

ON THE MODULI OF KÄHLER-EINSTEIN FANO MANIFOLDS

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ABSTRACT. We prove that Kähler-Einstein Fano manifolds with finite automorphism groups form Hausdorff moduli algebraic space with only quotient singularities. We also discuss the limits as \mathbb{Q} -Fano varieties which should be put on the boundary of its canonical compactification.

1. INTRODUCTION

After the well-known construction of moduli of stable curves, recently the construction of moduli of canonically polarized varieties (thus of general type) of higher dimension is established by the works of Shepherd-Barron, Kollár, Viehweg, Alexeev and subsequent authors, which depends on birational geometric techniques after the minimal model program, admitting *semi-log-canonical* singularities.

On the other hand, there are no general theory of constructing moduli of anti-canonically polarized varieties (i.e. of Fano type) nor of more general polarized varieties. In this exposition, we discuss a canonical way of constructing moduli using some differential geometry. That is, we will show the proof of the following theorem.

Theorem 1.1. *Kähler-Einstein Fano n -folds with finite automorphism groups form Hausdorff moduli algebraic space in the sense of Artin [Art71], which is actually a (complex) orbifold.*

As originally expected, the existence of Kähler-Einstein metrics on a Fano manifold is proved to be equivalent to the K-stability. The proof of stability of KE Fano manifolds is established by [Mab08], [Mab09], [Ber12] and for the converse direction, please consult [CDS12a], [CDS12b], [CDS13], [Tia12b] which discuss after the Donaldson's proposal of a new continuity method using conical singular Kähler-Einstein metrics [Don11]. Thanks to the equivalence, we can paraphrase as the following.

Theorem 1.2 (Re-statement). *Smooth K-stable Fano varieties form separated moduli algebraic space and moreover we have only quotient singularities on it.*

At the first talk at Kinosaki symposium (2010), the author suggested more general conjecture (K-moduli) - to construct moduli of K-(poly)stable varieties - which we review and discuss at the final section. So in a sense, this is a progress report on the conjecture.

This principle of constructing moduli of these varieties with canonical metrics has been often hided but existed behind the whole history of moduli construction in a sense. To see a bit of the connection, note that Kähler-Einstein metrics exist on smooth projective curves, smooth canonical models, and Calabi-Yau manifolds due to the Aubin-Yau existence theory ([Aub78], [Yau78]) while we know the existence of quasi-projective moduli of them by algebro-geometric independent work [Vie95]. What we discuss here is how to directly use Kähler-Einstein geometry for the construction of moduli as varieties or algebraic spaces.

Personally speaking, the author's original motivation for this program of constructing moduli is from his algebro-geometric observations on the K-stability [Od08], [Od11], which connected geometrically compactified moduli variety of general type varieties of the Kollár-Shepherd-Barron-Alexeev type (cf. e.g. [Kol10]) and the K-stability [Don02]. As originally expected, the existence of Kähler-Einstein metrics on a Fano manifold is proved to be equivalent to the K-stability. The proof of stability of KE Fano manifolds is established by [Mab08], [Mab09], [Ber12] and for the converse direction, please consult [CDS12a], [CDS12b], [CDS13], [Tia12b] which discuss after the Donaldson's proposal of a new continuity method using conical singular Kähler-Einstein metrics [Don11].

While the equivalence is now established, the K-polystability (or the existence of Kähler-Einstein metric) for a given or a concrete Fano manifold, and the behaviour with respect to complex deformation have both certain independent aspects and they are difficult interesting questions in general which remain. To that question, we can prove the following.

Theorem 1.3. *For a flat projective family of Fano manifolds $\pi: \mathcal{X} \rightarrow S$, then the subset:*

$$\{s \in S \mid \mathcal{X}_s \text{ is a KE Fano manifold with } \# \text{Aut}(X) < \infty\} \subset S$$

is a Zariski open subset of S .

Note that this Zariski openness question is also discussed in Donaldson's informal document [Don09] which takes a different approach under several conjectural hypotheses. Indeed, some discussion with Donaldson on this question is also reflected in this notes as we will explain

at each point. A fundamental remark is that this Zariski openness does *not* hold without the condition of discreteness of automorphism group.

