}(T^*(G/P)) \to \mathbf{P}(\bar{O})$. We compare this map with $\bar{\pi} : Z \to \mathbf{P}(X)$. The embeddings $\mathbf{P}(\bar{O}) \to \mathbf{P}(\mathfrak{g})$ and $\mathbf{P}(X) \to \mathbf{P}^{2n+r-1}$ determine the tautological line bundles $O_{\mathbf{P}(\bar{O})}(1)$ and $O_{\mathbf{P}(X)}(1)$. One can check that $\bar{\mu}^* O_{\mathbf{P}(\bar{O})}(1) = O_{\mathbf{P}(T^*(G/P))}(1)$ and $\bar{\pi}^* O_{\mathbf{P}(X)}(1) = O_{\mathbf{P}(T^*(G/P))}(1)$. Moreover both $\bar{\mu}$ and $\bar{\pi}$ coincide with the morphisms determined by the complete linear system $|O_{\mathbf{P}(T^*(G/P))}(1)|$. This implies that $(\mathbf{P}(X), O_{\mathbf{P}(X)}(1)) \cong (\mathbf{P}(\bar{O}), O_{\mathbf{P}(\bar{O})}(1))$ as polarized varieties. Therefore $X \cong \bar{O}$ as a $\mathbf{C}^*$-varieties.

The final task is to show that $\bar{O}$ is the nilpotent variety N of $\mathfrak{g}$. For simplicity we only discuss the case when $\mathfrak{g}$ is an exceptional simple Lie algebra. See [1] for other cases. We are now assuming that $\bar{O}$ is complete intersection in $\mathfrak{g}$. Then one can construct a G-equivariant morphism $f : \mathfrak{g} \to V$ from $\mathfrak{g}$ to a G-representation V with $\dim V = \mathrm{Codim}_{\mathfrak{g}} \bar{O}$ in such a way that $f^{-1}(0) = \bar{O}$. By the argument above the embedding $X \to \mathbf{C}^{2n+r}$ is identified with the embedding $\bar{O} \to \mathfrak{g}$. Recall that we have $\Sigma a_i = n + r$. As $a_i > 1$ for each i, we have $2r \leq n + r$; hence $r \leq n$. Note that $\dim \mathfrak{g} = 2n + r$ and $\mathrm{Codim}_{\mathfrak{g}} \bar{O} = r$. Thus we have

$$\dim V \leq 1/3 \cdot \dim \mathfrak{g}.$$

When $\mathfrak{g}$ is exceptional, there is no non-trivial irreducible G-representation that satisfies this inequality. Thus V is a direct sum of trivial G-representations. In particular, $\bar{O}$ is the common zeros of some adjoint invariant polynomials. On the other hand, N is the common zeros of *all* adjoint invariant polynomials. Hence $\bar{O}$ contains N. As all nilpotent orbits are contained in N, $\bar{O}$ is contained in N. Therefore $\bar{O} = N$.

References

[1] Y. Namikawa, *On the structure of homogeneous symplectic varieties of complete intersection*, arXiv: 1201.5444, to appear in Invent. Math.