代数幾何学シンポジウム記録
2013年度　pp.94-100

Dynamical Degrees and Applications

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Abstract. This is an extension of the author's talk at the Kinosaki Symposium of Algebraic Geometry, 2013.

One important tool in Complex Dynamics is dynamical degrees for dominant meromorphic selfmaps. They are bimeromorphic invariants of a meromorphic selfmap $f : X \to X$ of a compact Kähler manifold X. The p-th dynamical degree $\lambda_p(f)$ is the exponential growth rate of the spectral radii of the pullbacks $(f^n)^*$ on the Dolbeault cohomology group $H^{p,p}(X)$. For a surjective holomorphic map f, the dynamical degree $\lambda_p(f)$ is simply the spectral radius of $f^* : H^{p,p}(X) \to H^{p,p}(X)$. Fundamental results of M. Gromov and Y. Yomdin expressed the topological entropy of a surjective holomorphic map in terms of its dynamical degrees: $h_{top}(f) = \log \max_{0 \leq p \leq dim(X)} \lambda_p(f)$. Since then, dynamical degrees have played a more and more important role in dynamics of meromorphic maps. In many results and conjectures in Complex Dynamics in higher dimensions, dynamical degrees play a central role. Recently J. Silverman and subsequently Kawaguchi - Silverman proposed several deep conjectures concerning dynamical degrees and their arithmetic analogies to study dynamics of rational maps over a number field. This topic is a new and promising one. Many fundamental questions are still unanswered.

The plan of the talk is as follows:

First, I will recall the definition of dynamical degrees and relative dynamical degrees of a surjective holomorphic selfmap and more generally dominant meromorphic selfmaps of a compact Kähler manifold. Note that the definition of dynamical degrees consist of a limit, whose existence is a non-trivial fact.

Second, I will present two methods to define dynamical degrees. One is analytic, using regularization of positive closed currents. The other is purely algebraic, using Chow's lemma and intersection theory of algebraic cycles.

Third, I will present some properties of dynamical degrees and sketch of the proofs. These include: log concavity (proved using Hodge's index theorem); bimeromorphic invariance (quite easy); constraints they must satisfy if the map f has an invariant meromorphic fibration (this is joint work of Dinh, Nguyen and myself, proved using a semi-regularization of positive closed currents).

Fourth, I will present some applications. These include: Gromov and Yomdin's formula for topological entropy of a holomorphic map; Gromov and Dinh-Sibony's inequality for meromorphic maps; dynamical degrees of a pseudo-automorphism map of dimension ≤ 4; dynamical degrees of an automorphism of a complex 3-tori; constraint on geometry of cohomologically hyperbolic maps (these maps include polarized endomorphisms); invariant measures, periodic points, backward orbits of meromorphic maps; and primitive automorphisms of positive entropy on rational 3 folds.

Acknowledgements. The author would like to thank the organizers for the invitation to attend the Kinosaki's symposium. The event was very interesting and memorable, and the attendees were very friendly. The author is grateful to Keiji Oguiso for his generous help.

1. Definition of dynamical degrees

We first present some simple cases where dynamical degrees can be easily defined. Then we proceed to the general case.

1) The simplest case: holomorphic maps. If $f : X \to X$ is a surjective holomorphic map, we let $r_p(f)$ be the spectral radius of the linear map $f^* : H^{p,p}(X) \to H^{p,p}(X)$. We define the p-th dynamical degree of f to be $\lambda_p(f) = r_p(f)$. Since $(f^n)^* = (f^*)^n$ for all $n \geq 1$, we have that $\lambda_p(f) = r_p(f^n)^{1/n}$ for all $n \geq 1$.

2) One other simple case: Degree growth (first dynamical degree) of rational maps of projective spaces. Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be a dominant rational map. We can write $f = [P_0 : P_1 : \ldots : P_k]$ where $P_0, \ldots, P_k$ are homogeneous polynomials of the same degree; moreover, we can assume that $P_0, P_1, \ldots, P_k$ have no non-trivial common divisor. Then we define the degree $deg(f)$ of f to be the degree of any of the polynomials $P_0, \ldots, P_k$. Even though $deg(f^n)$ is not the same as $deg(f)^n$, it is easy to check that $deg(f^{m+n}) \leq deg(f^m)deg(f^n)$ for all $m, n \geq 1$. Therefore the limit $\lambda_1(f) = \lim_{n\to\infty} deg(f^n)^{1/n}$ exists and is called the degree growth of f. Note that $deg(f)$ is the same as degree of $f^{-1}(H)$, where H is a generic hyperplane.