Coming back to the moduli construction, the remained problem after Theorem 1.1, 1.2 is to construct the compactification of them by attaching *Gromov-Hausdorff limits*. Indeed the following result established those algebraicity.

Theorem 1.4. *Gromov-Hausdorff limit of Kähler-Einstein Fano n -folds is a KE \mathbb{Q} -Fano variety, i.e. log-terminal Fano varieties with a singular Kähler-Einstein metric.*

Note that the meaning of Gromov-Hausdorff convergence is enhanced there to encode the continuity of complex structures. See [DS12] for the details.

In the case of Del Pezzo surfaces, the above algebraicity was established by Tian two decades ago [Tia90]. In that case, we succeeded in the construction of the compactification in explicit way in the joint work with Cristiano Spotti and Song Sun [OSS12]. This is pioneered by Mabuchi-Mukai [MM93] where degree 4 case is written.

2. PROOFS

Proof of Theorem 1.3. To state the proof, we need to recall the following characterization of test configurations and introduce a certain modification of the K-stability notion.

Proposition 2.1 ([RT07, Proposition 3.7]). *Let X be a Fano manifold. Then for a large enough $m \in \mathbb{Z}_{>0}$, a one-parameter subgroup of $GL(H^0(X, \mathcal{O}_X(-mK_X)))$ is equivalent to the data of a test configuration $(\mathcal{X}, \mathcal{L})$ of $(X, -K_X)$ whose polarization \mathcal{L} is very ample (over \mathbb{A}^1) with exponent m .*

Definition 2.2. A \mathbb{Q} -Fano variety X is said to be K_m -stable (resp., K_m -semistable) if and only if $DF(\mathcal{X}, \mathcal{L}) > 0$ for any test configuration of the above type (i.e., with very ample \mathcal{L} of exponent m) which is non-trivial test configuration whose central fiber is normal.

Notice that what [LX11] proved can be rephrased as that a \mathbb{Q} -Fano variety is K-stable (resp., K-polystable, K-semistable) if and only if it is K_m -stable (resp., K_m -polystable, K_m -semistable) for sufficiently divisible $m \in \mathbb{Z}_{>0}$. The name “ K_m -stability” followed a suggestion of Professor Simon Donaldson in a discussion. The equivalent notion was also treated in [Tia12a].

Let us also recall the definition of the algebro-geometric counterpart of the α -invariant ([Tia87]) i.e., the so-called *global log-canonical threshold* ([Sho92]) and its variants as follows.

Definition 2.3. Suppose X is a projective manifold, E is an effective \mathbb{Q} -divisor on X , and L is an ample line bundle on X . Then we consider the following invariants.

$$\text{lct}(X, E) := \sup\{\alpha > 0 \mid (X, \alpha E) \text{ is log canonical}\},$$

$$\text{lct}_l(X; L) := \inf_{D \in |lL|} \text{lct}\left(X, \frac{1}{l}D\right) \text{ for } l \in \mathbb{Z}_{>0},$$

and

$$\text{glct}(X; L) := \inf_{l \in \mathbb{Z}_{>0}} \text{lct}_l(X; L),$$

which are called the *log canonical threshold* of X with respect to E , the *log canonical threshold* of X with respect to L with exponent l , and the *global log canonical threshold* of X with respect to L .

For a Fano manifold X , we simply write $\text{glct}(X) := \text{glct}(X; -K_X)$, $\text{lct}_l(X) := \text{lct}_l(X; -K_X)$.

According to the approximation theory of Kähler metrics by Bergman metrics, the last notion corresponds to α -invariant (cf. [Tia87], [CSD08, Appendix]).

We use the following general proposition in what follows.

Proposition 2.4. *Given a flat family of Fano manifolds $\mathcal{X} \rightarrow S$ and its relative pluri-anticanonical divisors $\mathcal{D} \in |-\lambda K_{\mathcal{X}/S}|$, we have uniform positive lower bounds for both $\text{lct}(\mathcal{X}_s; -K_{\mathcal{X}_s})$ and $\text{lct}(\mathcal{D}_s; -K_{\mathcal{X}_s}|_{\mathcal{D}_s})$ where s runs through S .*

proof of Proposition 2.4. First, let us recall the following basic lemma.

