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Classification of involutions on Enriques surfaces

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Terms

Involution

An automorphism of order 2 is called an involution.

Symplectic involution

An involution σ of a $K3$ surface X is called symplectic (resp. non-symplectic) if $\sigma^*(\omega_X) = \omega_X$ (resp. $\sigma^*(\omega_X) = -\omega_X$), where ω_X is a non-vanishing 2-form on X.

Enriques surface

An Enriques surface Y is a compact complex surface satisfying the following conditions:

(i) the geometric genus and the irregularity vanish,

(ii) the bi-canonical divisor on Y is linearly equivalent to 0.

Every Enriques surface Y is the quotient of a $K3$ surface X by a fixed point free involution ε.

Result and How to prove

Theorem (Ito-Ohashi)

Lattice types of involutions on Enriques surfaces are classified into 18 types.

How to prove

Let ι be an involution on Y. We get two lifted involutions g and $g \circ \varepsilon (= \varepsilon \circ g)$ on X. Here g is a symplectic involution. Hence we just have to classify the pair (g, ε).
The second cohomology group $L = H^2(X, \mathbb{Z})$ is equipped with lattice structure. By virtue of Torelli theorem, our problem is equivalent to the classification of the pair (g^*, ε^*), where g^* and ε^* are involutions on L induced by g and ε respectively.
For our purpose, we use the theory of the classification of involutions of a lattice with condition on a sublattice, due to Nikulin.

$$(Y, \iota) \underset{\text{quotient by } \varepsilon}{\overset{\text{lift}}{\rightleftarrows}} (X, g, \varepsilon) \underset{\text{Torelli theorem}}{\overset{\text{cohomology rep.}}{\rightleftarrows}} (L, g^*, \varepsilon^*)$$

Nikulin's theory

Let S be a fixed lattice and θ an involution of S. Nikulin gave the determining condition of a triple (L, φ, i) with the condition (S, θ) satisfying the following commutative diagram:

$$\begin{array}{ccc} L & \xrightarrow{\varphi} & L \\ {\scriptstyle i}\uparrow & & \uparrow{\scriptstyle i} \\ S & \xrightarrow[\theta]{} & S \end{array}$$

Here L is a unimodular lattice, φ is an involution of L, and $i\colon S \to L$ is a primitive embedding.

Our case

We identify $L = H^2(X, \mathbb{Z})$, $\varphi = \varepsilon^*$, $S = \{x \in L \mid g^*(x) = -x\} \simeq E_8(2)$ and $\theta = \varepsilon^*|_S$ to apply Nikulin's theory.

Related Topics

Involutions on $K3$ surfaces

Nikulin showed that finite symplectic automorphisms are determined by their order. Furthermore, he classified non-symplectic involutions into 75 types. Therefore involutions on $K3$ surfaces are classified into 76 types.

Numerically trivial involutions

Any involution f of a $K3$ surface X acts non-trivially on $H^2(X, \mathbb{Z})$. This is not true for an Enriques surface. Namely there exists an involution of an Enriques surface Y acting trivially on $H^2(Y, \mathbb{Q})$. Such an involution is called a numerically trivial involution. Mukai-Namikawa, Kondo and Mukai classified them into 3 types.

Examples

We also gave geometric realizations to all types. Most of them are by Horikawa construction.

Horikawa construction

Let ψ be an involution on $\mathbb{P}^1 \times \mathbb{P}^1$ given by $\psi\colon (u, v) \mapsto (-u, -v)$.
Let B be a curve on $\mathbb{P}^1 \times \mathbb{P}^1$ with the following properties:

(i) its bi-degree is $(4, 4)$ with at worst simple singularities,

(ii) it is preserved under ψ,

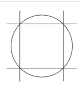
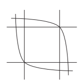
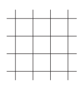
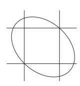
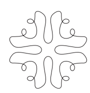
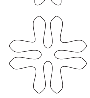
(iii) it does not pass through any of fixed points of ψ.

Then the minimal resolution X of the double cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched along B is a $K3$ surface. Moreover, ψ lifts to two involutions of X. One of them is a fixed point free involution ε. Therefore $Y = X/\varepsilon$ is an Enriques surface. Another one induces an involution ι on Y.

$$B \subset \mathbb{P}^1 \times \mathbb{P}^1 \xleftarrow{2:1} R \xleftarrow[\text{resolution}]{\text{minimal}} X \xrightarrow{/\varepsilon} Y$$

Examples

The column B stands for the branch locus stated above. The next column "fixed curve(s)" stands for the fixed curve(s) of ι in Y. Here we denote by $C^{(m)}$ a non-singular curve of genus m. The last column T_X stands for the transcendental lattice of the covering $K3$ surface generically.

No.	B	fixed curve(s)	T_X
[1]		$C^{(1)} + 4\mathbb{P}^1$	$U \oplus U(2)$
[2]		$4\mathbb{P}^1$	$U(2) \oplus U(2)$
[3]		$4\mathbb{P}^1$	$U(2) \oplus U(2)$
[4]		$C^{(1)} + 3\mathbb{P}^1$	$U \oplus U(2) \oplus A_1(2)$
[14]		$C^{(1)}$	$U \oplus U(2) \oplus A_1(2)^4$
[18]		$C^{(5)}$	$U \oplus U(2) \oplus E_8(2)$