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Virtual Hodge polynomials of the moduli spaces of representations for free monoids (joint work with Takeshi Torii)

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1. Introduction

This article is devoted to virtual Hodge polynomials of the moduli spaces of representations for free monoids. Especially, we deal with the moduli spaces of representations with Borel mold and 2-dimensional representations.

Let Γ be a monoid (or a group). Let k be a field. We say that $\rho : \Gamma \to \mathrm{M}_n(k)$ is an *n-dimensional representation* of Γ if ρ is a monoid homomorphism. We can regard k^n as a Γ-module by ρ.

Definition 1.1. We say that a representation $\rho : \Gamma \to \mathrm{M}_n(k)$ is *irreducible* if the Γ-module k^n has no non-trivial Γ-invariant subspace. We say that a representation $\rho : \Gamma \to \mathrm{M}_n(k)$ is *absolutely irreducible* if $\Gamma \xrightarrow{\rho} \mathrm{M}_n(k) \to \mathrm{M}_n(\overline{k})$ is irreducible, where $\overline{k}$ is an algebraic closure of k.

The following theorem gives us a characterization of absolutely irreducible representations.

Theorem 1.2. *For a representation $\rho : \Gamma \to \mathrm{M}_n(k)$, ρ is absolutely irreducible if and only if the subalgebra $k[\rho(\Gamma)] \subseteq \mathrm{M}_n(k)$ generated by $\rho(\Gamma)$ is equal to $\mathrm{M}_n(k)$.*

Definition 1.3. For representations $\rho_1, \rho_2 : \Gamma \to \mathrm{M}_n(k)$, we say that ρ_1 and ρ_2 are *equivalent* if there exists $P \in \mathrm{GL}_n(k)$ such that $P^{-1}\rho_1(\gamma)P = \rho_2(\gamma)$ for each $\gamma \in \Gamma$.

Definition 1.4. Let $\rho_1, \rho_2 : \Gamma \to \mathrm{M}_n(k)$ be representations. Let M_{ρ_1} and M_{ρ_2} be the Γ-modules induced by ρ_1 and ρ_2, respectively. Suppose that

$$M_{\rho_1} = V_1 \supset V_2 \supset V_3 \supset \cdots \supset V_s = 0$$
$$M_{\rho_2} = W_1 \supset W_2 \supset W_3 \supset \cdots \supset W_t = 0$$

are sequences of Γ-invariant subspaces such that V_i/V_{i+1} and W_j/W_{j+1} are irreducible for each i, j. We say that ρ_1 and ρ_2 are *S-equivalent* if $s = t$ and there exists $\sigma \in S_{s-1}$ such that $V_i/V_{i+1} \cong W_{\sigma(i)}/W_{\sigma(i)+1}$ for each i.

Our main goal is the following:

Goal : Construct the moduli of equivalence classes of representations of Γ.

However, by ordinal methods we can only construct the moduli of S-equivalence classes of representations. We can not construct the moduli of equivalence classes of representations. Hence we introduce a new notion for constructing the moduli of equivalence classes of representations.

Definition 1.5. Let $A, B \subseteq \mathrm{M}_n(k)$ be k-subalgebras. We say that A and B are *equivalent* if there exists $P \in \mathrm{GL}_n(k)$ such that $P^{-1}AP = B$.

Fix a subalgebra $A \subseteq \mathrm{M}_n(k)$. We regard the fixed subalgebra A as a mold, molding, or Igata (鋳型) for constructing the moduli of equivalence classes of representations.

Igata (mold) Program : Construct the moduli of equivalence classes of representations ρ of Γ such that $k[\rho(\Gamma)]$ is equivalent to the fixed subalgebra A.

2. Representations on schemes

For constructing the moduli of representations, we need to define representations on schemes.

Definition 2.1. Let X be a scheme. By a monoid homomorphism $\rho : \Gamma \to \mathrm{M}_n(\Gamma(X, \mathcal{O}_X))$, we understand *a representation of* Γ *on* X.

Proposition 2.2. *The following contravariant functor*

$$\begin{array}{cccc} \mathrm{Rep}_n(\Gamma): & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & X & \mapsto & \{\rho : \textit{ n-dim rep. of } \Gamma \textit{ on } X\ \} \end{array}$$

is representable by an affine scheme over $\mathbb{Z}$.

Proof. Let us consider the polynomial ring $P := \mathbb{Z}[a_{ij}(\gamma) \mid 1 \leq i, j \leq n, \gamma \in \Gamma]$, where each $a_{ij}(\gamma)$ is an indeterminate. Set $\sigma(\gamma) := (a_{ij}(\gamma))_{i,j} \in \mathrm{M}_n(P)$. Put $A_n(\Gamma) := P/I(\Gamma)$, where $I(\Gamma)$ is the ideal of P generated by all entries of $\sigma(\gamma)\sigma(\delta) - \sigma(\gamma\delta)$ and $\sigma(e) - I_n$ for all $\gamma, \delta \in \Gamma$. Then $\mathrm{Spec}A_n(\Gamma)$ represents the functor $\mathrm{Rep}_n(\Gamma)$. □

The group scheme PGL_n over $\mathbb{Z}$ acts on $\mathrm{Rep}_n(\Gamma)$ by $\rho \mapsto P^{-1}\rho P$ for $\rho \in \mathrm{Rep}_n(\Gamma)$ and $P \in \mathrm{PGL}_n$.

