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AN EFFECTIVE BIRATIONALITY OF PLURICANONICAL MAPS FOR A FAMILY OF CANONICALLY POLARIZED MANIFOLDS OVER A CURVE

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1. Introduction

We let B be a curve (i.e., a compact Riemann surface) of genus $g = g(B)$, and $S \subset B$ a finite set of points with $s = \#S$. A family of varieties $f : X \to B$, where X is smooth projective, is said to be *isotrivial*, if general fibers are isomorphic. We say $f : X \to B$ is *admissible over* (B, S), if f is smooth over $B \setminus S$ and non-isotrivial. We are interested in the case when general fibers F are canonically polarized i.e. K_F ample, with a Hilbert polynomial h i.e. $h(m) = \chi(F, mK_F) = \sum_{i=0}^{n} h^i(F, mK_F)$, where $n = \dim F$. Our main result is the following

Theorem 1.1. *For a given (B, S) and a Hilbert polynomial h as above, there exist effective positive integers $N = N(g, s, h)$ and $d = d(g, s, h)$ depending only on g, s and h with the following properties. For any admissible family $f : X \to B$ over (B, S) of canonically polarized manifolds with Hilbert polynomial h, there exists a rational map $\Phi : X \dashrightarrow \mathbb{P}^N$, which is birational onto its image and gives a regular embedding on $X \setminus f^{-1}(S)$, such that the degree of the image of X is bounded by d, i.e.,* $\deg \Phi(X) \leq d$.

A crucial point is the upper bound of the degree $\deg \Phi(X) \leq d$, which will lead a finiteness/boundedness of some families of varieties, as is generally known. In the case $g \geq 2$, one can take Φ to be a pluricanonical map $\Phi_{|m_0 K_X|}$ with $m_0 = O(n^3)$ depending only on $n = \dim F$. As we will explain, this Φ is given by a linear system of adjoint type related to relative pluri-canonical divisors in general.

As an application of this effective birationality, we can obtain an effective version of the so-called Shafarevich type conjecture, which is the main motivation of 1.1. By a technical reason, we suppose $S = \emptyset$ in the following theorem, although we still use the symbol S.

Theorem 1.2. *Let B and S be as above (but, read $S = \emptyset$ and $s = 0$).*

(1) For a given Hilbert polynomial h, there exist an effective positive number $C(g, s, h)$ depending only on g, s and h, such that the number of deformation types, of all admissible family $f : X \to B$ over (B, S) of canonically polarized manifolds with Hilbert polynomial h, is bounded from above by $C(g, s, h)$.

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(2) For a given positive integer v, there exist an effective positive number $C(g,s,n,v)$ depending only on g,s,n and v, such that the number of deformation types, of all admissible family $f : X \to B$ over (B,S) of n-dimensional canonically polarized manifolds with canonical volume $K_F^n = v$, is bounded from above by $C(g,s,n,v)$.

As $h(x) = (v/n!)x^n+$ (lower order terms), (2) clearly implies (1). However if we look at the estimates of $N(g,s,h)$ and $d(g,s,h)$ in 1.1 carefully how the estimates depend on h, we can see an estimate of the leading term of h (the volume $v = K_F^n$) is enough, and we can obtain (2) from (1). See §3 for some general effective bounds on Hilbert polynomials.

To obtain 1.2 from 1.1, we adapt an effective result of the complexity bound of Chow varieties due to Catanese [Cat92], Kollár [Kol96], Tsai, ..., it states that the number of irreducible components of $Chow_{k,\delta}(\mathbb{P}^N)$ (the parameter space of effective k-dimensional cycles of degree δ in $\mathbb{P}^N$) is bounded by an effective number $C(N,k,\delta)$.

In this notes, we do not try to give effective numbers explicitly, because these numbers can be huge in general, and as combinations of these numbers, the final estimate becomes messy. For example, the complexity bound of $Chow_{k,\delta}(\mathbb{P}^N)$ mentioned above is like

$$C(N,k,\delta) = \binom{(N+1)\delta}{N}^{(N+1)\left(\delta\binom{\delta+k-1}{k}+\binom{\delta+k-1}{k-1}\right)},$$

where $\binom{a}{b}$ is a binomial coefficient.