Lemma 2.5. *For any log pair (X, D) with \mathbb{Q} -Cartier effective \mathbb{Q} -divisor D ,*

$$\text{lct}(X, D) \geq 1/\text{mult}_p(D),$$

where if D is an effective \mathbb{Q} -divisor $\sum a_i D_i$ with prime divisors D_i and $a_i \in \mathbb{Q}_{>0}$, $\text{mult}_p(D) := \sum a_i \text{mult}_p(D_i)$.

This Lemma follows from the upper semicontinuity of the log canonical thresholds (cf. e.g., [Mus02]), applied to a degenerating family of D to a multiple of a smooth local divisor (cf. e.g., [Kol97] for more details).

On the other hand, supposing $-\lambda K_X$ is very ample for every Fano n -fold X , we have

$$\text{mult}_p(D) \leq \lambda^n (-K_X)^n$$

for all $p \in D$ with $D \in |-\lambda K_X|$. Indeed, if we take general members D_1, \dots, D_{n-1} among $|-\lambda K_X|$ which pass through the point $p \in X$ without singularity at p , the inequality follows from the standard estimate of the intersection number $(D_1 \dots D_{n-1} \cdot D) = \lambda^n (-K_X)^n$. The proof of Proposition 2.4 is completed. \square

On the other hand, let us recall the following results from [OS11].

Theorem 2.6 ([OS11, Theorem 4.1]). *Let X be a projective variety, D' is an effective \mathbb{Q} divisor and $0 \leq \beta \leq 1$. (i) Assume $(X, (1-\beta)D)$ is a log Calabi-Yau pair, i.e., $K_X + (1-\beta)D$ is numerically equivalent to the zero divisor and it is a semi-log-canonical pair (resp. kawamata-log-terminal pair). Then, $((X, D), L)$ is logarithmically K -semistable (resp. logarithmically K -stable) with respect to the boundary's parameter β .*

(ii) *Assume $(X, (1-\beta)D')$ is a semi-log-canonical model, i.e., $K_X + (1-\beta)D'$ is ample and it is a semi-log-canonical pair. Then, $((X, D), K_X + (1-\beta)D')$ and $\beta \in \mathbb{Q}_{>0}$ is log K -stable with respect to the boundary's parameter β .*

Theorem 2.7 (cf. [OS11, Corollary 5.5]). *Let X be an arbitrary \mathbb{Q} -Fano variety and D is an integral pluri-anti-canonical Cartier divisor $D \in |-\lambda K_X|$ with some $\lambda \in \mathbb{Z}_{>0}$. Then $(X, -K_X)$ is logarithmically K -stable (resp. logarithmically K -semistable) for cone angle $2\pi\beta$ with $\frac{\lambda-1}{\lambda} < \beta < (\lambda-1 + (\frac{n+1}{n})\min\{\text{glct}(X; -K_X), \text{glct}(D; -K_X|_D)\})/\lambda$ (resp. $\frac{\lambda-1}{\lambda} \leq \beta \leq (\lambda-1 + (\frac{n+1}{n})\min\{\text{glct}(X; -K_X), \text{glct}(D; -K_X|_D)\})/\lambda$.*

We make some remarks. Recall that Theorem 2.6 (i) extends and algebraically recovers [Sun11, Theorem 1.1], [Bre11, Theorem 1.1] and [JMR11, Theorem 2]. In other words, at least for the case where X is smooth, D is a smooth integral divisor, the above is expected to follow from their results as well. Regarding Theorem 2.7, the original version of [OS11, Corollary 5.5] treated only $\lambda = 1$ case, but it is recently noticed that it can be simply extended to the above form. We also remark that the case if X and D are both smooth and $\lambda = 1$, Theorem 2.7 also follows from [Ber10, Theorem 1.8] combined with [Ber12, Theorem 1.1].