Definition 2.3. Let $\rho : \Gamma \to \mathrm{M}_n(\Gamma(X, \mathcal{O}_X))$ be a representation of Γ on X. We say that ρ is *absolutely irreducible* if for each $x \in X$ the induced representation $\Gamma \xrightarrow{\rho} \mathrm{M}_n(\Gamma(X, \mathcal{O}_X)) \to \mathrm{M}(k(x))$ is absolutely irreducible, where $k(x)$ is the residue field of x. It is equivalent to that the subsheaf $\mathcal{O}_X[\rho(\Gamma)]$ of $\mathcal{O}_X$-algebras generated by $\rho(\Gamma)$ is equal to $\mathrm{M}_n(\mathcal{O}_X)$.

Definition 2.4.

$$\mathrm{Rep}_n(\Gamma)_{\mathrm{air}} := \{\ n\text{-dim absolutely irreducible rep. of } \Gamma\ \}$$

Note that $\mathrm{Rep}_n(\Gamma)_{\mathrm{air}}$ is an PGL_n-invariant open subscheme of $\mathrm{Rep}_n(\Gamma)$.

Definition 2.5. Let ρ_1, ρ_2 be n-dimensional representations of Γ on a scheme X. We say that ρ_1 and ρ_2 are *locally equivalent* (or $\rho_1 \sim \rho_2$) if there exist open covering $X = \cup_{i \in I} U_i$ and $P_i \in \mathrm{GL}_n(\mathcal{O}_X(U_i))$ such that $P_i^{-1} \rho_1(\gamma) P_i = \rho_2(\gamma)$ on U_i for each $\gamma \in \Gamma$ and $i \in I$.

Theorem 2.6 $(\cdots,$ K.Saito [11], N- [4] $)$**.** *There exists a universal geometric quotient*

$$\mathrm{Ch}_n(\Gamma)_{\mathrm{air}} := \mathrm{Rep}_n(\Gamma)_{\mathrm{air}}/\mathrm{PGL}_n$$

for arbitrary monoid (or group) Γ. *The quotient* $\mathrm{Rep}_n(\Gamma)_{\mathrm{air}} \to \mathrm{Ch}_n(\Gamma)_{\mathrm{air}}$ *is a* PGL_n*-fibre bundle. Moreover,* $\mathrm{Ch}_n(\Gamma)_{\mathrm{air}}$ *is the coarse moduli scheme over* $\mathbb{Z}$ *associated to the contravariant functor*

$$\begin{array}{rccl} \mathrm{EqAIR}_n(\Gamma): & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & X & \mapsto & \{\ n\text{-}dim.\ abs.\ irr.\ rep.\ of\ \Gamma\ on\ X\}/\sim . \end{array}$$

In other words, there exists a natural transformation $\tau : \mathrm{EqAIR}_n(\Gamma) \to h_{\mathrm{Ch}_n(\Gamma)_{\mathrm{air}}}$ *satisfying the following two conditions:*

(i) *For any scheme* Z, τ *induces the following isomorphism*

$$\tau : \mathrm{Hom}(\mathrm{EqAIR}_n(\Gamma), h_Z) \cong \mathrm{Hom}(h_{\mathrm{Ch}_n(\Gamma)_{\mathrm{air}}}, h_Z).$$

(ii) *For any algebraically closed field* Ω, *the morphism*

$$\tau : \mathrm{EqAIR}_n(\Gamma)(\Omega) \to h_{\mathrm{Ch}_n(\Gamma)_{\mathrm{air}}}(\Omega)$$

is bijective.

Here we denote $\mathrm{Hom}(-, Z)$ *by* h_Z *for a scheme* Z.

3. Borel mold

We construct several moduli spaces of representations by using the notion of molds on schemes. More precisely, see [5], [7], [8], and [9].

Definition 3.1. Let $\mathcal{A} \subseteq \mathrm{M}_n(\mathcal{O}_X)$ be a subsheaf of $\mathcal{O}_X$-algebras on a scheme X. We say that $\mathcal{A}$ is *a mold (or Igata)* if $\mathcal{A}$ and $\mathrm{M}_n(\mathcal{O}_X)/\mathcal{A}$ are locally free sheaves on X.

