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UNIRTAIONALITY OF UENO-CAMPANA'S THREEFOLDS

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1. Introduction

This is a brief report of my talk at Kinosaki, which is based on my joint works with Professor Fabirizio Catanese and Doctor Tuyen Truong ([OT13], [COT13]), motivated by the algebro-geometric aspect of complex dynamics of several variables. Unless stated otherwise, we shall work over the complex number field $\mathbf{C}$. I would like to thank the organizers for invitation and their hospitalities during my stay at Kinosaki.

2. Ueno-Campana's threefolds

Let $E_\eta = \mathbf{C}/(\mathbf{Z} + \mathbf{Z}\eta)$ be the elliptic curve of period η. There are exactly two elliptic curves with a Lie automorphism other than ± 1. Namely, the elliptic curves of periods $\sqrt{-1}$ and $\omega = (-1+\sqrt{-3})/2$, the primitive forth and third root of unity.

Let X_4 be the canonical resolution of the quotient threefold

$$E_{\sqrt{-1}} \times E_{\sqrt{-1}} \times E_{\sqrt{-1}}/\langle \mathrm{diag}\,(\sqrt{-1}, \sqrt{-1}, \sqrt{-1})\rangle\ ,$$

i.e., the blow up at the maximal ideals of singular points of type $1/2(1,1,1)$ and $1/4(1,1,1)$.

Let X_6 be the canonical resolution of the quotient threefold

$$E_\omega \times E_\omega \times E_\omega/\langle \mathrm{diag}\,(-\omega, -\omega, -\omega)\rangle\ ,$$

i.e., the blow up at the maximal ideals of singular points of type $1/2(1,1,1)$, $1/3(1,1,1)$ and $1/6(1,1,1)$.

We call X_4 and X_6 *Ueno-Campana's threefolds* in this report. Both threefolds are smooth and of Kodaira dimension $-\infty$. In particular, they are uni-ruled, by the minimal model theory for threefolds.

The canonical resolution X_3 of

$$E_\omega \times E_\omega \times E_\omega/\langle \mathrm{diag}\,(\omega, \omega, \omega)\rangle\ ,$$

is a Calabi-Yau threefold in the strict sense. Here an n-dimensional smooth projective manifold M is called a Calabi-Yau manifold of strict sense, if M is simply-connected, $H^0(M, \Omega^k) = 0$ for $0 \le k \le n$ and has a nowhere vanishing global holomorphic n-form ω_M. X_3 is rigid, i.e., has no small deformation, but plays important roles in the classification of Calabi-Yau threefolds in the view of the second Chern class.

Remark 2.1. The Kodaira dimensions of the canonical resolution of $E_{\sqrt{-1}}^n/\langle\sqrt{-1}I_n\rangle$ ($n \ge 4$) and the canonical resolution of $E_\omega^n/\langle -\omega I_n\rangle$ ($n \ge 6$) are 0. So, they are not uni-ruled. Moreover, they are not smooth Calabi-Yau manifolds in the strict sense, either, as the

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singular point of type $1/2(1, 1, \cdots, 1)$ is terminal. Similarly, the canonical resolution of $E_{\omega}^{n}/\langle \omega I_n \rangle$ $(n \geq 4)$ and $E_{\eta}^{n}/\langle -I_n \rangle$ $(n \geq 3)$ are not Calabi-Yau manifolds in the strict sense, either.

3. Main results and relevant results, questions

The main theorem I have explained is the following:

Theorem 3.1. *(1) X_4 is uni-rational and X_6 is rational.*

(2) Moreover, X_4, X_6 and also X_3 admit primitive biregular automorphisms of positive entropy.

Uni-rationality of X_4 is proved by [COT13] and rationality of X_6 and the second statement (2) are proved by [OT13]. We should also emphasize that *Colliot-Thélène* [CTh13] *finally proved that X_4 is actually rational*, using the equation we will explan later.

Here, we recall some notions in the main theorem:

Definition 3.2. (1) A birational selfmap f of a manifold M is *imprimitive* if f comes from the lower dimensional manifolds, or more precisely, there are a dominant rational map $\varphi : M \to B$ with $0 < \dim B < \dim M$ and a dominant rational map $f_B : B \cdots \to B$ such that $\varphi \circ f = f_B \circ \varphi$. A birational selfmap that is not imprimitive is *primitive*. This notion is inroduced by De-Qi Zhang [Zh99].