3) For a general dominant meromorphic map $f : X \to X$ we can define the pullback map $f^* : H^{p,q}(X) \to H^{p,q}(X)$. This is done as follows: If θ is a smooth closed (p, q) form then $f^*(\theta)$ is a closed (p, q) current with L^1 coefficients, which on the open set U where f is defined is the same as the usual pullback. Note that in general we don't have $(f^n)^* = (f^*)^n$. T.-C. Dinh showed that the spectral radius $r_{p,q}$ can be bound by the spectral radius $r_p := r_{p,p}$ and $r_q := r_{q,q}$. This is proved by using the Kunneth's formula. Hence the growth of the pullback of f^n on the total cohomology group $H^*(X)$ is the same as the growth of the pullback of f^n on $H^{p,p}(X)$ for $p = 0, \ldots, dim(X)$. Russakovskii and Shiffman (the case of compact projective spaces $\mathbb{P}^N$) and Dinh and Sibony (the case of general compact Kähler manifolds) defined dynamical degrees as follows:

$$\lambda_p(f) = \lim_{n\to\infty} r_p(f^n)^{1/n},$$

where $r_p(f^n)$ is the spectral radius of the linear map $(f^n)^* : H^{p,p}(X) \to H^{p,p}(X)$. The existence of the limit is non-trivial, and the proof uses regularization of positive closed currents, which I will present later.

4) Relative dynamical degrees. For a dominant meromorphic map $f : X \to X$ which preserves a dominant meromorphic map $\pi : X \to Y$, i.e.there is a dominant meromorphic map $g : Y \to Y$ such that $\pi \circ f = g \circ \pi$, Dinh and Nguyen defined relative dynamical degrees $\lambda_j(f|\pi)$, where $0 \leq j \leq dim(X) - dim(Y)$. Roughly speaking, these measure the growth of the pullback of $(f^n)^*$ on the cohomology groups of a generic fiber of π. The actual definition is more complicated. These dynamical degrees are birationally equivalent, in the sense that if $\pi' : X' \to Y'$ is birationally equivalent to $\pi : X \to Y$, and $f' : X' \to X'$ is the induced map then $\lambda_j(f'|\pi') = \lambda_j(f|\pi)$ for all j. In the simplest case where π is holomorphic, then these dynamical degrees can be

defined as follows. Choose $||.||$ to be any norm on the vector space $H^*(X)$. Let $k = dim(X)$ and $l = dim(Y)$, let ω_X be a Kähler form on X and ω_Y is a Kähler form on Y. Then for any $0 \leq p \leq k - l$ we define

$$\lambda_p(f|\pi) = \lim_{n\to\infty} ||(f^n)^*(\omega_X) \wedge \pi^*(\omega_Y^l)||^{1/n}.$$

Again, the existence of the limit is non-trivial.

2. Proofs of the existence of the limit

Here, I present two methods to prove the existence of the limit in the definition of dynamical degrees.

1) Analytic method. This is given by Russakovskii and Shiffman (the case of projective space) and Dinh and Sibony (the case of compact Kähler manifold). This uses semi-regularization of positive closed currents. Examples of positive closed currents include currents of integration over a complex variety and positive closed smooth forms. If X is a compact Kähler manifold of dimension k with a Kähler form ω_X, then for any positive closed (p,p) current T we can define its mass by $||T|| =< T, \omega_X^{k-p} >$. The mass depends only on the cohomology class of T. Then it is easy to check that there is a constant $C > 0$ independent of the function f so that $C^{-1}||f^*(\omega_X^p)|| \leq r_p(f) \leq C||f^*(\omega_X^p)||$. The semi-regularization of positive closed currents we need to use are as follows: If T is a positive closed (p,p) current, then there is a sequence of positive closed smooth (p,p) forms T_n such that $||T_n|| \leq C||T||$ for every n, and the limit point T' satisfies $T' \geq T$. When $X = \mathbb{P}^N$ is a projective space then this can be done by using that $\mathbb{P}^N$ has a lot of automorphisms. For a general compact Kähler manifold, this semi-regularization theorem is proved by Dinh and Sibony.

This semi-regularization can be used to prove the following. Let $f : X \to X$ be a dominant meromorphic map, and let U be the maximal open set of X such that $f|_U$ is locally invertible. Then for any positive closed (p,p) current S we have $||(f|_U)^*(S)|| \leq C||f^*(\omega_X^p)||$, here $C > 0$ is independent of f. Applying this to $f = f^n$ and $S = (f^m)^*(\omega_X^p)$ we obtain that $||(f^{n+m})^*(\omega_X^p)|| \leq C||(f^n)^*(\omega_X^p)||||(f^m)^*(\omega_X^p)||$, and from this obtain the existence of the limit.