Definition 3.2. Let $\mathcal{A}, \mathcal{B} \subseteq \mathrm{M}_n(\mathcal{O}_X)$ be molds on X. We say that $\mathcal{A}$ and $\mathcal{B}$ are *locally equivalent* if there exist open covering $X = \cup_{i\in I} U_i$ and $P_i \in \mathrm{GL}_n(\mathcal{O}_X(U_i))$ such that $P_i^{-1} \cdot \mathcal{A}|_{U_i} \cdot P_i = \mathcal{B}|_{U_i}$ for each $i \in I$.

Definition 3.3. Let ρ be a representation of Γ on X. We say that ρ *has a mold* $\mathcal{A}$ if $\mathcal{A} = \mathcal{O}_X[\rho(\Gamma)]$.

Borel mold is a typical example. The moduli of representations with Borel mold can be described by configuration spaces.

Definition 3.4. Let $\mathcal{B}_n := \left\{ \begin{pmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & * \end{pmatrix} \right\} \subset \mathrm{M}_n(\mathbb{Z})$ be the mold which consists of upper triangular matrices on $\operatorname{Spec}\mathbb{Z}$. For a mold $\mathcal{A}$ on X, we say that $\mathcal{A}$ is *a Borel mold* on X if $\mathcal{A}$ and $\mathcal{B}_n \otimes_{\mathbb{Z}} \mathcal{O}_X$ are locally equivalent on X.

Let $\mathrm{Rep}_n(\Gamma)_B := \{$ n-dim rep. with Borel mold of Γ $\}$. Note that $\mathrm{Rep}_n(\Gamma)_B$ is a PGL_n-invariant subscheme of $\mathrm{Rep}_n(\Gamma)$.

Theorem 3.5 (N- [5])**.** *There exists a universal geometric quotient*

$$\mathrm{Ch}_n(\Gamma)_B := \mathrm{Rep}_n(\Gamma)_B/\mathrm{PGL}_n$$

for arbitrary monoid (or group) Γ*. The quotient* $\mathrm{Rep}_n(\Gamma)_B \to \mathrm{Ch}_n(\Gamma)_B$ *is a* PGL_n*-fibre bundle which has locally trivialization with respect to Zariski topology. Moreover,* $\mathrm{Ch}_n(\Gamma)_B$ *is the fine moduli scheme over* $\mathbb{Z}$ *associated to the sheafification (with respect to Zariski topology) of the contravariant functor*

$$\begin{array}{cccc} \mathrm{EqB}_n(\Gamma): & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & X & \mapsto & \left\{ \begin{array}{l} n\text{-dimensional representation} \\ \text{with Borel mold of } \Gamma \text{ on } X \end{array} \right\} \Big/ \sim . \end{array}$$

Remark 3.6. By a *generalized representation with Borel mold* of degree n on a scheme X, we understand pairs $\{(U_i, \rho_i)\}_{i\in I}$ of an open set U_i and a representation with Borel mold $\rho_i : \Gamma \to \mathrm{M}_n(\Gamma(U_i, \mathcal{O}_X))$ such that $X = \cup_{i\in I} U_i$ and for each $i, j \in I$, ρ_i and ρ_j are locally equivalent on $U_i \cap U_j$. We say that generalized representations with Borel mold $\{(U_i, \rho_i)\}_{i\in I}$ and $\{(V_j, \rho_j)\}_{j\in J}$ are *equivalent* if $\{(U_i, \rho_i)\}_{i\in I} \cup \{(V_j, \rho_j)\}_{j\in J}$ is a generalized representations with Borel mold again. For any scheme X, $\mathrm{EqB}_n(\Gamma)(X)(= h_{\mathrm{Ch}_n(\Gamma)_B}(X))$ is equal to the set of equivalence classes of generalized representations of degree n on X.

Let $\Gamma_m := \langle \alpha_1, \alpha_2, \ldots, \alpha_m \rangle$ be the free monoid of rank m. Put $\mathrm{Rep}_n(m)_B := \mathrm{Rep}_n(\Gamma_m)_B$ and $\mathrm{Ch}_n(m)_B := \mathrm{Ch}_n(\Gamma_m)_B$. Denote by $\mathrm{Rep}_n(m)_B(\mathbb{C})$ and $\mathrm{Ch}_n(m)_B(\mathbb{C})$ the sets of $\mathbb{C}$-valued points of $\mathrm{Rep}_n(m)_B$ and $\mathrm{Ch}_n(m)_B$, respectively.

