代数幾何学シンポジウム記録
2013年度　pp.25-32

DEFORMATIONS OF IRRATIONAL AFFINE RULED SURFACES

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Abstract. This note consists in a report of my talk about deformations of affine surfaces possessing $\mathbb{A}^1$-fibrations at Kinosaki conference, which concerns a joint work with Adrien Dubouloz (L'Institut de Mathématiques, l'Université de Bourgogne) [DuKi13]. Certainly the structure of affine surfaces admitting $\mathbb{A}^1$-fibrations themselves are well understood nowadays, but that of deformations of such surfaces is quite complicated to grasp. We mention a factorization theorem in case of a deformation of *irrational* affine surfaces with $\mathbb{A}^1$-fibrations by noting that such a factorization is not feasible in general in case of a deformation of rational ones.

1. Introduction & Motivation

1.1. All varieties treated in this article are defined over the field of complex numbers $\mathbb{C}$. As a general perception, projective varieties covered by images of projective line $\mathbb{P}^1$, so-called *uniruled* varieties, play important roles in the theory of birational geometry as outputs of minimal model program. As an affine counterpart of this concept, affine algebraic varieties swept out by images of the affine line $\mathbb{A}^1$, so-called *affine uniruled* varieties, should play still important roles in order to deal with problems on polynomial rings by means of geometry.

Example 1.1. As an example which justifies an importance to observe affine ruled varieties, we can say about the Zariski cancellation problem which asks, in terms of geometry, whether or not an affine algebraic variety X of dimension n satisfying the condition $X \times \mathbb{A}^1 \cong \mathbb{A}^{n+1}$ is isomorphic to the affine n-space $\mathbb{A}^n$. In case of $n = 1$, it is an easy exercise to confirm that it is true. Whereas even in case of $n = 2$, the problem is never trivial. From the condition $X \times \mathbb{A}^1 \cong \mathbb{A}^3$, it is easy to see that the coordinate ring $A = \Gamma(\mathscr{O}_X)$ is UFD, invertible regular functions on X are non-zero constants and X contains plenty of affine lines. In fact, these three conditions are enough to ascertain that such an X is isomorphic to $\mathbb{A}^2$ due to Fujita, Miyanishi-Sugie (cf. [MS80], [KM99]). Namely, they show that a given affine algebraic surface $X = \mathrm{Spec}(A)$ is isomorphic to the affine plane $\mathbb{A}^2$ if and only if the following three conditions are simultaneously satisfied:

(i) A is UFD,

(ii) $A^\times$ coincides with non-zero constants, and

(iii) X is affine uniruled.

For $n \geqq 3$, the problem is still open being lack of a nice characterization of $\mathbb{A}^n$ as an affine uniruled variety. Actually, as in case of dimension two, the same properties (i), (ii) and (iii) as above are satisfied in this case also, but these three are not enough to deduce that $X \cong \mathbb{A}^n$.[1]

1.2. The third condition (iii) in Example 1.1 means, by definition, that for a general point $x \in X$ there exists an algebraic curve $x \in C_x \subseteq X$ whose normalization $\widetilde{C_x}$ is isomorphic to the affine line $\mathbb{A}^1$. Hence even with conditions (i) and (ii), it seems that X in question is not so close to the affine plane $\mathbb{A}^2$. But, for a given smooth affine surface Y, the following four conditions (a), (b), (c) and (d) are equivalent to each other due to [MS80] and [KM99]:

(a) Y is affine uniruled,

(b) the log Kodaira dimension $\overline{\kappa}(Y)$ of Y is equal to $-\infty$,

(c) Y has an $\mathbb{A}^1$-fibration over a smooth algebraic curve,

(d) Y contains an open affine subset $U \subseteq Y$ of the form $U \cong V \times \mathbb{A}^1$, where V is an affine curve.

The author was supported by a Grant-in-Aid for Scientific Research of JSPS No. 24740003.

[1] Nevertheless, it is worthwhile to recall that there are several nice characterizations of the affine 3-space $\mathbb{A}^3$ as *affine ruled* variety, see e.g. [Mi84a]. For the definition of affine ruledness, see Definition 1.2.

In particular, the existence of plenty of affine lines on Y implies that of a cylinder $U \cong V \times \mathbb{A}^1$ there. The analogue of this a little bit strange looking peculiarity of smooth affine surfaces is no longer true in case of higher dimension. For the later use, we prepare the notion arising from the property (d).

Definition 1.2. An affine algebraic variety X is said to be *affine ruled* if X contains an open affine subset $U \subseteq X$ such that $U \cong V \times \mathbb{A}^1$ with a suitable affine variety V.

By definition, it is obvious that affine ruledness implies affine uniruledness, and as mentioned above, the converse also holds true in case of smooth affine surfaces. It is worthwhile to notice that this equivalence holds no longer in case of normal affine surfaces. For instance, affine cones over a non-rational projective curve are affine uniruled, but not affine ruled.