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2. Effective Birationality

We shall prove 1.1. We explain how to obtain the map Φ in §2.1, and the degree bound $\deg \Phi(X) \le d$ in §2.2.

2.1. **Birationality.** We use the following set up.

Setup 2.1. Let $f : X \to B$ be any admissible family over (B,S) of canonically polarized n-folds with Hilbert polynomial h. We take a positive integer $m_0 = O(n^3)$ so that $|m_0K_F|$ is very ample for any smooth fiber F (this is Angehrn=Siu's bound [AS95]), and an ample divisor A on B with $\deg A = a \ge 2$. We let

$$L = f^*(K_B + A) + m_0K_{X/B},$$
$$E = f_*\mathcal{O}_X(L) = \mathcal{O}_B(K_B + A) \otimes f_*\mathcal{O}_X(m_0K_{X/B}).$$

Both L and E are "adjoint type". We note that E is a vector bundle of rank $r := h(m_0)$. We let $\pi : \mathbb{P}(E) \to B$ be the $\mathbb{P}^{r-1}$-bundle associated to E, $\mathcal{O}(1)$ the universal quotient line bundle for π, and H a divisor on $\mathbb{P}(E)$ with $\mathcal{O}_{\mathbb{P}(E)}(H) = \mathcal{O}(1)$. □

We also use the following notations. Let $\omega_B = \mathcal{O}_B(K_B)$, $\omega_{X/B}^{m_0} = \mathcal{O}_X(m_0K_{X/B})$, $\mathcal{A} = \mathcal{O}_B(A), \mathcal{L} = \mathcal{O}_X(L)$. We denote, as usual, by $\Phi_{|L|} : X \dashrightarrow \mathbb{P}^{N_0}$ the rational map associated to the complete linear system $|L|$, and by $\Phi_{|L|}(X)$ the closure $\overline{\Phi_{|L|}(X \setminus \mathrm{Bs}\,|L|)} \subset \mathbb{P}^{N_0}$, where $\mathrm{Bs}\,|L|$ is the base locus of the linear system.

We will use the following fundamental facts.

Fact 2.2. (1) $f_*\mathcal{O}_X(m_0K_{X/B})$ is ample (it is nef in general over curves), due to Kawamata [Kaw82], Viehweg, Kollár, ..., because of the ampleness of K_F and the non-isotriviality of f.

(2) rank $f_*\mathcal{O}_X(m_0K_{X/B}) = h^0(F, m_0K_F) = h(m_0)$.

(3) deg $f_*\mathcal{O}_X(m_0K_{X/B}) \leq \delta(g,s,h,m_0)$. Here, for every integer $m \geq 2$, we set

$$\delta(g,s,h,m) = (n(2g-2+s)+s)\cdot m \cdot (m^nK_F^n+1)\cdot h(m).$$

This is an essential term in our effective estimate, and comes from a theorem of Bedulev-Viehweg [BV00, 1.4(c)]. We note that $2g-2+s>0$ by [BV00, 1.4(a)]. □

We shall denote vaguely various effective positive integers depending only on g, s and h by $N(g,s,h), d(g,s,h), C(g,s,h)$ for example. Because of $m_0 = O(n^3)$ and $n = \deg h$, we can regard "$\delta(g,s,h,m_0) = \delta(g,s,h)$".

The next proposition gives a more explicit form of 1.1. In the case $g \geq 2$, we can take $A = (m_0-1)K_B$ above, then $L = m_0K_X$ and $\Phi_{|L|}$ is the m_0-th pluricanonical map. Hence if we put $a = (m_0-1)(2g-2)$, we have the bounds with respect to $\Phi_{|m_0K_X|}$. In any case, every smooth fiber F is embedded by $|m_0K_F|$.

Proposition 2.3. *In Setup 2.1, one has:*

(1) $h^0(X,\mathcal{L}) = h^0(\mathbb{P}(E),\mathcal{O}(1))$, and $N_0 := h^0(X,\mathcal{L})-1 \leq N(g,s,h)$.