Remark 3.7 ([7])**.** Let us consider the case $n = 1$ or the case $n \geq 2$ and $m \geq 2$. Then $\mathrm{Rep}_n(m)_B(\mathbb{C})$ and $\mathrm{Ch}_n(m)_B(\mathbb{C})$ are non-empty connected complex smooth manifolds. There exist fibre bundle structures

$$\begin{array}{ccccc} Y_R & \to & \mathrm{Rep}_n(m)_B(\mathbb{C}) & \to & \mathrm{F}_n(\mathbb{C}^m) \\ Y_C & \to & \mathrm{Ch}_n(m)_B(\mathbb{C}) & \to & \mathrm{F}_n(\mathbb{C}^m) \end{array}$$

with fibers Y_R and $Y_C \cong (\mathbb{CP}^{m-2})^{n-1} \times \mathbb{C}^{m(n-1)(n-2)/2} \simeq (\mathbb{CP}^{m-2})^{n-1}$. Here $\mathrm{F}_n(\mathbb{C}^m) := \{(p_1, p_2, \ldots, p_n) \mid p_i \text{ are } n\text{-distinct points of } \mathbb{C}^m\}$. The smooth algebraic variety Y_R also has a fibre bundle structure

$$Y_B \to Y_R \to \mathrm{Flag}(\mathbb{C}^n)$$

with fibre $Y_B \cong (\mathbb{C}^m - \mathbb{C}^1)^{n-1} \times \mathbb{C}^{m(n-1)(n-2)/2} \simeq (S^{2m-3})^{n-1}$, where $\mathrm{Flag}(\mathbb{C}^n)$ is the flag variety which consists of complete flags in $\mathbb{C}^n$.

We can determine not homotopy types, but rational homotopy types of $\mathrm{Rep}_n(m)_B(\mathbb{C})$ and $\mathrm{Ch}_n(m)_B(\mathbb{C})$.

Theorem 3.8 (Torii [8])**.** *$\mathrm{Rep}_n(m)_B(\mathbb{C})$ and $\mathrm{Ch}_n(m)_B(\mathbb{C})$ are rationally homotopy equivalent to $\mathrm{F}_n(\mathbb{C}^m) \times Y_R$ and $\mathrm{F}_n(\mathbb{C}^m) \times Y_C$, respectively. Moreover, the Sullivan's minimal models $\mathcal{M}(\mathrm{Rep}_n(m)_B(\mathbb{C}))$ and $\mathcal{M}(\mathrm{Ch}_n(m)_B(\mathbb{C}))$ with mixed Hodge structure are equivalent to the tensor products of $\mathcal{M}(\mathrm{F}_n(\mathbb{C}^m)) \otimes \mathcal{M}(Y_R)$ and $\mathcal{M}(\mathrm{F}_n(\mathbb{C}^m)) \otimes \mathcal{M}(Y_C)$, respectively.*

Let X be a smooth complex algebraic variety. Let $a^{p,q}(H^m(X;\mathbb{Q}))$ be the dimension of the (p,q)-component of the pure Hodge structure $\mathrm{Gr}^W_{p+q}(H^m(X;\mathbb{Q}))$ of weight $p+q$. Set $h^{p,q} := \sum_m (-1)^m a^{p,q}(H^m(X;\mathbb{Q}))$. The virtual Hodge polynomial of X is defined by

$$\mathrm{VHP}(X) := \sum_{p,q} h^{p,q} x^p y^q.$$

Let $\mathrm{VHP}_c(X)$ be the virtual Hodge polynomial of X based on compact support cohomology. If $\dim X = m$, then

$$\mathrm{VHP}(X)(x,y) = (xy)^m \mathrm{VHP}_c(X)(x^{-1}, y^{-1}).$$

Theorem 3.9 (Torii and N- [7], [8]). *For simplicity, $z = xy$.*

$$\mathrm{VHP}_c(\mathrm{Rep}_n(m)_B(\mathbb{C})) = \frac{z^{m(n-1)(n-2)/2}(z^m - z)^{n-1}\prod_{k=0}^{n-1}(z^m - k)\prod_{k=1}^{n}(z^k - 1)}{(z-1)^n}.$$

$$\mathrm{VHP}_c(\mathrm{Ch}_n(m)_B(\mathbb{C})) = \frac{z^{(m-1)(n-1)(n-2)/2}(z^{m-1} - 1)^{n-1}\prod_{k=0}^{n-1}(z^m - k)}{(z-1)^{n-1}}.$$

Remark 3.10. Let X be a separated scheme of finite type over $\mathbb{Z}$. If there exists $P(t) \in \mathbb{Z}[t]$ such that $|X(\mathbb{F}_q)| = P(q)$ for all finite fields $\mathbb{F}_q$, then $\mathrm{VHP}_c(X) = P(z)$, where $z = xy$. For details, see [1, §6].

The next theorem can be obtained independently from the result on virtual Hodge polynomials. By counting rational points directly we can obtain the following:

Theorem 3.11 (Torii and N- [10]).

$$|\mathrm{Rep}_n(m)_B(\mathbb{F}_q))| = \frac{q^{m(n-1)(n-2)/2}(q^m - q)^{n-1}\prod_{k=0}^{n-1}(q^m - k)\prod_{k=1}^{n}(q^k - 1)}{(q-1)^n}.$$

$$|\mathrm{Ch}_n(m)_B(\mathbb{F}_q))| = \frac{q^{(m-1)(n-1)(n-2)/2}(q^{m-1} - 1)^{n-1}\prod_{k=0}^{n-1}(q^m - k)}{(q-1)^{n-1}}.$$

4. The degree 2 case

This section is devoted to the moduli of representations of degree 2. For details, see [9] and [10]. Recall that k-subalgebras A and B of $\mathrm{M}_n(k)$ are equivalent if there exists $P \in \mathrm{GL}_n(k)$ such that $P^{-1}AP = B$.