1.3. On the other hand, once we consider the case of higher dimension, the affine uniruledness guarantees no longer affine ruledeness in general even for smooth ones.

Example 1.3. Let $S \subseteq \mathbb{P}^3$ be a smooth cubic hypersurface and $X := \mathbb{P}^3 \backslash S$ the complement with respect to S. It follows then that X is affine uniruled (see 1.4 for this), whereas it is not affine ruled (cf. [DuKi12]). This fact is not trivial to ascertain, and indeed we need to depend on a result due to Clemens and Griffiths about the non-rationality of smooth cubic threefolds (cf. [ClGr72]). We shall sketch the proof for it. For a detailed proof, we shall refer readers to [DuKi12]: supposing on the contrary that X is affine ruled, we can find an effective $\mathbb{G}_a$-action on X.[2] Letting $\mu : \widetilde{X} \to X$ be an étale triple covering associated to a generator of $\mathrm{Pic}(X) \cong \mathbb{Z}/3\mathbb{Z}$, we know that the upper variety $\widetilde{X}$ is obtained as a complement of a hyperplane section in a smooth cubic threefold $\widetilde{V} \subseteq \mathbb{P}^4$. Since μ is étale, the $\mathbb{G}_a$-action on X can be lifted to that on $\widetilde{X}$ uniquely. Let $\widetilde{p} : \widetilde{X} \to \widetilde{Y}$ be the corresponding quotient map. Then it is well known that there exists an open affine subset $\widetilde{V} \subseteq \widetilde{Y}$ such that its inverse image $\widetilde{U} := \widetilde{p}^{-1}(\widetilde{V}) \subseteq \widetilde{X}$ is the fiber product of $\widetilde{V}$ with the affine line $\mathbb{A}^1$, i.e., $\widetilde{U} \cong \widetilde{V} \times \mathbb{A}^1$. On the other hand, by virtue of [ClGr72], it follows that $\widetilde{V}$ is unirational but not rational. In consideration of the property $\widetilde{U} \cong \widetilde{V} \times \mathbb{A}^1$, we know that $\widetilde{V}$ is unirational as well as $\widetilde{U}$. Notice that in case of dimension two over $\mathbb{C}$, the unirationality and the rationality coincide to each other. In particular, $\widetilde{V}$ is rational, so that $\widetilde{U}$ is also rational. But this is absurd as $\widetilde{V}$ is not rational.

1.4. In Example 1.3, we asserts that $X = \mathbb{P}^3 \backslash S$ is affine uniruled. In order to relate this fact with deformations of affine ruled surfaces, we shall verify this fact from now on, which is not so difficult to see taking the equivalence (b) $\Longleftrightarrow$ (c) into account. As well known, a smooth cubic surface S does contain twenty seven lines, so let us denote by l one of them. Let us investigate the base point free linear pencil $\mathscr{L}$ of conics on S that is determined as follows:

$$\left| \mathscr{O}_{\mathbb{P}^3}(1) \otimes \mathscr{I}_l \right||_S = \mathscr{L} + l.$$

It is easy to see that there exists a special member in $\mathscr{L}$, which is either a smooth conic meeting l tangentially or a union of lines intersecting l at a common point. In any case, let $H \subseteq \mathbb{P}^3$ be a hyperplane such that $H|_S - l$ yields a spacial member of $\mathscr{L}$ as above. On the other hand, letting Λ be the linear pencil on $\mathbb{P}^3$ spanned by S and $3H$, we shall look into the morphism $\varphi : X \to \mathbb{A}^1$, which is realized as a restriction of the rational map $\Phi_\Lambda : \mathbb{P}^3 \dashrightarrow \mathbb{P}^1$ determined by Λ onto X. The general fiber $\varphi^*(a)$ of φ is nothing but a complement of a smooth cubic surface in Λ with respect to a union of the special member of $\mathscr{L}$ and l. Then a straightforward computation in aid of adjunction says that $\overline{\kappa}(\varphi^*(b)) = -\infty$, which means that $\varphi^*(b)$ admits an $\mathbb{A}^1$-fibration. As a consequence, it follows that X is affine uniruled, as desired.