2) Algebraic method. For the case of maps over fields of characteristic 0, this method is given by myself recently. The main idea is to use Chow's moving lemma to show that under the same notations as above, if S is a variety of codimension p then there is a pencil $Z \to \mathbb{P}^1$ of varieties of codimension p, such that a generic fiber Z_t is rationally equivalent to $Cdeg(S)f^*(\omega_X^p)$ where $C > 0$ is independent of f and the special fiber Z_0 contains the closure of $(f|_U)^*(S)$. The case of maps over a field of positive characteristic is proved in my ongoing joint work with C. Favre, where we give applications for maps over non-Archimedean fields.

3. Some properties

Let X be a compact Kähler manifold of dimension k and let $f : X \to X$ be a dominant meromorphic map.

1) From definition, it can be seen easily that $\lambda_0(f) = 1$ and $\lambda_k(f) =$ the topological entropy of f.

2) Log-concavity: $\lambda_p(f)^2 \geq \lambda_{p+1}(f)\lambda_{p-1}(f)$. This is proved by using Hodge index theorem: If α and β are nef classes on a compact Kähler manifold of dimension k then $(\alpha^p.\beta^{k-p})^2 \geq (\alpha^{p+1}\beta^{k-p-1}).(\alpha^{p-1}.\beta^{k-p+1})$. Let Γ be a resolution of the graph of f^n, and let $\pi, g : \Gamma \to X$ be the two induced maps such that $f = g\circ\pi^{-1}$. We then apply Hodge index theorem to $\alpha = \pi^*(\omega_X)$ and $\beta = g^*(\omega_X)$.

3) Birational invariance: If $\pi : X' \to X$ is a birational map and $f' = \pi^{-1}\circ f\circ\pi$ then $C^{-1}||f^*(\omega_X)|| \leq ||f'^*(\omega_{X'})|| \leq C||f^*(\omega_X)||$ here $C > 0$ is independent of f. From this we obtain that $\lambda_p(f) = \lambda_p(f')$ for any $0 \leq p \leq k = dim(X)$.

4) Good behavior under meromorphic fibrations: Property 3) can be extended to the case $\pi : X \to Y$ is any dominant meromorphic map preserved by f. Here we don't assume that a generic fiber of π is finite. More precisely, if $g : Y \to Y$ is a dominant meromorphic map such that $\pi\circ f = g\circ\pi$, then we have a relation: $\lambda_p(f) = \max_{0,p-k+l\leq j\leq l,p}\lambda_j(g)\lambda_{p-j}$. We can see this relation easily in the special case $X = Y\times Z$, $f = (g, h)$ a product map, and $\pi : X \to Y$ is the projection to Y; by using the Kunneth formula for the cohomology groups of X and Dinh's bound above. In the general case, the main tool is an analogous Kunneth formula, which now is an inequality rather than an equality. More precisely, if T is a positive closed (p, p) current which is smooth on a Zariski open dense set and has no mass on proper subvarieties, then in cohomology:

$$\{T\} \leq A \sum_{\max\{0,p-k+l\}\leq j\leq\min\{l,p\}} \alpha_j(T)\{\pi^*(\omega_Y^j)\} \smile \{\omega_X^{p-j}\},$$

where

$$\alpha_j(T) := \langle T, \pi^*(\omega_Y^{l-j}) \wedge \omega_X^{k-l-p+j}\rangle. \tag{1}$$

If $X = Y\times\mathbb{P}^{k-l}$, where Y is projective, Dinh and Nguyen proved (1) using Kunneth's formula and the fact that $\mathbb{P}^{k-l}$ has a lot of automorphisms. They then prove the relation for dynamical degrees in case X and Y are projective, using that any dominant meromorphic map $\pi : X \to Y$ is, upto a finite covering, the canonical projection $Y\times\mathbb{P}^{k-l}$.