(2) *Entropy* is an important invariant that measures how fast two general points spread out under the action of $\langle f \rangle$. The original definition is a completely topological one, but when f is a biregular automorphism of a compact Kähler manifold, a fundamental theorem of Gromov and Yomdin says that the entropy of f is

$$\log \max_{0 \leq k \leq \dim M} d_k(f) \ ,$$

where $d_k(f)$ is the k-th dymanical degree, i.e., the spectral radius of $f^*|H^{2k}(M, \mathbf{Z})$. (For more details and for dynamical degrees of a rational selfmap, see Section 3 of [OT13] and references therein and also a report of Doctor Tuyen Truong in this volume.) Note that if f is of positive entropy, then f is not in the identity component $\mathrm{Aut}^0(M)$ and it is also of infinite order.

In complex dynamics of several variables, the following problem is one of the most basic problems and it is completely open:

Problem 3.3. Find (many) examples of rational manifolds and Calabi-Yau manifolds admitting *primitive* biregular automorphisms of *positive entropy.*

Our main theorem provides the *first* explicit answer to this question.

The following important result of De-Qi Zhang [Zh99] explains the reason why our examples are (uni-)rational manifolds and Calabi-Yau manifolds:

Theorem 3.4. *Let M be an n-dimensional projective manifold with a birational automorphism of infinite order. Then M is either rationally-connected, birational to a complex torus, or birational to a minimal Calabi-Yau variety, i.e., a normal projective variety W with $\mathbf{Q}$-factorial terminal singularities such that K_W is numerically trivial and $H^1(\mathcal{O}_W) = 0$, provided that the minimal model program works in the category of n-dimensional normal* *projective varieties with $\mathbf{Q}$-factorial terminal singularities and that the weak abundance conjecture holds in dimension n, that is, there is a positive integer m such that $H^0(mK_X) \neq 0$ for each n-dimensional normal projective variety X with $\mathbf{Q}$-factorial terminal singularities and with nef K_X. Note that the result is unconditional when $n \leq 3$.*

Rational manifolds and uni-rational manifolds are important examples of rationally connected manifolds and Calabi-Yau manifolds in the strict sense are also important examples of minimal Calabi-Yau varieties.

Our theorem (1) together with a result of Colliot-Thélène [CTh13] give the final affirmative answer to the following question asked by Ueno [Ue75][1] and Campana [Ca12]:

Question 3.5. Is X_4 rational or unirational?

Campana [Ca12] himself showed that X_4 is rationally connected. He then asked this question above explicitly.

4. Brief outline of proof of the main theorem

We shall give a brief outline of proof of the main theorem (1). See a report of Doctor Tuyen Truong (in this volme) for outline of proof of the main theorem (2). One can find a complete proof of the main theorem in [OT13], [COT13], [CTh13].

The most crucial part of the main theorem (1) is to find simple affine hypersurface models of X_4 and X_6.

Let (x, y) be the affine coordinates of $\mathbf{C}^2$. Consider the affine plane curve $C(\sqrt{-1})$ and its automorphism g_4 defined by

$$y^2 = x(x^2 - 1)\ ,\ g_4^*(x, y) = (-x, \sqrt{-1}y)$$

and the affine plane curve $C(\omega)$ and its automorphism g_6 defined by

$$y^2 = x^3 - 1\ ,\ g_6^*(x, y) = (-\omega x, -y)\ .$$

Then the pair $(E_{\sqrt{-1}}, \sqrt{-1})$ of the elliptic curve $E_{\sqrt{-1}}$ and its automorphism $\sqrt{-1}$ is birational to the pair $(C(\sqrt{-1}), g_4)$. Similarly, $(E_\omega, -\omega)$ is birational to $(C(\omega), g_6)$. Thus, the rational function fields $\mathbf{C}(X_4)$ and $\mathbf{C}(X_6)$ are isomorphic to the invariant fields $\mathbf{C}(C(\sqrt{-1})^3)^{\tilde{g_4}}$ and $\mathbf{C}(C(\omega)^3)^{\tilde{g_6}}$ respectively. Here $\tilde{g_4}$ and $\tilde{g_6}$ are the diagonal actions of g_4 and g_6 on the product threefolds $C(\sqrt{-1})^3$ and $C(\omega)^3$. As one can imagine, the fields $\mathbf{C}(C(\sqrt{-1})^3)^{\tilde{g_4}}$ and $\mathbf{C}(C(\omega)^3)^{\tilde{g_6}}$ are computable from the explicit equations of $C(\sqrt{-1})$, $C(\omega)$ and the explicit forms of the actions $\tilde{g_4}$ and $\tilde{g_6}$. Actually, we have:

