# Weak BAB conjecture for log del Pezzo surfaces and Mori fiber spaces

代数幾何学シンポジウム記録
2013年度　pp.157-157

Chen Jiang
Graduate School of Mathematical Sciences, The University of Tokyo

## Definitions

We work over complex number field.
Let $X$ be a normal projective variety and $\Delta$ be an $\mathbb{Q}$-divisor on $X$ with coefficients in $[0,1]$ such that $K_X+\Delta$ is $\mathbb{Q}$-Cartier. We say that $(X,\Delta)$ is a *log Fano variety* if $-(K_X+\Delta)$ is ample. In dimension two, it is called log del Pezzo surface. Let $f: Y \to X$ be a log resolution of $(X,\Delta)$, write

$$K_Y = f^*(K_X+\Delta) + \sum a_i F_i,$$

where $F_i$ is a prime divisor. For some $\epsilon \in (0,1]$, the pair $(X,\Delta)$ is called *$\epsilon$-kawamata log terminal* ($\epsilon$-*klt*, for short) if $a_i > -1+\epsilon$ for all $i$, or *$\epsilon$-log canonical* ($\epsilon$-*lc*, for short) if $a_i \geqslant -1+\epsilon$ for all $i$.

## Boundedness on log Fano varieties

One of the most interesting conjecture in minimal model theory is the following B-A-B Conjecture due to A. Borisov, L. Borisov and V. Alexeev.

### (Birational) BAB conjecture

Fix $0<\epsilon<1$, an integer $n>0$, and consider the set of all $n$-dimensional $\epsilon$-klt log Fano varieties $(X,\Delta)$. The set of underlying varieties $\{X\}$ is (birationally) bounded.

The BAB Conjecture is still open in dimension three and higher. We are mainly interested in the following weak conjecture for anti-canonical volumes which is a consequence of BAB Conjecture.

### Weak BAB conjecture

Fix $0<\epsilon<1$, an integer $n>0$, and consider the set of all $n$-dimensional $\epsilon$-klt log Fano varieties $(X,\Delta)$. The volume $\mathrm{Vol}(-(K_X+\Delta)) = (-(K_X+\Delta))^n$ is bounded from above by a fixed number $M(n,\epsilon)$ depending only on $n$ and $\epsilon$.

## Weak BAB conjecture in dimension two

In dimension two, the conjecture is well-researched in history. We give an optimal bound.

### Theorem A

Let $(X,\Delta)$ be an $\epsilon$-lc weak log del Pezzo surface. Then the anti-canonical volume $\mathrm{Vol}(-(K_X+\Delta)) = (K_X+\Delta)^2$ satisfies

$$(K_X+\Delta)^2 \leqslant \max\left\{9, \lfloor 2/\epsilon \rfloor + 4 + \frac{4}{\lfloor 2/\epsilon \rfloor}\right\},$$

where $\lfloor\ \rfloor$ means round down.
Moreover, the equality holds if and only if one of the following holds:
(1) $\epsilon > \frac{2}{5}$ and $(X,\Delta)$ is $(\mathbb{P}^2, 0)$;
(2) $\epsilon \leqslant \frac{1}{2}$ and $(X,\Delta)$ is $(\mathbb{F}_n, (1-\frac{2}{n})S_n)$ or $(\mathrm{PC}_n, 0)$, where $n = \lfloor 2/\epsilon \rfloor$, $\mathbb{F}_n$ is the $n$-th Hirzebruch surface, $S_n \subset \mathbb{F}_n$ is the unique curve with negative self-intersection and $\mathrm{PC}_n$ is the projective cone over a rational normal curve of degree $n$.

## BAB conjecture for Mori fiber spaces

A variety $X$ is said to be with a *Mori fiber space* structure if $X$ has $\mathbb{Q}$-factorial terminal singularities and there is a morphism $f: X \to S$ such that $-K_X$ is ample on fibers, $\rho(X/S) = 1$ and $\dim X > \dim S$. We have the following facts.

### Fact

BAB $\Longrightarrow$ BAB for Mfs $\Longrightarrow$ birational BAB
'weak BAB' for Mfs $\Longrightarrow$ weak BAB

So it is very interesting to investigate the boundedness of Mori fiber spaces.

## 3-fold Mori fiber spaces

There are three possible Mori fiber space structures on a 3-fold log Fano pair $(X,\Delta)$.
**Case 0:** $\dim S = 0$
In this case $X$ is a terminal Fano 3-fold with Picard number one. The BAB conjecture is true for this class of varieties by Kawamata. And the optimal bound for the anti canonical volume is given by Prokhorov.
**Case 1:** $\dim S = 1$
In this case $S$ is just $\mathbb{P}^1$ and the general fiber is smooth del Pezzo surface. We can prove BAB conjecture for $\mathbb{P}^2$-bundles over $\mathbb{P}^1$, or more general, $\mathbb{P}^n$-bundles over $\mathbb{P}^1$. The method is to compute discrepancies for some special centers to get some bound.

### Theorem B

$X = \mathbb{P}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_n))$ with $0 \le a_1 \le \cdots \le a_n$, and $(X,\Delta)$ is $\epsilon$-klt log Fano for some $\Delta$. Then $a_n < 2/\epsilon$.

**Case 2:** $\dim S = 2$
In this case $X$ is a conic bundle over $S$. We can prove weak BAB conjecture for the case $S = \mathbb{P}^2$. The idea is that if the volume is big then we can construct non-klt centers and using connected lemma to get special non-klt centers which is very big on the fibers.

### Theorem C

Let $X \to \mathbb{P}^2$ be a conic bundle, and $(X,\Delta)$ is $\epsilon$-klt log Fano for some $\Delta$. Then $-(K_X+\Delta)^3 \le (12/\epsilon)^3$.