The above two theorems imply that

Corollary 2.8. *For an arbitrary smooth Fano manifold X and a smooth pluri-anticanonical divisor $D \in |-\lambda K_X|$, $(X, -K_X)$ is logarithmically K -stable (resp. logarithmically K -semistable) for cone angle $2\pi\beta$ with $0 < \beta < (\lambda-1 + (\frac{n+1}{n})\min\{\text{glct}(X; -K_X), \text{glct}(D; -K_X|_D)\})/\lambda$ (resp. $0 \leq \beta \leq (\lambda-1 + (\frac{n+1}{n})\min\{\text{glct}(X; -K_X), \text{glct}(D; -K_X|_D)\})/\lambda$.*

Combining the above lower boundedness of log canonical thresholds and Corollary 2.8, we can take uniform $\beta_\infty (> 0)$ such that every Fano manifolds \mathcal{X}_s ($s \in S$) are log K -stable for cone angle $2\pi\beta_\infty$ with respect to the divisor Cartier \mathcal{D}_s .

We use the following theorem for the proof as well.

Theorem 2.9. [CDS12b, Theorem 1], [CDS13, Theorem 2, the paragraph below it] *For a Fano manifold X , if X does not admit Kähler-Einstein metric, the following holds. For a smooth divisor $D \in |-\lambda K_X|$ with some $\lambda \in \mathbb{Z}_{>0}$, we have a non-trivial normal log test configuration $(\mathcal{X}, \mathcal{D})$ of $(X, (1 - \beta_\infty)D)$ with cone angle $2\pi\beta_\infty$ ($1 - \frac{1}{\lambda} < \beta_\infty \leq 1$) such that $\text{DF}_{\beta_\infty}((\mathcal{X}, \mathcal{D}); -K_{\mathcal{X}}) = 0$ and \mathcal{X}_0 is a \mathbb{Q} -Fano variety. Moreover, the \mathbb{Q} -Gorenstein index of \mathcal{X}_0 can be taken uniformly if we have a fixed dimension $\dim(X)$, λ and a lower bound for β_∞ which is bigger than $1 - \frac{1}{\lambda}$.*

Note that the bound of index of \mathcal{X}_0 is equivalent to that they are all parametrized in a fixed Hilbert scheme, by the recent boundedness result [HMX12, Corollary 1.8]. Note that the analogical bound of \mathbb{Q} -Gorenstein index of the limit as Theorem 2.4 was indeed already proved earlier in non-log setting as well ([DS12]).

Thanks to the existence of the uniform β_∞ , in turn Theorem 2.9 implies that there is a large enough but fixed m such that for any $s \in S$, \mathcal{X}_s is Kähler-Einstein Fano manifold if and only if it is K_m -stable. Take a Hilbert scheme H which exhausts all \mathcal{X}_s , embedded by $| -mK_{\mathcal{X}_s} |$. Automatically we have CM line bundle on H which we denote by Λ_{CM} and there are actions of $\text{SL}(h^0(-mK_{\mathcal{X}_s}))$ on both H and Λ_{CM} (linearization). Denote the locus in H which parametrizes Kähler-Einstein Fano manifold \mathcal{X}_s with $s \in S$ by H_{KE} , and the locus which parametrizes normal fibers by H_{normal} . It is well known that H_{normal} is a Zariski open subset of H as a general phenomenon.

Let us fix a maximal torus T of $\text{SL}(h^0(-mK_{\mathcal{X}_s}))$. Recall the following known fact for usual setting in Geometric Invariant Theory [Mum65] which is for *ample* linearized line bundles.

Lemma 2.10 (cf. [Mum65, Chapter 2, Proposition 2.14]). *Consider the action of a reductive algebraic group G on a polarized projective scheme (H, Λ) i.e. H is a projective scheme and Λ is an ample linearized line bundle. Let us denote the corresponding GIT weights function with respect to a one parameter subgroup $\varphi: \mathbb{G}_m \rightarrow G$ at $h \in H$ by $\lambda(h, \varphi; \Lambda)$. Then there are some finite linear functions $\{l_i\}_{i \in I}$ on $\text{Hom}_{\text{alg.grp}}(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{R}$ with rational coefficients, indexed by a finite set I such that the weight function $\lambda(h, -; \Lambda)$, regarded as a function from $\text{Hom}_{\text{alg.grp}}(\mathbb{G}_m, T)$ to \mathbb{Z} , extends to a piecewise linear rational function of the form*

$$\sup\{l_j(-) \mid j \in J_h\}$$

with some $J_h \subset I$. Moreover, $\psi: H \rightarrow 2^I$ which maps h to J_h , is constructible in the sense that for any $J \in I$, $\{h \in H \mid J_h = J\}$ is constructible.