(2) $\Phi_{|L|} : X \dashrightarrow \mathbb{P}^{N_0}$ gives an embedding on $X \setminus f^{-1}(S)$.

(3) $\Phi_{|H|} : \mathbb{P}(E) \to \mathbb{P}^{N_0}$ gives an embedding.

(4) The natural homomorphism $\pi^*E \to \mathcal{L}$ is surjective on $X \setminus f^{-1}(S)$, and the induced rational map $\varphi_0 : X \dashrightarrow \mathbb{P}(E)$ gives an embedding on $X \setminus f^{-1}(S)$ with $\Phi_{|L|} = \Phi_{|H|}\circ\varphi_0$.

(5) $\deg \Phi_{|L|}(X) \leq d(g,s,h)$.

$$\begin{array}{ccccc} X & \xrightarrow[\text{rat'l}]{\varphi_0} & \mathbb{P}(E) & \xrightarrow[\text{emb.}]{\Phi_{|H|}} & \mathbb{P}^{N_0} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi} & & \\ B & = & B & & \end{array}$$

Proof. (0) We first note that $E = f_*\mathcal{L} = \omega_B \otimes \mathcal{A} \otimes f_*\omega_{X/B}^{m_0}$ commutes with arbitrary base change on $B\setminus S$ (cf. [Vie95, 2.40]). In our case, this is simply due to [Har77, III.12.11] and $H^i(F,\mathcal{L}|_F) \cong H^i(F,\omega_{X/B}^{m_0}|_F) \cong H^i(F,\omega_F^{m_0}) = 0$ for any $i>0$ and any smooth fiber F. In particular, the base change map: $f_*\mathcal{L}\otimes\mathcal{O}_B/m_P^k \to H^0(X_P, \mathcal{L}\otimes\mathcal{O}_X/\mathcal{I}_{X_P}^k)$ is an isomorphism for any point $P \in B\setminus S$ and for any positive integer k, where m_P (respectively $\mathcal{I}_{X_P}$) is the ideal sheaf of P in B (respectively X_P in X).

(1) It is immediate that $h^0(X,\mathcal{L}) = h^0(B,E) = h^0(\mathbb{P}(E),\mathcal{O}(1))$. We shall estimate $h^0(B,\omega_B\otimes\mathcal{A}\otimes f_*\omega_{X/B}^{m_0}) = N_0-1$. The key ingredient is an estimate of deg $f_*\omega_{X/B}^{m_0}$ due to Bedulev-Viehweg. We apply 2.2 to obtain

$$\deg f_*\omega_{X/B}^{m_0} \leq (n(2g-2+s)+s)\cdot m_0\cdot(m_0^nK_F^n+1)\cdot h(m_0) = \delta(g,s,h,m_0).$$

On the other hand, as $f_*\omega_{X/B}^{m_0}$ is ample by 2.2, the vector bundle $\mathcal{A}\otimes f_*\omega_{X/B}^{m_0}$ is also ample, and in particular $H^1(B,\omega_B\otimes\mathcal{A}\otimes f_*\omega_{X/B}^{m_0})=0$. Then, by Riemann-Roch on B, we have

$$\begin{aligned} h^0(B,\omega_B\otimes\mathcal{A}\otimes f_*\omega_{X/B}^{m_0}) &= \deg(\omega_B\otimes\mathcal{A}\otimes f_*\omega_{X/B}^{m_0})+(1-g)\operatorname{rank}(\omega_B\otimes\mathcal{A}\otimes f_*\omega_{X/B}^{m_0}) \\ &= \deg f_*\omega_{X/B}^{m_0}+(2g-2+a)h(m_0)+(1-g)h(m_0). \end{aligned}$$

Combining with the estimate for $\deg f_*\omega_{X/B}^{m_0}$, we have our estimate for N_0.