Proposition 4.1. *Let $k = \overline{k}$. Let $A \subseteq \mathrm{M}_2(k)$ be a k-subalgebra. Then A is equivalent to one of the following:*

(i) $\mathrm{M}_2(k)$

(ii) $\mathcal{B}_2(k) = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$ *Borel mold*

(iii) $D_2(k) = \left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}$ *semi-simple mold*

(iv) $U_2(k) = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \mid a, b \in k \right\}$ *unipotent mold*

(v) $k \cdot I_2 = \left\{ \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} \mid a \in k \right\}$ *scalar mold*

In the cases (i) $A = \mathrm{M}_2(k)$ and (ii) $A = \mathcal{B}_2(k)$, we have constructed the moduli of equivalence classes of representations with mold A in the previous section. In the case (ii) $A = \mathcal{B}_2(k)$, we have computed the virtual Hodge polynomials of the moduli of equivalence classes of representations with mold A for free monoids. In this section, we deal with another molds of degree 2.

4.1. **Semi-simple mold.**

Definition 4.2. Let ρ be a 2-dimensional representation of Γ on X. We say that ρ is *a representation with semi-simple mold* if $\mathcal{O}_X[\rho(\Gamma)]$ is a rank 2 mold on X and for each $x \in X$ there exists $\gamma \in \Gamma$ such that $\mathrm{tr}(\rho(\gamma))^2 - 4\det(\rho(\gamma)) \neq 0$ in the residue field $k(x)$ of x.

Definition 4.3. For a monoid (or group) Γ, we define $\mathrm{Rep}_2(\Gamma)_{s.s.} :=$ $\{$ 2-dim rep. with semi-simple mold of Γ $\}$. Note that $\mathrm{Rep}_2(\Gamma)_{s.s.}$ is a PGL_2-invariant subscheme of $\mathrm{Rep}_2(\Gamma)$.

Theorem 4.4 (N- [9])**.** *There exists a universal geometric quotient*

$$\mathrm{Ch}_2(\Gamma)_{s.s.} := \mathrm{Rep}_2(\Gamma)_{s.s.}/\mathrm{PGL}_2$$

for arbitrary monoid (or group) Γ. *Moreover,* $\mathrm{Ch}_2(\Gamma)_{s.s.}$ *is the fine moduli scheme over* $\mathbb{Z}$ *associated to the sheafification (with respect to Zariski topology) of the contravariant functor*

$$\begin{array}{rccc} \mathrm{EqSS}_2(\Gamma): & (\mathbf{Sch})^{op} & \to & (\mathbf{Sets}) \\ & X & \mapsto & \left\{ \begin{array}{c} \textit{2-dim. rep. with semi-simple} \\ \textit{mold of } \Gamma \textit{ on } X \end{array} \right\} \Big/ \sim . \end{array}$$

Theorem 4.5 (Torii and N- [10])**.** *For simplicity,* $z = xy$. *For free monoids* $\Gamma = \Gamma_m$,

$$\begin{aligned} \mathrm{VHP}_c(\mathrm{Rep}_2(m)_{s.s.}(\mathbb{C})) &= z^{m+2}(z^m - 1) \\ \mathrm{VHP}_c(\mathrm{Ch}_2(m)_{s.s.}(\mathbb{C})) &= z^m(z^m - 1). \end{aligned}$$

Theorem 4.6 (Torii and N- [10])**.**

$$\begin{aligned} |\mathrm{Rep}_2(m)_{s.s.}(\mathbb{F}_q)| &= q^{m+2}(q^m - 1) \\ |\mathrm{Ch}_2(m)_{s.s.}(\mathbb{F}_q)| &= q^m(q^m - 1). \end{aligned}$$

4.2. **Unipotent mold of** ch $\neq 2$. We need to divide (iv) unipotent molds into two cases. One is the case ch $\neq 2$, and another is the case ch $= 2$. One of the reasons why we need to consider two cases is that we can construct "good" moduli spaces of representations with unipotent mold by dividing two cases. By constructing "good" moduli spaces, we understand constructing smooth moduli spaces at least for free monoids.

Definition 4.7. Let ρ be a 2-dimensional representation of Γ on a scheme X over $\mathbb{Z}[1/2]$. We say that ρ is *a representation with unipotent mold* if $\mathcal{O}_X[\rho(\Gamma)]$ is a rank 2 mold on X and $\mathrm{tr}(\rho(\gamma))^2 - 4\det(\rho(\gamma)) = 0$ for each $\gamma \in \Gamma$.