The last line is not stated in [Mum65, Proposition 2.14] but follows rather easily from the proof and the arguments written there.

Recall that the CM line bundle [PT06] (cf. also [FS90]) is a line bundle defined on the base scheme for an arbitrary polarized family and the formal GIT weight of the CM line bundle on the Hilbert scheme is exactly the corresponding Donaldson-Futaki invariant. However, unfortunately we can not directly apply to the CM line bundle Λ_{CM} which is *not* necessarily ample in general. Nevertheless, Lemma 2.10 can be still extended to the case for our H and the CM line bundle Λ_{CM} on it, though $\lambda(t, -; \Lambda_{CM})$ is not necessarily convex. To see that, first we take a Plücker polarization Λ_{Pl} on H which is known to be very ample, yielding the Plücker embedding of Hilbert scheme. Take sufficiently large $l \in \mathbb{Z}_{>0}$ such that $\Lambda_{CM} \otimes \Lambda_{Pl}^{\otimes l}$ is also very ample. Then the point is that $\lambda(h; -; \Lambda_{CM}) = \lambda(h; -; \Lambda_{CM} \otimes \Lambda_{Pl}^{\otimes l}) - l\mu(h; -; \Lambda_{Pl})$. Therefore Lemma 2.10 extends to the case with our setting that Hilbert scheme is H , $L = \Lambda_{CM}$ and $G = \mathrm{SL}(h^0(-mK_{X_s}))$ as follows.

Lemma 2.11. *There are some finite linear rational functions $\{l_i\}_{i \in I'}$ on $\mathrm{Hom}_{\mathrm{alg}, \mathrm{grp}}(\mathbb{G}_m, T) \otimes_{\mathbb{Z}} \mathbb{R}$ with rational coefficients, indexed by a finite set I' such that the followings hold. For each $[X \subset \mathbb{P}]$, there is $J_X \subset I'$ such that its Donaldson-Futaki invariants with respect to one parameter subgroups of $G := \mathrm{SL}(h^0(-mK_{X_s}))$ regarded as a function from $\mathrm{Hom}_{\mathrm{alg}, \mathrm{grp}}(\mathbb{G}_m, T)$ to \mathbb{Z} , extends to a piecewise linear rational function whose pieces are $\{l_j\}_{j \in J_X}$ with some $J_X \subset I'$. Moreover, ψ is constructible in the sense that for any subset $J_X \subset I'$, $\{[X \subset \mathbb{P}] \in H \mid J_X = J\}$ is constructible.*

Next, let us discuss from another viewpoint which is more birational-geometric. This is related to what Donaldson called “splitting of orbits” in [Don09]. Considering the family over S and trivialize the locally free coherent sheaf $\pi_*\mathcal{O}(\mathcal{D})$ on a Zariski open covering $\{S_i\}_i$ we have the corresponding morphism $(\sqcup S_i \times T) \rightarrow H$, which can be regarded as a rational map $(\sqcup S_i \times (\mathbb{P}^1)^r) \dashrightarrow H$. For simplicity, re-set $S := \sqcup S_i$ from now on which does not lose generality from our assertion of the theorem. Thinking of a resolution of indeterminacy, we get some blow up of $S \times (\mathbb{P}^1)^r$ along a closed subscheme inside $S \times ((\mathbb{P}^1)^r \setminus T)$. The blow up is generically isomorphic over $S \times ((\mathbb{P}^1)^r \setminus T)$, which means the GIT weights with respect to any fixed one parameter subgroups in T stay constant for some Zariski open dense subset S' of S . Replace S by $S \setminus S'$ then we can construct a constructible stratification of S such that each stratum’s degenerations with respect to any fixed one parameter subgroup of G fits into a flat family again. This implies that those corresponding Donaldson-Futaki invariants are constant on each

strata by [PT06]. We can also prove it by using Wang's interpretation of Donaldson-Futaki invariant [Wan12].