Using $\deg E=\deg f_*\omega_{X/B}^{m_0}+\deg(\omega_B\otimes A)\operatorname{rank} f_*\omega_{X/B}^{m_0}$ and the same reasoning as above, we have

$$\deg E\le\delta(g,s,h,m_0)+(2g-2+a)h(m_0).$$

(2) Let P and Q be two points on B, not necessarily distinct. By the same token as above, we have $H^1(B,\omega_B\otimes\mathcal{A}\otimes f_*\omega_{X/B}^{m_0}\otimes\mathcal{O}_B(-P-Q))=0$. Then the restriction map

$$(*)\qquad H^0(X,\mathcal{L})\cong H^0(B,E)\longrightarrow H^0(B,E\otimes\mathcal{O}_B/(m_P\cdot m_Q))$$

is surjective. Let us now suppose $P,Q\in B\setminus S$ and $P\neq Q$. Then by the base change property,

$$H^0(B,E\otimes\mathcal{O}_B/(m_P\cdot m_Q))\cong H^0(X_P,\omega_{X_P}^{m_0})\oplus H^0(X_Q,\omega_{X_Q}^{m_0}).$$

Since $|m_0K_{X_P}|$ and $|m_0K_{X_Q}|$ are very ample, we can see, by varying P and Q in $B\setminus S$ with $P\neq Q$ in the surjection $(*)$, that the map $\Phi_{|L|}:X\dashrightarrow\mathbb{P}^{N_0}$ is regular on $X\setminus f^{-1}(S)$, and bijective on $X\setminus f^{-1}(S)$ onto its image. Moreover, on every smooth fiber F, the restriction $\Phi_{|L|}|_F:F\to\mathbb{P}^{N_0}$ gives an embedding by $|m_0K_F|$. We can adapt a similar argument to separate tangent vectors, at least over $X\setminus f^{-1}(S)$.

(3) Recall $r=\operatorname{rank}E=h(m_0)$. We note the base change property for $E=\pi_*\mathcal{O}(1)$, due to the fact that $H^1(\pi^{-1}(P),\mathcal{O}(1))=H^1(\mathbb{P}^{r-1},\mathcal{O}_{\mathbb{P}^{r-1}}(1))=0$ for any $P\in B$. Again, recall that $H^1(B,E\otimes\mathcal{O}_B(-P-Q))=0$ for any $P,Q\in B$, not necessarily distinct. Hence the restriction map

$$(*')\qquad H^0(\mathbb{P}(E),\mathcal{O}(1))\cong H^0(B,E)\longrightarrow H^0(B,E\otimes\mathcal{O}_B/(m_P\cdot m_Q))$$

is surjective for any $P,Q\in B$. On every $\pi^{-1}(P)$, we have $H^0(\pi^{-1}(P),\mathcal{O}(1))=H^0(\mathbb{P}^{r-1},\mathcal{O}_{\mathbb{P}^{r-1}}(1))$, and see that $|H_{|\pi^{-1}(P)}|$ is very ample. The remaining arguments to obtain the very ampleness of $|H|$ are the same as in (2) above.

(4) On $X\setminus f^{-1}(S)$, we have $\Phi_{|L|}=\Phi_{|H|}\circ\varphi_0$, because of $(\Phi_{|H|}\circ\varphi_0)^*\mathcal{O}_{\mathbb{P}^{N_0}}(1)=\varphi_0^*(\Phi_{|H|}^*\mathcal{O}_{\mathbb{P}^{N_0}}(1))=\varphi_0^*\mathcal{O}(1)=\mathcal{L}$ over $X\setminus f^{-1}(S)$. Since $\Phi_{|L|}$ gives an embedding on $X\setminus f^{-1}(S)$, so does φ_0.

(5) This degree bound will be given separately in 2.7. □

2.2. **Degree bound.** We devote this subsection to proving the effective degree bound of $\Phi_{|L|}(X)\subset\mathbb{P}^{N_0}$, stated in Proposition 2.3(5). The argument in the previous subsection is not very new. While the one in this subsection is original. We first fix some notations and make remarks.

Remark 2.4. (1) We let $X' := \varphi_0(X) \subset \mathbb{P}(E)$ with reduced structure, and let $f' : X' \to B$ be the induced morphism. We denote by $\mathcal{I}_{X'} \subset \mathcal{O}_{\mathbb{P}(E)}$ the ideal sheaf of X', and let $\mathcal{I}_{X'}(k) = \mathcal{I}_{X'} \otimes \mathcal{O}_{\mathbb{P}(E)}(k)$ for every integer k.