Definition 4.8. $\mathrm{Rep}_2(\Gamma)_u := \{$ 2-dim rep. with unipotent mold of Γ $\}$. Note that $\mathrm{Rep}_2(\Gamma)_u$ is a $\mathrm{PGL}_2 \otimes \mathbb{Z}[1/2]$-invariant subscheme of $\mathrm{Rep}_2(\Gamma) \otimes \mathbb{Z}[1/2]$.

Theorem 4.9 (N- [9]). *There exists a universal geometric quotient*

$$\mathrm{Ch}_2(\Gamma)_u := \mathrm{Rep}_2(\Gamma)_u / (\mathrm{PGL}_2 \otimes \mathbb{Z}[1/2])$$

for arbitrary monoid (or group) Γ. *Moreover,* $\mathrm{Ch}_2(\Gamma)_u$ *is the fine moduli scheme over* $\mathbb{Z}[1/2]$ *associated to the sheafification (with respect to Zariski topology) of the contravariant functor*

$$\begin{array}{rccl} \mathrm{EqU}_2(\Gamma): & (\mathbf{Sch}/\mathbb{Z}[1/2])^{op} & \to & (\mathbf{Sets}) \\ & X & \mapsto & \{ \textit{ rep. with unip. mold on } X \}/\sim . \end{array}$$

Theorem 4.10 (Torii and N- [10]). *For simplicity,* $z = xy$. *For free monoids* $\Gamma = \Gamma_m$,

$$\begin{aligned} \mathrm{VHP}_c(\mathrm{Rep}_2(m)_u(\mathbb{C})) &= z^m(z+1)(z^m-1) \\ \mathrm{VHP}_c(\mathrm{Ch}_2(m)_u(\mathbb{C})) &= \frac{z^m(z^m-1)}{z-1}. \end{aligned}$$

Theorem 4.11 (Torii and N- [10]). *If* 2 *does not divide* q, *then*

$$\begin{aligned} |\mathrm{Rep}_2(m)_u(\mathbb{F}_q)| &= q^m(q+1)(q^m-1) \\ |\mathrm{Ch}_2(m)_u(\mathbb{F}_q)| &= \frac{q^m(q^m-1)}{q-1}. \end{aligned}$$

4.3. **Unipotent mold of** ch $= 2$.

Definition 4.12. Let X be a scheme over $\mathbb{F}_2$. Let ρ be a 2-dimensional representation of Γ on X. We say that ρ is *a representation with unipotent mold over* $\mathbb{F}_2$ if $\mathcal{O}_X[\rho(\Gamma)]$ is a rank 2 mold on X and $\mathrm{tr}(\rho(\gamma)) = 0$ for each $\gamma \in \Gamma$.

Definition 4.13. For a monoid (or group) Γ, we define $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2} :=$ $\{$ 2-dim rep. with unipotent mold over $\mathbb{F}_2$ of Γ $\}$. We see that $\mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2}$ is a $\mathrm{PGL}_2 \otimes \mathbb{F}_2$-invariant subscheme of $\mathrm{Rep}_2(\Gamma) \otimes \mathbb{F}_2$.

Theorem 4.14 (N- [9]). *There exists a universal geometric quotient*

$$\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2} := \mathrm{Rep}_2(\Gamma)_{u/\mathbb{F}_2}/(\mathrm{PGL}_2 \otimes \mathbb{F}_2)$$

for arbitrary monoid (or group) Γ. *Moreover,* $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ *is the fine moduli scheme over* $\mathbb{F}_2$ *associated to the sheafification (with respect to Zariski topology) of the contravariant functor*

$$\begin{array}{cccc} \mathrm{EqU}_{2/\mathbb{F}_2}(\Gamma): & (\mathbf{Sch}/\mathbb{F}_2)^{op} & \to & (\mathbf{Sets}) \\ & X & \mapsto & \left\{ \begin{array}{c} \textit{rep. with unip. mold} \\ \textit{over } \mathbb{F}_2 \textit{ on } X \end{array} \right\} \Big/ \sim . \end{array}$$

Theorem 4.15 (Torii and N- [10]). *If* 2 *divides* q, *then*

$$\begin{aligned} |\mathrm{Rep}_2(m)_{u/\mathbb{F}_2}(\mathbb{F}_q)| &= q^m(q+1)(q^m-1) \\ |(\mathrm{Ch}_2(m)_{u/\mathbb{F}_2}(\mathbb{F}_q)| &= \frac{q^m(q^m-1)}{q-1}. \end{aligned}$$

Remark 4.16. As in Remark 3.6, we can define 2-dimensional generalized representations with semi-simple mold (unipotent mold, or unipotent mold over $\mathbb{F}_2$) of Γ on a scheme X over $\mathbb{Z}$ (over $\mathbb{Z}[1/2]$ or $\mathbb{F}_2$, respectively). We can also define equivalence classes of generalized representations with given mold. We see that $\mathrm{EqSS}_2(\Gamma)(X)$, $\mathrm{EqU}_2(\Gamma)(X)$, and $\mathrm{EqU}_{2/\mathbb{F}_2}(\Gamma)(X)$ are the set of equivalence classes of generalized representations with the corresponding mold of Γ on X. Then $\mathrm{Ch}_2(\Gamma)_{s.s.}$, $\mathrm{Ch}_2(\Gamma)_u$, and $\mathrm{Ch}_2(\Gamma)_{u/\mathbb{F}_2}$ represent $\mathrm{EqSS}_2(\Gamma)$, $\mathrm{EqU}_2(\Gamma)$, and $\mathrm{EqU}_{2/\mathbb{F}_2}(\Gamma)$, respectively.

