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On Blow-Analytic Equivalence of Plane Curves

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Motivation

What is a "good" equivalence choice for the classification of real singularities?

- $\mathcal{C}^0$-equivalence is very weak,
- $\mathcal{C}^r$-equivalence ($r > 0$) is too strict:
  for example, the Whitney family

$$(\{xy(x-y)(x-\lambda y)=0\},0)$$

generates a continuous $\mathcal{C}^1$-moduli.

Blow-analytic equivalence: stronger than $\mathcal{C}^0$-equivalence, more flexible than analytic type classifications.

Definition

$f : \mathbb{R}^2 \to \mathbb{R}$ is a **blow-analytic function** if there exists a composition of simple blow-ups $\beta = \beta_1 \circ \cdots \circ \beta_n : X \to \mathbb{R}^2$ such that $f \circ \beta$ is analytic.
$h : \mathbb{R}^2 \to \mathbb{R}^2$ is a **blow-analytic homeomorphism** if h and h^{-1} have blow-analytic components.

Goal

Classify germs of plane real curves up to blow-analytic homeomorphism.
Two plane curves $(C,0),(D,0)$ are **blow-analytically equivalent** if there exists a blow-analytic homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ that carries $(C,0)$ to $(D,0)$.

Blow-Analytic Invariants

$(C,0)$: a germ of a real plane curve with an isolated singularity at the origin.

- Take $\beta : X \to \mathbb{R}^2$, a resolution which is
  -**simple**: composition of simple blow-ups,
  -**good**: $\beta^{-1}(C)$ is simple normal crossing (components are smooth and meet transversally).

- Construct the **weighted dual graph** Γ associated to X:

Γ		X
vertex v_i	$\longleftrightarrow$	except. curve E_i
edge v_i-v_j	$\longleftrightarrow$	$E_i \cdot E_j = 1$
	$\mathrm{w}(v_i) = E_i^2 \mod 2$	

Notation:
$\circ$: odd except. curve,
$\bullet$: even except. curve,
$\times$: non-compact comp.

- Define:
  A: the intersection matrix associated to Γ,
  i.e. $A = (a_{ij})$, where $a_{ii} = \mathrm{w}(v_i)$ and $a_{ij} = E_i \cdot E_j \mod 2$.
  Δ: the minimal subtree connecting the non-compact components in Γ.
  Γ': the subgraph of Γ spanned by the vertices not in Δ.
  A': the intersection matrix associated to Γ'.

INVARIANTS:

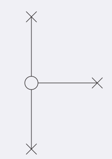
$\mu = \operatorname{corank} A \longleftrightarrow$ surface
$\mu' = \operatorname{corank} A' \longleftrightarrow$ embedded curve

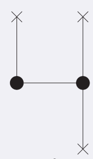
Explicit Results

Theorem. [1] *All unibranched germs of plane curves are blow-analytically equivalent to a line.*

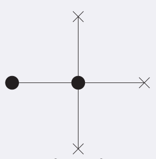
Theorem. [2] *Bibranched germs of plane curves are blow-analytically equivalent in the level of the graph Γ if and only if they have the same μ'.*

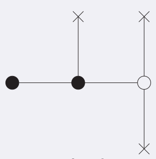
Theorem. [2] *A germ of a tribranched plane curve with $\mu' = 0$ is blow-analytically equivalent to one of the following:*

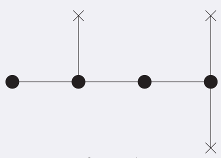

$(\{xy(x-y)=0\},0)$

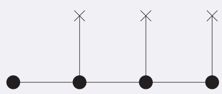

$(\{xy(x-y^2)=0\},0)$

Theorem. [3] *A germ of a tribranched plane curve with $\mu' = 1$ is blow-analytically equivalent in the level of the graph Γ to one of the following:*


$(\{y(y-x^2)(y+x^2)=0\},0)$


$(\{x(y-x)(y^2-x^3)=0\},0)$


$(\{y(y-x^2)(y-x^4)=0\},0)$


$(\{y(y-x^2)(y^2-x^5)=0\},0)$

A Finite Classification

Main Theorem. [3] *The number of blow-analytic equivalence classes of plane n-branched germs of curves with $\mu' = k$ is finite, for any n, k in $\mathbb{N}$.*

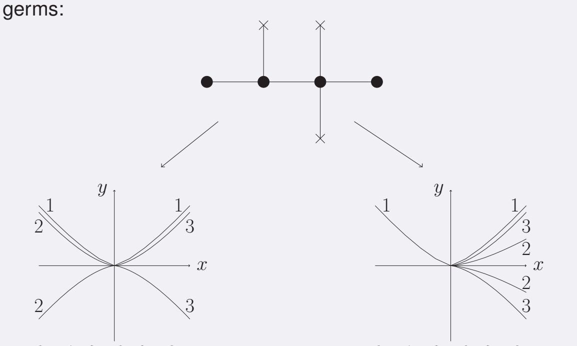
The number of possible non-equivalent Γ can be easily estimated by means of graph theory.
In the tribranched case this is lower or equal than

$$(k^3 - 2k^2 - k + 11)2^{k-2} \qquad (\forall k \in \mathbb{N}).$$

From Graphs to Germs

Dual graphs Γ carry less information than resolutions of singularities X.

In some cases, a graph corresponds to two or more blow-analytically distinct curve germs:



$(\{(y^3-x^4)(y^2+x^3)(y^2-x^3)=0\},0)$ $(\{(y^3-x^4)(2y^2-x^3)(y^2-x^3)=0\},0)$

INVARIANT:

Order of intersection of the demibranches with the boundary of the exceptional set.

We can obtain information about the blow-analytic type of a germ without having to resolve the singularity.

Proposition. *A bibranched germ of plane curve is blow-analytically equivalent to $(\{xy=0\},0)$ if and only if the order of its demibranches is 1212.*

Proposition. *If an n-branched germ of plane curve has order of demibranches $12\ldots n12\ldots n$, then $\mu' = 0$.*

Conjecture. *An n-branched germ of plane curve is blow-analytically equivalent to*

$$(\{\prod_{i=1}^{n}(x-\lambda_i y)=0\},0) \qquad (\forall\, i \neq j,\ \lambda_i \neq \lambda_j)$$

if and only if the order of its demibranches is $12\ldots n12\ldots n$.

References

[1] M. Kobayashi, T-C. Kuo, *On Blow-Analytic Equivalence of Embedded Curve Singularities*. Proc. Jpn. Acad. **73**, 97-99 (1997)
[2] M. Kobayashi, *On Blow-Analytic Equivalence of Branched Curves in $\mathbb{R}^2$*. (preprint)
[3] C. Valle, *On the Blow-analytic Equivalence of Tribranched Plane Curves*. (preprint)