(2) Since H is very ample on $\mathbb{P}(E)$ and $\Phi_{|L|} = \Phi_{|H|} \circ \varphi_0$, we have $\deg \Phi_{|L|}(X) = X' \cdot H^{n+1}$. Thus we shall estimate the intersection number $X' \cdot H^{n+1}$.

(3) In the course of the proof of Proposition 2.3, we observed that $\deg f_* \omega_{X/B}^m \leq \delta(g, s, h, m)$ for any $m \geq m_0$, $r = \dim \mathbb{P}(E) = \operatorname{rank} E = h(m_0)$, and the top self-intersection number $H^r = \deg E \leq \delta(g, s, h, m_0) + (2g - 2 + a)h(m_0)$. □

$$\begin{array}{ccccc} X & \xrightarrow[\text{rat'l}]{\varphi_0} & \mathbb{P}(E) & \xrightarrow[\text{emb.}]{\Phi_{|H|}} & \mathbb{P}^{N_0} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \pi} & & \\ B & = & B & & \end{array} \quad ; \quad \begin{array}{ccccc} X & \xrightarrow{\varphi_0} & X' & \subset \mathbb{P}(E) \subset & \mathbb{P}^{N_0} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} & \downarrow{\scriptstyle \pi} & \\ B & = & B & B & \end{array}$$

Let us introduce a key invariant of a Hilbert polynomial, which is not familiar in geometric situations.

Definition 2.5. Let $Y \subset \mathbb{P}$ be a closed subscheme of dimension n in a projective space $\mathbb{P}$. We denote by $\mathcal{O}(1)$ the ample line bundle on Y which is the restriction of $\mathcal{O}_{\mathbb{P}}(1)$. Let $P(x) \in \mathbb{Q}[x]$ be the Hilbert polynomial of Y with respect to $\mathcal{O}(1)$, i.e., $P(m) = \chi(Y, \mathcal{O}_Y(m))$ holds for all sufficiently large integers m. By a theorem of Gotzmann [Got78] ([Laz04a, 1.8.35], [BH93, 4.3.2]), there exists a unique finite sequence of integers $a_1 \geq a_2 \geq \ldots \geq a_\ell \geq 0$ such that

$$P(x) = \binom{x + a_1}{a_1} + \binom{x + a_2 - 1}{a_2} + \ldots + \binom{x + a_\ell - (\ell - 1)}{a_\ell}.$$

We will refer to the integer ℓ as the *length* of the binomial sum expression of $P(x)$. □

Recall that $\binom{x}{a} = \frac{1}{a!} x(x-1)\ldots(x-a+1)$, which is a polynomial of degree a for a positive integer a, and $\binom{x}{0} = 1$. If we write $P(x) = p_n x^n + p_{n-1} x^{n-1} + \ldots + p_1 x + p_0$ with $p_i \in \mathbb{Q}$, we can write $a_1, \ldots, a_\ell$ and ℓ in terms of $p_n, \ldots, p_0$ and n in recursive relations. For example, the sequence starts with $a_j = n$ for $1 \leq j \leq n! p_n$, and $a_j < n$ for $j > n! p_n$. We can also give an effective bound of ℓ in terms of $p_n, \ldots, p_0$ and n, see 3.1. We particularly denote by

$$\ell_0$$

the length of the binomial sum expression of the Hilbert polynomial $h_0(m) = h(m_0 m)$. We can say $\ell_0 = \ell_0(g, s, h)$ is effectively bounded from g, s and h.

We now come to the main point of the whole story. To bound the degree $X' \cdot H^{n+1}$, we aim to find hypersurfaces in $\mathbb{P}(E)$ with "degree bound." The precise statement is

Lemma 2.6. *Let $P_0 \in B$ be a point. Then there exists an effective integer $d_0 = d_0(g, s, h, m_0)$ depending only on g, s, h and m_0 such that $\mathcal{I}_{X'}(\ell_0) \otimes \pi^* \mathcal{O}_B(d_0 P_0)$ is generated by global sections.*

Because we know well the intersection theory on $\mathbb{P}(E)$ at least for H and $\pi^*\mathcal{O}_B(P_0)$, if we know 2.6, we can estimate the degree in the following form as a consequence.