4.4. **Scalar mold.** For scalar mold, we see that

$$\mathrm{Rep}_2(\Gamma)_{sc} = \mathrm{Ch}_2(\Gamma)_{sc} = \mathrm{Rep}_1(\Gamma) \ \ (= \mathrm{Ch}_1(\Gamma) := \mathrm{Rep}_1(\Gamma)//\mathrm{PGL}_1)$$

for arbitrary Γ. For $\Gamma = \Gamma_m$, we have

$$\begin{array}{ccccc} \mathrm{VHP}_c(\mathrm{Rep}_2(m)_{sc}(\mathbb{C})) & = & \mathrm{VHP}_c(\mathrm{Ch}_2(m)_{sc}(\mathbb{C})) & = & z^m \\ |\mathrm{Rep}_2(m)_{sc}(\mathbb{F}_q)| & = & |\mathrm{Ch}_2(m)_{sc}(\mathbb{F}_q)| & = & q^m. \end{array}$$

4.5. **Absolutely irreducible representations.** Note that

$$\mathrm{M}_2(\mathbb{F}_q)^m = \coprod_{*=\mathrm{sc,ss,u,B,air}} \mathrm{Rep}_2(m)_*(\mathbb{F}_q).$$

Hence

$$|\mathrm{Rep}_2(m)_{\mathrm{air}}(\mathbb{F}_q)| = q^{4m} - \sum_{*=sc,ss,u,B} |\mathrm{Rep}_2(m)_*(\mathbb{F}_q)|.$$

Theorem 4.17 (Torii and N- [10]). *For simplicity, $z = xy$. For $\Gamma = \Gamma_m$,*

$$\mathrm{VHP}_c(\mathrm{Rep}_2(m)_{\mathrm{air}}(\mathbb{C})) = z^{2m+1}(z^m - 1)(z^{m-1} - 1),$$
$$\mathrm{VHP}_c(\mathrm{Ch}_2(m)_{\mathrm{air}}(\mathbb{C})) = \frac{z^{2m}(z^m - 1)(z^{m-1} - 1)}{z^2 - 1}.$$

Theorem 4.18 (Torii and N- [10]).

$$|\mathrm{Rep}_2(m)_{\mathrm{air}}(\mathbb{F}_q))| = q^{2m+1}(q^m - 1)(q^{m-1} - 1),$$
$$|\mathrm{Ch}_2(m)_{\mathrm{air}}(\mathbb{F}_q))| = \frac{q^{2m}(q^m - 1)(q^{m-1} - 1)}{q^2 - 1}.$$

Corollary 4.19. *The number of equivalence classes of 2-dimensional absolutely irreducible representations over $\mathbb{F}_q$ of the free algebra $\mathbb{F}_q\langle X_1, X_2, \ldots, X_m\rangle$ is*

$$\frac{q^{2m}(q^m - 1)(q^{m-1} - 1)}{q^2 - 1}.$$

Remark 4.20. These results are compatible with Hua's result on the number of absolutely indecomposable representations of quivers over $\mathbb{F}_q$. Let S_m be the quiver with one vertex and m edge loops. Let $\mathrm{AID}_{S_m}(n, q)$ be the number of isomorphism classes of n-dimensional absolutely indecomposable representations of S_m over $\mathbb{F}_q$. By [2, Theorem 4.6], we have

$$\mathrm{AID}_{S_m}(2, q) = \frac{q^{2m-1}(q^{2m} - 1)}{q^2 - 1}.$$

This number is equal to

$$|\mathrm{Ch}_2(m)_{\mathrm{air}}(\mathbb{F}_q)| + |\mathrm{Ch}_2(m)_B(\mathbb{F}_q)| + |\mathrm{Ch}_2(m)_u(\mathbb{F}_q)|.$$

Remark 4.21. Let $A_n(\Gamma)$ be the affine ring of $\mathrm{Rep}_n(\Gamma)$ (*cf.* Proposition 2.2). Let $A_n(\Gamma)^{\mathrm{PGL}_n}$ be the PGL_n-invariant ring of $A_n(\Gamma)$. We set $\mathrm{Ch}_n(\Gamma) := \mathrm{Spec} A_n(\Gamma)^{\mathrm{PGL}_n}$. For $\Gamma = \Gamma_m$ and $n = 2$, we have

$$\begin{aligned} |\mathrm{Ch}_2(m)(\mathbb{F}_q)| &= |\mathrm{Ch}_2(m)_{\mathrm{air}}(\mathbb{F}_q)| + |\mathrm{Ch}_2(m)_{s.s.}(\mathbb{F}_q)| + |\mathrm{Ch}_2(m)_{sc}(\mathbb{F}_q)| \\ &= \frac{q^{2m+2}(q^{2m-3} - q^{m-2} - q^{m-3} + 1)}{q^2 - 1} \end{aligned}$$

by considering closed PGL_2-orbits of $\mathrm{Rep}_2(m)$.