Claim 2.12. *There is a finite set I of one parameter subgroups $\{\varphi_i\}: \mathbb{G}_m \rightarrow T$ ($i \in I$) such that the following is true. There is a stratification of H_{normal} by constructible subsets $\{H_J\}_{J \subset I}$, such that for any $t \in H_J \subset H_{normal}$, \mathcal{X}_t is K_m -stable (resp., K_m -semistable) if and only if $DF(\varphi_j; [\mathcal{X}_t \subset \mathbb{P}]) > 0$ (resp., $DF(\varphi_j; [\mathcal{X}_t \subset \mathbb{P}]) \geq 0$) for all $j \in J$ where $DF(\varphi_j; [\mathcal{X}_t \subset \mathbb{P}])$ actually only depends on j and J . Here $DF(\varphi_j; [\mathcal{X}_t \subset \mathbb{P}])$ means the Donaldson-Futaki invariant of the test configuration of $(\mathcal{X}_t, \mathcal{O}(1) = \mathcal{O}(-mK_{\mathcal{X}_t}))$ associated to the one parameter subgroup via Proposition 2.1.*

Summarizing up, we proved that the locus of Fano manifolds which are K_m -stable (resp., K_m -semistable, K_m -unstable, not K_m -stable) with respect to the maximal torus T 's action are all constructible subsets. Then the constructibility of K_m -stable locus in H follows from C. Chevalley's lemma (cf. [Har77]) which states that the image of constructible set by an algebro-geometric morphism is again constructible. Indeed, we can apply it to the group action morphism $SL(h^0(-mK_{\mathcal{X}_s})) \times H \rightarrow H$ and the constructible subset of H which parametrizes Fano manifolds which are not K_m -stable with respect to the action of T . Then, it shows that the subset of H which parametrizes Fano manifolds with discrete automorphism groups which are not K_m -stable is again constructible.

On the other hand, using the deformation theory of Kähler metrics with constant scalar curvature it was shown in [LS94] that

$$\{s \in S \mid \mathcal{X}_s \text{ is a KE Fano manifold with } \# \text{Aut}(X) < \infty\} \subset S$$

is an open subset with respect to the Euclidean topology. More precisely speaking, we need to take resolution of singularities of S to apply [LS94]. Thus the locus of Kähler-Einstein Fano manifolds with finite automorphism groups should be a Zariski open subset. We complete the proof of the main theorem. \square

We prove Theorem 1.1 using Theorem 1.3 as follows.

proof of Theorem 1.1 (using Theorem 1.3). Consider the Hilbert scheme H which exhausts all m -pluri-anticanonically embedded Fano n -folds with uniform $m \in \mathbb{Z}_{>0}$. By applying the previous statement of Theorem 1.3 to a Zariski open covering of universal family over H , we obtain that the locus which parametrizes Kähler-Einstein Fano manifolds form a Zariski open locus. Therefore the natural moduli

functor $\mathcal{F}: (Sch/\mathbb{C})^{op} \rightarrow (Sets)$ whose B -valued points set is defined by

$$\{\text{flat proj. } f: \mathcal{Y} \rightarrow B \mid [\mathcal{Y}_b \subset \mathbb{P}(H^0(-mK_{\mathcal{Y}_b}))] \in H_{KE} \text{ for } \forall b \in B\} / \cong$$

naturally gives rise to Deligne-Mumford stack which is simply a quotient stack $[H_{KE}/\text{PGL}(h^0(-mK_{\mathcal{X}_s}))]$. Then the Keel-Mori theorem [KM97, Corollary 1.3] implies that we have a coarse moduli algebraic space for it. The separatedness follows from a standard argument using the uniqueness of the limits as Riemannian manifolds (cf. e.g. [CS10, Lemma 8.1]) and the Bando-Mabuchi theorem [BM85], or [Don01].

Moreover, the moduli algebraic stack (in the sense of Deligne-Mumford) is smooth as the deformation is unobstructed. It is because of the vanishing of obstruction group $H^2(X, T_X) = H^2(X, \Omega^{n-1} \otimes K_X^{-1}) = 0$ thanks to the Akizuki-Nakano's generalization of the Kodaira vanishing [AN54] combined with the finiteness of isotropy groups. Therefore the coarse moduli algebraic space has only quotient singularities. This completes the proof of Theorem 1.1. \square

3. REVISITING THE GENERAL CONJECTURE

The moduli space constructed in Theorem 1.1 is expected to have canonical projective compactification by attaching (singular) \mathbb{Q} -Fano varieties on the boundary (cf. [OSS12]) but let us review this in broader context as in [Od10]. The author expects that K-semistable polarized varieties have coarse projective moduli in general. Including what we have already mentioned, the inspirations are mainly from the following backgrounds.