Corollary 2.7. *The degree is bounded by*

$$\deg \Phi_{|L|}(X) = X' \cdot H^{n+1} \leq \big(\ell_0+1\big)^{r-n-1}H^r + (r-n-1)d_0\big(\ell_0+1\big)^{r-n-2} = d(g,s,h)$$

with $r = \operatorname{rank} E = h(m_0)$ *and* $H^r = \deg E$.

Let us discuss the global generation Lemma 2.6. We note that $\varphi_0 : X \dashrightarrow X'$ is biregular over $B \setminus S$, and that X' may be singular along $f'^{-1}(S)$. On the other hand, $\mathcal{O}_{X'}(1) := \mathcal{O}(1)|_{X'}$ is very ample, and $f' : X' \to B$ is a flat family of subschema of $\mathbb{P}^{r-1}$ with Hilbert polynomial $\chi(X'_P, \mathcal{O}_{X'_P}(m))$ ([Har77, III.9.7, III.9.9]), where $X'_P = f'^*P$ is the scheme theoretic fiber for $P \in B$. Since $\mathcal{O}_{X'_P}(1) \cong \omega_{X_P}^{m_0}$ if $P \in B \setminus S$, the Hilbert polynomial $\chi(X'_P, \mathcal{O}_{X'_P}(m))$ is $h_0(m) = h(m_0 m)$. For the original $f : X \to B$, although *smooth* fibers have the same Hilbert polynomial $h(m)$, we did not have a natural way to make *all* fibers have the same Hilbert polynomial.

The next lemma, essentially due to Gotzmann, on Castelnuovo-Mumford regularity will give a surprising input in our effective estimate.

Lemma 2.8. *In 2.4, for every scheme theoretic fiber* $X'_P = f'^*P$ *over* $P \in B$*, the ideal sheaf* $\mathcal{I}_{X'_P} \subset \mathcal{O}_{\mathbb{P}^{r-1}}$ *is* ℓ_0*-regular. In particular,*

(1) $\mathcal{I}_{X'_P}(\ell_0)$ *is generated by global sections in* $H^0(\mathbb{P}^{r-1}, \mathcal{I}_{X'_P}(\ell_0))$*,*

(2) $\pi_*\mathcal{I}_{X'}(\ell_0)$ *commutes with arbitrary base change,*

(3) $R^1\pi_*(\mathcal{I}_{X'}(\ell_0)) = 0$*, and*

(4) the natural sequence $0 \to \pi_*\mathcal{I}_{X'}(\ell_0) \to \pi_*\mathcal{O}(\ell_0) \to f'_*\mathcal{O}_{X'}(\ell_0) \to 0$ *is exact.*

Proof. Every fiber of $f' : X' \to B$ has the same Hilbert polynomial $h_0(m)$. By a theorem of Gotzmann [Got78] ([Laz04a, 1.8.35], [BH93, 4.3.2]), every $\mathcal{I}_{X'_P}$is ℓ_0-regular. By definition, $\mathcal{I}_{X'_P}$is ℓ_0-regular if $H^i(\mathbb{P}^{r-1}, \mathcal{I}_{X'_P}(\ell_0 - i)) = 0$ for all $i > 0$ ([Laz04a, 1.8.1]). As a consequence, for every $k \geq \ell_0$, $\mathcal{I}_{X'_P}(k)$ is generated by global sections, and $\mathcal{I}_{X'_P}$ is k-regular ([Laz04a, 1.8.3]). From this, we obtain that, for any $P \in B$, $\mathcal{I}_{X'_P}(\ell_0)$ is generated by global sections, and $H^1(\mathbb{P}^{r-1}, \mathcal{I}_{X'_P}(\ell_0)) = 0$ by the (ℓ_0+1)-regularity. In particular, the direct image sheaf $\pi_*\mathcal{I}_{X'_P}(\ell_0)$ commutes with arbitrary base change, and hence every fiber at $P \in B$ is naturally isomorphic to $H^0(\mathbb{P}^{r-1}, \mathcal{I}_{X'_P}(\ell_0))$. The vanishing $R^1\pi_*(\mathcal{I}_{X'}(\ell_0)) = 0$ is a consequence of $H^1(\mathbb{P}^{r-1}, \mathcal{I}_{X'_P}(\ell_0)) = 0$ for any $P \in B$. □

We are now ready to prove Lemma 2.6.