Theorem 4.1. *(1) $\mathbf{C}(X_4)$ is isomorphic to*

$$\mathbf{C}(t, s, z, w)\ ,$$

with a single equation

$$(t^2 - z)(s^2 - w^3) = (s^2 - w)(t^2 - z^3)\ ,$$

More precisely, $\mathbf{C}(s, z, w)$ is purely transcendental over $\mathbf{C}$ of transcendental degree 3 and $\mathbf{C}(X_4)$ is isomorphic to

$$\mathbf{C}(s, z, w)[T]/I$$

[1] Strictly speaking, he just mentions that it is unknown if X_4 is uni-rational or not.

where I is the principal ideal generated by

$$(T^2 - z)(s^2 - w^3) - (s^2 - w)(T^2 - z^3) \ ,$$

(an irreducible quadratic polynomial) in the polynomial ring $\mathbf{C}(s, z, w)[T]$ *over* $\mathbf{C}(s, z, w)$.

(2) $\mathbf{C}(X_6)$ *is isomorphic to*

$$\mathbf{C}(t, s, z, w) \ ,$$

with a single equation

$$(w^3 - 1)(t^2 - 1) = (z^3 - 1)(s^2 - 1) \ .$$

More precisely, $\mathbf{C}(s, z, w)$ *is purely transcendental over* $\mathbf{C}$ *of transcendental degree* 3 *and* $\mathbf{C}(X_4)$ *is isomorphic to*

$$\mathbf{C}(s, z, w)[T]/I$$

where I is the principal ideal generated by

$$(w^3 - 1)(T^2 - 1) - (z^3 - 1)(s^2 - 1) \ ,$$

in the polynomial ring $\mathbf{C}(s, z, w)[T]$ *over* $\mathbf{C}(s, z, w)$.

Let $\mathbf{C}^4$ is the affine 4-plane with affine coordinates (t, s, z, w). The above theorem means that X_4 is birational to the 3-dimensional affine hypersurface H_4 defined by

$$(t^2 - z)(s^2 - w^3) = (s^2 - w)(t^2 - z^3) \ ,$$

and X_6 is birational to the 3-dimensional affine hypersurface H_6 defined by

$$(w^3 - 1)(t^2 - 1) = (z^3 - 1)(s^2 - 1) \ .$$

The projection $p_{34} : (t, s, z, w) \mapsto (z, w)$ defines the conic bundle structures on H_4 an H_6:

$$p_{34} : H_4 \to \mathbf{C}^2 \ , \ p_{34} : H_6 \to \mathbf{C}^2 \ .$$

It is clear that $(t, s, z, w) = (1, 1, z, w)$ is a section of $p_{34} : H_6 \to \mathbf{C}^2$, and therefore H_6 is rational. Unfortunately, p_4 does not admit a rational section. However, if one can take the base change $\pi : \mathbf{C}^2 \to \mathbf{C}^2$ given by

$$\pi^* z = \tilde{z}^2 \ , \ \pi^* w = \tilde{w}^2 \ ,$$

then the induced conic bundle $H_4 \times_{\mathbf{C}^2} \mathbf{C}^2 \to \mathbf{C}^2$ admits a section $(t, s, \tilde{z}, \tilde{w}) = (\tilde{z}, \tilde{w}, \tilde{z}, \tilde{w})$, and therefore H_4 is uni-rational.

Remark 4.2. Colliot-Thélène [CTh13] shows that the conic bundle $p_{34} : H_4 \to \mathbf{C}^2$ is also birational to the conic bundle $p_{34} : (H_4)' \to \mathbf{C}^2$ where $(H_4)'$ is the affine hypersurface defined by

$$t^2 - zs^2 - w = 0 \ ,$$

by showing that these two conic bundles define the same element of the Brauer group $\mathrm{Br}\,(\mathbf{C}(z, w))$ (See Proposition (2.2) in [CTh13] how to check this). It is clear that $(H_4)'$ is rational as $w = t^2 - zs^2$, whence so is H_4.

Remark 4.3. By the proof above and Remark 4.2, H_4 is rational over any field k containing $\sqrt{-1}$ of characteristic $\neq 2$ and H_6 is rational over any field k containing ω of characteristic $\neq 2, 3$.

Question 4.4. (1) It would be interesting to connect X_4 and X_6 to $\mathbf{P}^3$ by explicit blowings-up and blowings-down along smooth centers. Especially, it is quite interesting to see if one can obtain X_4 and/or X_6 only by blowings-up of $\mathbf{P}^3$ along smooth centers or not.

(2) It would be also interesting to find the conic bundle structures on X_4 and X_6 purely geometrical ways. For instance, how can one describe the fibers of p_{34} in X_4 and X_6 in purely geometrical terms of X_4 and X_6?

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