- The *moment map picture* on the infinite-dimensional space of complex structures by Fujiki, Donaldson ([Fuj90], [Don97], [Don04]).
- The observations ([Od08], [Od11]) that the projective moduli of general type varieties with semi-log-canonical singularities, constructed by using the *minimal model program* (cf. [Kol10]) and some other projective moduli corresponds to K-stability.
- The algebraicity of *Gromov-Hausdorff limits* of Kähler-Einstein Fano manifolds ([DS12]) and the Gromov (pre)compactness theorem (cf. [MM93], [Spo12], [OSS12]).

Or we can also say that the Yau-Tian-Donaldson conjecture itself is another background, as GIT polystably is originally designed by Mumford [Mum65] to make them form projective moduli although the K-stability is *not* quite a GIT stability notion in that original sense.

As a preparation for making the expectation rigorous, we introduce the following terminology. Let us call a numerical equivalence class of ample line bundle, a *weak polarization* and a pair of projective variety and weak polarization a *weakly polarized variety*.

Then the following is a modified version of the conjecture stated in the Kinoshita symposium 2010.

Conjecture 3.1 (K-moduli cf. [Od10]). *The moduli functor of K-semistable weakly polarized varieties has proper coarse moduli algebraic space¹ \bar{M}_{cscK} . It set-theoretically parametrizes the set of all smooth K-polystable weakly polarized varieties and certain equivalence classes of singular weakly polarized varieties with singular cscK metrics. The connected components of \bar{M}_{cscK} are all projective schemes, because the CM line bundle descends to ample line bundles on them.*

We refer to [OSS12] for detailed discussions on our \mathbb{Q} -Fano varieties case which largely settled (smoothable) del Pezzo surfaces case explicitly. In that paper, we have constructed explicit moduli for all degrees as the Gromov-Hausdorff compactifications. Indeed, the author expects more generally the K-moduli can be constructed as a “*Gromov-Hausdorff compactification*” in appropriately rescaled sense, of moduli of smooth cscK polarised varieties.

The point of view with Gromov-Hausdorff limit had been missed in the original K-moduli expectation [Od10] and inspired by differential geometric colleagues later (e.g. [DS12], [Spo12]), which the author is truly grateful. Especially the memorable collaboration with Spotti and Sun [OSS12] makes the author more confident on the general conjecture above. Moreover, very recently the wonderful work of Berman-Guenancia [BG13] -extending hyperbolic metrics on stable curves- appeared which constructed singular Kähler-Einstein (thus cscK) metrics on semi-log-canonical varieties with ample canonical classes. We note that allowing just the semi-log-canonical singularities should be more general phenomenon for all classes of K-semistable (weakly) polarized varieties [Od08].

4. ABOUT THE LIMITS

An interesting problem is to recover or to have better understanding of the Gromov-Hausdorff limits of K-stable punctured family or maximally destabilizing test configurations of K-unstable varieties via the minimal model program along the line of [Od08], [LX11]. (The

¹With infinitely many connected components.

author expects that the destabilizing test configuration of non-slc varieties [Od08] via MMP is maximal when we normalize Donaldson-Futaki invariants.)

The following characterization for special case is essentially obtained in [OS12], which the author was not aware for a while.

Theorem 4.1 (CM minimizing for (special) Fano varieties). *Suppose $\pi: \mathcal{X} \rightarrow C \ni 0$ is a \mathbb{Q} -Gorenstein flat projective family of n -dimensional \mathbb{Q} -Fano varieties over a projective curve C and the alpha invariant of the central fiber $\text{glct}(\mathcal{X}_0; -K_{\mathcal{X}_0})$ is at least (resp. bigger than) $\frac{n}{n+1}$. Regarding π as a polarized family but putting anticanonical line bundle on \mathcal{X} , any other flat polarized family $(\mathcal{X}', \mathcal{L}') \rightarrow C$ which is isomorphic to $\mathcal{X} \rightarrow C$ away from the central fiber, we have*

$$\text{deg}(CM(\mathcal{X}, -\mathcal{K}_{\mathcal{X}/C})) \leq (\text{resp. } <) \text{deg}(CM(\mathcal{X}', \mathcal{L}')).$$