The Weil zeta functions of $\mathrm{Rep}_2(m)_{\mathrm{air}}$ and $\mathrm{Ch}_2(m)_{\mathrm{air}}$ are

$$Z(\mathrm{Rep}_2(m)_{\mathrm{air}}, q, t) := \exp\Big(\sum_{n=1}^{\infty} \frac{|\mathrm{Rep}_2(m)_{\mathrm{air}}(\mathbb{F}_{q^n})|}{n} t^n\Big)$$
$$= \frac{(1-q^{3m+1}t)(1-q^{3m}t)}{(1-q^{4m}t)(1-q^{2m+1}t)},$$
$$Z(\mathrm{Ch}_2(m)_{\mathrm{air}}, q, t) := \exp\Big(\sum_{n=1}^{\infty} \frac{|\mathrm{Ch}_2(m)_{\mathrm{air}}(\mathbb{F}_{q^n})|}{n} t^n\Big)$$
$$= \frac{\prod_{i=1}^{[\frac{m}{2}]}(1-q^{2m+2i-2}t)}{\prod_{i=1}^{[\frac{m}{2}]}(1-q^{4m-2i-1}t)}.$$

The Hasse-Weil zeta functions of $\mathrm{Rep}_2(m)_{\mathrm{air}}$ and $\mathrm{Ch}_2(m)_{\mathrm{air}}$ are

$$\zeta(\mathrm{Rep}_2(m)_{\mathrm{air}}, s) := \prod_p Z(\mathrm{Rep}_2(m)_{\mathrm{air}}, p, p^{-s})$$
$$= \frac{\zeta(s-4m)\zeta(s-2m-1)}{\zeta(s-3m-1)\zeta(s-3m)},$$
$$\zeta(\mathrm{Ch}_2(m)_{\mathrm{air}}, s) := \prod_p Z(\mathrm{Ch}_2(m)_{\mathrm{air}}, p, p^{-s})$$
$$= \frac{\prod_{i=1}^{[\frac{m}{2}]}\zeta(s-4m+2i+1)}{\prod_{i=1}^{[\frac{m}{2}]}\zeta(s-2m-2i+2)},$$

where $\zeta(s)$ is the Riemann zeta function.

The completions of these zeta functions are defined as

$$\hat{\zeta}(\mathrm{Rep}_2(m)_{\mathrm{air}}, s) := \frac{\hat{\zeta}(s-4m)\hat{\zeta}(s-2m-1)}{\hat{\zeta}(s-3m-1)\hat{\zeta}(s-3m)},$$
$$\hat{\zeta}(\mathrm{Ch}_2(m)_{\mathrm{air}}, s) := \frac{\prod_{i=1}^{[\frac{m}{2}]}\hat{\zeta}(s-4m+2i+1)}{\prod_{i=1}^{[\frac{m}{2}]}\hat{\zeta}(s-2m-2i+2)},$$

where $\hat{\zeta}(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$ is the completion of the Riemann zeta function. Since $\hat{\zeta}(1-s) = \hat{\zeta}(s)$, the following functional equations hold:

$$\begin{aligned}\hat{\zeta}(\mathrm{Rep}_2(m)_{\mathrm{air}}, 6m+2-s) &= \hat{\zeta}(\mathrm{Rep}_2(m)_{\mathrm{air}}, s)\\ \hat{\zeta}(\mathrm{Ch}_2(m)_{\mathrm{air}}, 6m-2-s) &= \hat{\zeta}(\mathrm{Ch}_2(m)_{\mathrm{air}}, s)^{-1}.\end{aligned}$$

Remark 4.22. In the degree 3 case, there exist 26 types of equivalence classes of k-subalgebras of $\mathrm{M}_3(k)$ for algebraically closed fields k. We will discuss this topic in another paper.

In the degree ≥ 4 case, there are special types of irreducible representations, which is called "thick" and "dense". The moduli of equivalence classes of absolutely thick representations (or absolutely dense representations) are open subschemes of $\mathrm{Ch}_n(\Gamma)_{\mathrm{air}}$. In [6], Omoda classified thick (or dense) finite dimensional representations of complex simple Lie groups. It is very interesting to describe the moduli spaces of non-thick absolutely irreducible representations for a given groups, which we will discuss in the future.

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