Proof of Lemma 2.6. (1) We first establish "how negative" $\pi_*\mathcal{I}_{X'}(\ell_0)$ is. Let $\pi_*\mathcal{I}_{X'}(\ell_0) \to \mathcal{M}$ be a quotient line bundle with kernel $\mathcal{N}$. We claim $\deg \mathcal{M} > -d_0$ for some $d_0 = d_0(g,s,h,m_0)$, i.e., there exists a uniform effective bound.

Since $\mathcal{N}$ can be seen as a subbundle of $\pi_*\mathcal{O}(\ell_0) = S^{\ell_0}(E)$ and $S^{\ell_0}(E)$ is ample, we have $\deg \mathcal{N} < \deg \pi_*\mathcal{O}(\ell_0)$. Then $\deg \mathcal{M} = \deg \pi_*\mathcal{I}_{X'}(\ell_0) - \deg \mathcal{N} = \deg \pi_*\mathcal{O}(\ell_0) - \deg f'_*\mathcal{O}_{X'}(\ell_0) - \deg \mathcal{N} > -\deg f'_*\mathcal{O}_{X'}(\ell_0) \geq -\deg f_*\mathcal{L}^{\otimes \ell_0}$. For the second equality, we used the exact sequence in Lemma 2.8, and for the last inequality we used the following fact that there exists a natural injective homomorphism $f'_*\mathcal{O}_{X'}(k) \to f_*\mathcal{L}^{\otimes k}$ for every $k \geq 1$, which is isomorphic on $B \setminus S$. Thus, it is enough to show that $\deg f_*\mathcal{L}^{\otimes \ell_0} \leq d_0$.

Since $f_*\mathcal{L}^{\otimes \ell_0} = (\omega_B \otimes \mathcal{A})^{\otimes \ell_0} \otimes f_*\omega_{X/B}^{m_0\ell_0}$, we have $\deg f_*\mathcal{L}^{\otimes \ell_0} = \deg f_*\omega_{X/B}^{m_0\ell_0} + \ell_0(2g - 2 + a) \operatorname{rank} f_*\omega_{X/B}^{m_0\ell_0}$. The key is again [BV00, 1.4(c)], and we have $\deg f_*\omega_{X/B}^{m_0\ell_0} \leq \delta(m_0\ell_0)$ by Remark 2.4. Since $\operatorname{rank} f_*\omega_{X/B}^{m_0\ell_0} = h(m_0\ell_0)$, we have $\deg f_*\mathcal{L}^{\otimes \ell_0} \leq d_0$ for some $d_0 = d_0(g, s, h, m_0)$.

(2) Now, in view of (1), $\omega_B^{-1} \otimes \pi_*\mathcal{I}_{X'}(\ell_0) \otimes \mathcal{O}_B((d_0 + 2g)P_0 - P - Q)$ is ample for any $P, Q \in B$ by Hartshorne's theorem [Har71] ([Laz04b, 6.4.15]), because any quotient line bundle has positive degree. Thus, we have a vanishing $H^1(B, \pi_*\mathcal{I}_{X'}(\ell_0) \otimes \pi^*\mathcal{O}_B((d_0 + 2g)P_0 - P - Q)) = 0$ for any $P, Q \in B$. Hence the restriction map

$$H^0(\mathbb{P}(E), \mathcal{I}_{X'}(\ell_0) \otimes \mathcal{O}_B((d_0 + 2g)P_0)) \longrightarrow H^0(\mathbb{P}^{r-1}, \mathcal{I}_{X'_P}(\ell_0)) \oplus H^0(\mathbb{P}^{r-1}, \mathcal{I}_{X'_Q}(\ell_0))$$

is surjective, where $P \neq Q$ in this expression. Here we used Lemma 2.8 that $\pi_*\mathcal{I}_{X'}(\ell_0)$ commutes with arbitrary base change. Since $\mathcal{I}_{X'_P}(\ell_0)$ and $\mathcal{I}_{X'_Q}(\ell_0)$ are generated by global sections by Lemma 2.8, we also have the global generation of $\mathcal{I}_{X'}(\ell_0) \otimes \pi^*\mathcal{O}_B((d_0 + 2g)P_0)$ on $\mathbb{P}(E)$. The d_0 in the statement is $d_0 + 2g$ in the last sentence. □

