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On the classification of ACM line bundles on quartic hypersurfaces on $\mathbb{P}^3$

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1 ACM bundles

We work over the complex number field $\mathbb{C}$. Let X be a smooth hypersurface on $\mathbb{P}^n$ and $\mathcal{O}_X(1)$ be the very ample line bundle given by a hyperplane section of X. Then, an *Arithmetically Cohen-Macaulay* (*ACM* for short) bundle is defined as follows.

Definition 1.1. A vector bundle $\mathcal{E}$ on X is ACM if $H^i(X, \mathcal{E}(l)) = 0$ for all $1 \leq i \leq \dim(X) - 1$ and $l \in \mathbb{Z}$, where $\mathcal{E}(l) = \mathcal{E} \otimes \mathcal{O}_X(l)$.

It is well known that if a vector bundle $\mathcal{E}$ on X splits, that is, $\mathcal{E}$ is a direct sum of ACM line bundles, then $\mathcal{E}$ is an ACM bundle. If $X = \mathbb{P}^n$ $(n \geq 1)$, the converse assertion of it is also correct [1]. However, in the case where X is a hypersurface of degree $d \geq 2$ on $\mathbb{P}^n$, there exists a counterexample.

Question. What is the condition for an ACM bundle to be indecomposable?

2 Previous results

In the past decades, several authors have studied about indecomposable ACM bundles on a hypersurface in $\mathbb{P}^n$. For example, Knörrer [2] has completely classified ACM bundles on quadrics. A variety which has only finitely many indecomposable ACM bundles (up to twist) is said to *finite representation type*. The projective varieties of finite representation type have been completely classified into a small list [5].

On the other hand, a variety is said to *wild representation type* if there exist n-dimensional families of non-isomorphic indecomposable ACM bundles for arbitrarily large n. For example, the family of higher rank indecomposable ACM bundles on a smooth cubic surface in $\mathbb{P}^3$ is constructed by Casanellas and Hartshorne [3].

Theorem 2.1. Let X be a nonsingular cubic surface on $\mathbb{P}^3$. Then for every $r \geq 2$, there are stable Ulrich bundles of rank r with $c_1 = rH$ and $c_2 = \frac{1}{2}(3r^2 - r)$. They form a smooth open subset of an irreducible component of dimension $r^2 + 1$ of the moduli space $M_X^s(r; c_1, c_2)$ of rank r stable vector bundles with Charn classes c_1, c_2 on X.

3 Quartic hypersurface case

In this section, we consider the splitting problem of ACM bundles on quartic hypersurfaces. In the case where X is a quartic hypersurface, if an ACM vector bundle on X splits, then the components of the direct sum are ACM line bundles. Therefore, in order to investigate a sufficient condition for an ACM bundle to be indecomposable, we consider the classification of ACM line bundles on a quartic hypersurface.

Definition 3.1. A vector bundle $\mathcal{E}$ on a smooth projective hypersurface X is said to *initialized* if it satisfies the condition $H^0(X, \mathcal{E}(-1)) = 0$ and $H^0(X, \mathcal{E}) \neq 0$.

Main result. Let X be a smooth quartic hypersurface on $\mathbb{P}^3$, and let D be a non-zero effective divisor on X. Then the following conditions are equivalent.

(i) $\mathcal{O}_X(D)$ is an ACM and initialized line bundle.
(ii) For any general member $C \in |\mathcal{O}_X(1)|$, one of the following cases occurs.
(a) $D^2 = -2$ and $1 \leq C.D \leq 3$.
(b) $D^2 = 0$ and $3 \leq C.D \leq 4$.
(c) $D^2 = 2$ and $C.D = 5$.
(d) $D^2 = 4$, $C.D = 6$ and $|D - C| = |2C - D| = \phi$.

In the proof of the main result, we may concretely construct the divisors D as in the above result. Moreover, if we give a quartic form f on the homogeneous coordinates of $\mathbb{P}^3$, we have a further detailed result.

Corollary. Let $f = f(x_1.x_2, x_3)$ be a general ternary form, $X_f := \{w^4 = f(x_1, x_2, x_3)\} \subset \mathbb{P}^3$, and let L be a non-trivial line bundle on X_f. Then the following conditions are equivalent.

(i) L is an ACM and initialized line bundle.
(ii) L is given by a (-2)-curve or an elliptic curve on X_f.

The quartic hypersurface as in Corollary is given by a double covering of a DelPezzo surface of degree 2 and the Picard lattice of it is a 2-*elementary* lattice. Therefore, there is a unique non-symplectic involution on X_f which acts trivially on the Picard lattice of X_f [4]. Since the hyperplane section of X_f is linearly equivalent to the fixed locus of the involution, the intersection number of $\mathcal{O}_{X_f}(1)$ and L as in Corollary is uniquely determined.

References

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