For the case of a dominant meromorphic map of compact Kähler manifolds $\pi : X \to Y$, it is not known whether π is upto a finite covering, a canonical projection. In joint work with Dinh and Nguyen, we instead proceed as follows. Let T be a positive closed current on X, and let Δ_X be the diagonal. Then we have $T = (\pi_2)_*(\pi_1^*(T)\wedge[\Delta_X])$ (here π_1 and π_2 are the projections), and hence it is enough to prove a similar formula for $T = [\Delta_X]$ and π is replaced by the product $\pi\times\pi : X\times X \to Y\times Y$. To this end, we observe that Δ_X is a subvariety of $(\pi\times\pi)^{-1}(\Delta_Y)$. Then we extend the semi-regularization of Dinh and Sibony to the form: if V is a submanifold of W and T is a positive closed current on V, then T can be semi-regularized by currents of the form $\iota^*(\theta_n)$, where $\iota : V \subset W$ is the inclusion of V in W and θ_n is a positive closed smooth form on W. Dinh and Sibony's regularization corresponds to the case $V = W$.

4. Applications

1) Gromov - Yomdin formula to compute topological entropy of holomorphic maps: Let $f : X \to X$ be a surjective holomorphic maps. Then f is in particular continuous, and by the classical ergodic theory we can define the topological entropy of f. Its topological entropy

measures how the map separates the orbits of distinct points, and hence is an indication of the complexity of f. The larger topological entropy, the more complexity. Its definition is given by:

$$h_{top}(f) = \sup_{\epsilon>0}(\limsup_{n\to\infty} \log \max\{\#F : F \text{ is an } (n,\epsilon) set\}).$$

Here a set F is called an (n,ϵ) set if any two distinct points x and y in F are (n,ϵ) separated, that is the distance between the two n-orbits $(x, f(x), f^2(x), \ldots, f^n(x))$ and $(y, f(y), f^2(y), \ldots, f^n(y))$ is at least ϵ. Since X is compact, any (n,ϵ) set is finite.

Surprisingly, we have the following simple formula to compute topological entropy $h_{top}(f) = \log \max_{1\leq p\leq k} \lambda_p(f)$. Yomdin proved the inequality $h_{top}(f) \geq \max \log_{1\leq p\leq k} \lambda_p(f)$, and his proof is valid for C^∞ maps. Gromov proved the other inequality $h_{top}(f) \geq \max \log_{1\leq p\leq k} \lambda_p(f)$. The argument of Gromov goes as follows. Let $\Gamma_n \in X^n = X \times X \times \ldots \times X$ (n times), be the graph of f^n, i.e. the set of points of the form $(x, f(x), \ldots, f^n(x))$. If we consider the product metric on X^n, then any (n,ϵ) set F gives n points in Γ_n such that the distance between any two points are at least ϵ. Now a classical theorem says that for any ball B_ϵ of radius ϵ in X^n we have $Vol(\Gamma_n \cap B_\epsilon) \geq C_\epsilon$, where C_ϵ is independent of n. From this we obtain

$$h_{top}(f) \leq \limsup_{n\to\infty} \frac{1}{n} \log Vol(\Gamma_n).$$

Thus it remains to show that $\limsup_{n\to\infty} Vol(\Gamma_n)^{1/n} = \max_{1\leq p\leq k} \lambda_p(f)$. But by definition, $Vol(\Gamma_n) \sim \sum_{0\leq p\leq k} ||(f^n)^*(\omega_X^p)||$, and we are done.

For a dominant meromorphic map $f : X \to X$ we can define topological entropy using Gromov's idea of considering the (n,ϵ) sets on the graphs Γ_n. Here $\Gamma_n =$ the closure of the points $(x, f(x), f^2(x), \ldots, f^n(x))$ where x is such that all $f(x), \ldots, f^n(x)$ are well-defined. Yomdin's inequality fails for a dominant meromorphic map, for example (Guedj's) for the rational map $f(z, w) = (z^d, w + 1)$ on $\mathbb{P}^2$ for $d \geq 2$. This map has zero topological degree, but has $\lambda_1(f) = \lambda_2(f) = d > 1$.

Meanwhile, Gromov's inequality still holds, and was proved by Dinh and Sibony. Here the main tool is again regularization of positive closed currents. Using this we can bound the mass of currents of the form $T_1 \wedge T_2 \wedge \ldots \wedge T_j|_U$ by the product of the masses of each T_j, where U is a Zariski open dense set on which all T_j's are smooth. Applying this to $T_j = (f^{n_j})^*(\omega_X^{p_j})$ and U a Zariski open dense set over which all maps f^{n_j} are holomorphic gives us the desired inequality.