3. Effective bounds on Hilbert polynomials

In this final section, we just mention some effective bounds regarding Hilbert polynomials, which were mentioned in the argument in §2.

3.1. **The bound on length.** We give an effective bound for (ℓ_0 for example) the length of the binomial sum expression defined in 2.5 in a general context.

Let $Y \subset \mathbb{P}$ be a closed subscheme of dimension n in a projective space $\mathbb{P}$. For the Hilbert polynomial $P(x)$ of Y with respect to $\mathcal{O}(1)$, by Gotzmann [Got78] ([Laz04a, 1.8.35], [BH93, 4.3.2]), there exists a unique sequence of integers $a_1 \geq a_2 \geq \ldots \geq a_\ell \geq 0$ such that

$$P(x) = \binom{x + a_1}{a_1} + \binom{x + a_2 - 1}{a_2} + \ldots + \binom{x + a_\ell - (\ell - 1)}{a_\ell}.$$

We write $P(x) = p_n x^n + p_{n-1}x^{n-1} + \ldots + p_1 x + p_0$ with $p_i \in \mathbb{Q}$. Noting $\binom{x+a-j}{a} = x^a/a! +$ (lower order terms), we see that the sequence starts with $a_j = n$ for $1 \leq j \leq n!p_n$, and $a_j < n$ for $j > n!p_n$. In view of this, we set $\ell_{n+1} = 0$, and

$$\ell_k = \max\{j \geq 0;\ a_j \geq k\}$$

for $k = n, n-1, \ldots, 0$. Then $0 = \ell_{n+1} < \ell_n = n!p_n \leq \ell_{n-1} \leq \ldots \leq \ell_1 \leq \ell_0$, and ℓ_0 is the length of $P(x)$. By an elementary argument to comparing coefficients of the polynomial and the binomial sum, we obtain

Proposition 3.1. *One can compute $\ell_n, \ell_{n-1}, \ldots, \ell_0$ recursively in terms of $p_n, p_{n-1}, \ldots, p_0$ and n. If one prefers an explicit effective bound, one has for example*

$$\ell_0 \leq \sum\nolimits_{k=0}^{n} \gamma_k \mu_P^{(k+1)!},$$

where $\gamma_0 = 1, \gamma_1 = 2$, $\gamma_k = k^{k+1}\gamma_{k-1}^{k+1} = k^{k+1}(k-1)^{k(k+1)}\dots 3^{4\cdot 5\dots k(k+1)}(2^{3\cdot 4\dots k(k+1)})^2$ *for* $k \geq 2$, *and* $\mu_P = \max\{n!p_n, |(n-1)!p_{n-1}|, \dots, |p_0|, n\}$.

3.2. **The bound on coefficients.** At least for a Hilbert polynomial of a canonically polarized manifold, a length bound can be reduced to the volume bound.

Proposition 3.2. *Let* F *be a canonically polarized manifold of dimension* n*, and let* $\chi(F, \mathcal{O}_F(xK_F)) = \sum_{i=n,\dots,1,0} h_i x^i \in \mathbb{Q}[x]$ *be the Hilbert polynomial. Then* $h_n = K_F^n/n!$ *and*

$$|h_{n-k}| < n!\, a_1 \cdots a_n\, m_n^k\, (1+m_n)^{nk}\, K_F^n$$

for $k = 0, 1, \dots, n$*, where* $m_n = 1 + \frac{1}{2}(n+1)(n+2)$ *and* $a_p = 2^{p(p+3)/2-2}/p!$ *for* $p \geq 1$.

The proof is done by induction on the dimension by cutting out by very ample pluricanonical divisors.

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