In general, the author expect that the - a priori differential geometric notion - the Gromov-Hausdorff limit of punctured K-stable family of \mathbb{Q} -Fano varieties is, algebraically, characterized as *what minimizes the CM degree* as above, and the construction or above characterization is to be able to prove via the minimal model program or differential geometrically Kähler-Ricci flow. An easier version may be trivial family of K-semistable Fano varieties case (which, at least in smooth case by [CDS13], [Tia12b], isotrivially degenerates to K-polystable Fano manifolds).

For the proof of Theorem 4.1, first we take a resolution of indeterminacy of $\mathcal{X} \dashrightarrow \mathcal{X}'$ as blow up $\pi: \mathcal{B} \rightarrow \mathcal{X}$ where $K_{\mathcal{X}}^{\otimes(-r)}(-E)$ with the Cartier exceptional divisor E is the pullback of $\mathcal{L}'^{\otimes(-r)}$. Then we can estimate

$$CM(\mathcal{X}', \mathcal{L}') - CM(\mathcal{X}, K_{\mathcal{X}}^{\otimes(-r)}) = CM(\mathcal{B}, \pi^* K_{\mathcal{X}}^{\otimes(-r)}(-E)) - CM(\mathcal{X}, \mathcal{L}),$$

the difference of CM lines of \mathcal{X} and \mathcal{X}' , in the completely similar way as in [OS12] in which they did estimate for the case \mathcal{X} is $\mathcal{X}_0 \times \mathbb{P}^1$ via the basics of discrepancy.

Analogously, for Calabi-Yau varieties and general type varieties, if we apply the estimates of Donaldson-Futaki invariants at [Od11] similarly to the differences of CM degrees of a family and its blow up, which we can obtain as a resolution of rational map $\mathcal{X} \dashrightarrow \mathcal{X}'$, we can (re)prove the following. The semi-log-canonical models (“general type”) case was obtained by X. Wang - C. Xu earlier [WX12], so in a sense, our results are its generalization to different types of varieties.

Theorem 4.2 (Analogues). (i) (Calabi-Yau case) *Suppose $\pi: \mathcal{X} \rightarrow C \ni 0$ is a \mathbb{Q} -Gorenstein flat projective family of*

n-dimensional semi-log-canonical models over a projective curve C . Regarding π as a polarized family but putting canonical line bundle on \mathcal{X} , any other flat polarized family $(\mathcal{X}', \mathcal{L}') \rightarrow C$ which is isomorphic to $\mathcal{X} \rightarrow C$ away from the central fiber, we have

$$\deg(\mathrm{CM}(\mathcal{X}, -\mathcal{K}_{\mathcal{X}/C})) \leq (\text{resp. } <) \deg(\mathrm{CM}(\mathcal{X}', \mathcal{L}')).$$

- (ii) (Canonical model case: [WX12, Theorem 6]) Suppose $\pi: \mathcal{X} \rightarrow C \ni 0$ is a \mathbb{Q} -Gorenstein flat projective (“KSBA”) family of *n*-dimensional semi-log-canonical models over a projective curve C . Regarding π as a polarized family but putting canonical line bundle on \mathcal{X} , any other flat polarized family $(\mathcal{X}', \mathcal{L}') \rightarrow C$ which is isomorphic to $\mathcal{X} \rightarrow C$ away from the central fiber, we have

$$\deg(\mathrm{CM}(\mathcal{X}, -\mathcal{K}_{\mathcal{X}/C})) \leq (\text{resp. } <) \deg(\mathrm{CM}(\mathcal{X}', \mathcal{L}')).$$

Corollary 4.3 (well known? e.g. [Bou13]). *If a punctured family of polarized log terminal Calabi-Yau varieties can be completed with a log terminal CY filling, there is no other semi-log-canonical CY filling.*

In particular, log terminal polarized Calabi-Yau varieties have Hausdorff moduli.

We end this short article by saying regretfully that the above CM minimizing result was once claimed with Richard Thomas for any K-(semi)stable limit (family) whose proof had a gap. We still hope a possibility to recover that in general situation.

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