2) Constraints on the geometry of manifolds with a dynamical degree larger than other dynamical degrees: One interesting class of meromorphic maps are those with a dynamical degree larger than other dynamical degrees. These include the class of polarized endomorphisms. Guedj called these maps cohomologically hyperbolic, and it was conjectured that if X has such maps then the Kodaira dimension of X is 0 or $-\infty$ and the Albanese map is surjective. The idea of the proof is to use my joint result with Dinh and Nguyen on dynamical degrees of semi-conjugate maps, to show that if f is cohomologically hyperbolic and semi-conjugates with a map $g : Y \to Y$ then g must be itself cohomologically hyperbolic. Now the map from X to its image $Kod(X)$ under the Kodaira map is preserved by any selfmap $f : X \to X$, and the map $g : Kod(X) \to Kod(X)$ is the restriction of a linear map on a projective space, hence all of the dynamical degrees of g are 1. For the assertion about the Albanese map, we use that if the

Albanese map is not surjective then the image of X by the Albanese map has positive Kodaira dimension, and this contradicts the previous result.

3) Counting periodic points of meromorphic maps whose topological degree is larger than other dynamical degrees: These maps include polarized endomorphisms. We say that x is an isolated periodic point of f if there is a neighborhood of x on which f is holomorphic and x is a usual isolated fixed point of f. My other joint work with Dinh and Nguyen shows that in this case the isolated periodic points are Zariski dense, the number of them are $\lambda_k(f)^n + o(\lambda_k(f)^n)$, and most of the isolated periodic points are repelling. We also prove that the backward orbit of a generic point is Zariski dense. In case of polarized endomorphisms, these results were proved by Briend-Duval.

Two main ideas are used:

For upper bound of the number of periodic points: We use a recent theory of Dinh and Sibony on tangent currents. These are the analytic analog of the deformation by the diagonal in intersection theory.

For lower bound, we need to construct enough good inverse branches of the maps f^n. This is done by considering a positive closed $(1,1)$ current constructed from the indeterminate and exceptional sets of f.

4) Detect that a map preserves no non-trivial fibration. One interesting problem is to construct primitive automorphisms of positive entropy on (uni)rational 3 folds. One way to construct automorphisms of positive entropy is as follows. Let A be a complex 3-torus and $f : A \to A$ be an automorphism with $\lambda_1(f) \neq \lambda_2(f)$ (there are many problems of such examples). Let G be a finite subgroup of $Aut(A)$ and let $Y = A/G$.

Claim. Assume that f descends to an automorphism g of Y. Then f lifts to an automorphism of a smooth model X of Y, and this map is of positive entropy and is primitive in the sense that it preserves no meromorphic fibration.

Proof of the claim. By Hironaka's equivariant resolution of singularity, any automorphism g on Y lifts to an automorphism h of a desingularization X of Y. Then we have a generically finite meromorphic map $\pi : A \to Z$ such that $\pi \circ f = h \circ g$.

Apply 4) of Section 3, we find that $\lambda_p(f) = \lambda_p(h)$ for any p. Then Gromov-Yomdin's formula shows that the topological entropy of h is positive.

It remains to show that h is primitive. This follows from the following result, which was given by Oguiso and I: If $f : X \to X$ is a dominant meromorphic map with $\lambda_1(f) > \lambda_2(f)$, here X has dimension ≥ 2, then f is primitive. The latter result is a consequence of 4) of Section 3. $\square$

So if we can show that there are such quotients with interesting geometric properties (such as rational or unirational), then we have examples of manifolds with interesting geometric properties and automorphisms. Catanese, Oguiso and I showed that an example, which is a finite quotient of a complex 3-torus $X_4 = E^3_{\sqrt{-1}}/ < diag(\sqrt{-1}, \sqrt{-1}, \sqrt{-1}) >$, constructed in 1975 by Kenji Ueno, is unirational. We showed that X_4 is birationally equivalent to the quintic 3-fold H with the equation $a_2^2 b_3(1-b_3^2) = a_3^2 b_2(1-b_2^2) + b_2 b_3(b_2^2 - b_3^2)$. H is a conic bundle over $K(b_2, b_3)$.

Previously, Oguiso and I showed that a similar construction ($X_6 = E_\omega^3/ < diag(-\omega, -\omega, -\omega) >$, here $\omega^2 + \omega + 1 = 0$) is a rational 3-fold having primitive automorphisms of positive entropy. This gives, for the first time, examples of primitive automorphisms with positive entropy.

Remark. Applying Brauer's group theory, Colliot-Thélène showed that H is birationally equivalent to the conic bundle $a_2^2 - b_2 a_3^2 - b_3 = 0$. The latter is clearly rational, and so is X_4. Hence, X_4 is the second known rational smooth 3-fold with primitive automorphisms of positive entropy.