1.5. With the notation as in 1.4, we can find an open affine subset U of X, which is covered by mutually disjoint affine lines, more precisely to say, there exists a two-dimensional family $\mathscr{F} = \{C_\gamma\}$ of affine lines such that for any point $x \in U$ we can find a unique $\mathbb{A}^1 \cong C_\gamma \in \mathscr{F}$ such that C_γ passes through x. In consideration of this geometric fact, it seems to be reasonable to expect that the family $\mathscr{F}$ yields an $\mathbb{A}^1$-fibration $\psi : U \to V$ over a surface to factor $\varphi|_U$, i.e., the following diagram:

$$(*) \quad \varphi|_U = \phi \circ \psi : U \xrightarrow{\psi} V \xrightarrow{\phi} W,$$

[2] Note that $\mathrm{Pic}(X) \cong \mathbb{Z}/3\mathbb{Z}$, in particular, it is finite. Thence the affine ruledness on X is equivalent to the existence of an effective $\mathbb{G}_a$-action on X (see [Ki13, Remark 4.4], see also [KPZ09, KPZ11]).

where W is an open subset of the base curve $\mathbb{A}^1$ of φ. Provided such a factorization $(*)$, the result in [KaMi78] ascertains that we can find an open affine subset $V_0 \subseteq V$ such that $U_0 := \psi^{-1}(V_0) \subseteq U$ is of the form $U_0 \cong V_0 \times \mathbb{A}^1$. But this means that X is affine ruled, which is a contradiction to Example 1.3 above.

Instead of $X = \mathbb{P}^3 \backslash S$, we shall consider the restriction $f := \Phi_\Lambda|_{(\mathbb{P}^3\backslash H)} : \mathbb{P}^3\backslash H \cong \mathbb{A}^3 \to \mathbb{A}^1$ of the rational map Φ_Λ onto the complement $\mathbb{P}^3\backslash H \cong \mathbb{A}^3$. By the same fashion, we know that a general fiber of f is a smooth affine surface admitting an $\mathbb{A}^1$-fibration. Although the affine 3-space $\mathbb{A}^3$ is clearly affine ruled, f is neither factored by an $\mathbb{A}^1$-fibration over a surface even if we restrict f onto an open dense subset of $\mathbb{A}^3$.

1.6. We deal with more concrete examples to focus on what happens.

Example 1.4. Let $f(x, y, z) := xy(x + y) + z(z + 1) \in \mathbb{C}[x, y, z]$, and let us observe the polynomial map defined by $f(x, y, z)$, i.e.,

$$f : \mathbb{A}^3 \ni (a, b, c) \mapsto f(a, b, c) \in \mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[f]).$$

We can confirm that a general fiber of f is a smooth affine surface with log Kodaira dimension $-\infty$, in particular, it possesses an $\mathbb{A}^1$-fibration. Whereas, f can never be factorized by means of an $\mathbb{A}^1$-fibration even if we take an open dense subset of X and the restriction of f onto it however (see 1.5, or [Ki13]).

Example 1.5. Instead, let $f(x, y, z) := x^2z^2 - 2xy^2z + y^4 - z^3 \in \mathbb{C}[x, y, z]$ and let us consider the corresponding polynomial map $f : \mathbb{A}^3 \to \mathbb{A}^1$ determined by $f(x, y, z)$. Then the fiber $f^*(\alpha) \subseteq \mathbb{A}^3$ is defined in the affine 3-space $\mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x, y, z])$ by the polynomial:

$$f(x, y, z) - \alpha = (xz - y^2)^2 - z^3 - \alpha \in \mathbb{C}[xz - y^2, z].$$

The inclusions $\mathbb{C}[f] \subseteq \mathbb{C}[xz - y^2, z] \subseteq \mathbb{C}[x, y, z]$ induce morphisms:

$$f = h \circ g : \mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x, y, z]) \xrightarrow{g} \mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[xz - y^2, z]) \xrightarrow{h} \mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[f]).$$

Notice that the sub-algebra $\mathbb{C}[xz - y^2, z] \subseteq \mathbb{C}[x, y, z]$ coincides with the kernel $\mathrm{Ker}(\delta)$ of the locally nilpotent derivation δ on $\mathbb{C}[x, y, z]$ of the form:

$$\delta = 2y\frac{\partial}{\partial x} + z\frac{\partial}{\partial y}.$$

In other words, the morphism g gives rise to an $\mathbb{A}^1$-fibration over the affine plane $\mathbb{A}^2$, which is obtained as a quotient map with respect to an effective $\mathbb{G}_a$-action on $\mathbb{A}^3$. Summarizing, a general fiber $f^*(\alpha)$ of f admits an $\mathbb{A}^1$-fibration, furthermore, f can be decomposed by means of an $\mathbb{A}^1$-fibration associated to an effective $\mathbb{G}_a$-action.

1.7. As demonstrated in 1.4, 1.5 and 1.6, the structure of a deformation $f : X \to B$ of affine ruled surfaces seems to be chaotic from the point of view about the possibility of a factorization of f by making use of an $\mathbb{A}^1$-fibration even in case of $X \cong \mathbb{A}^3$. Then we propose the following problem:

Problem 1.6. Let $f : X \to B$ be a morphism from a normal affine algebraic threefold (with certain kinds of mild singularities, e.g., $\mathbb{Q}$-factorial, terminal singularities) onto a smooth algebraic curve such that a general fiber of f is an affine ruled affine surface. Then, under which conditions about a general fiber of f or about the ambient space X, can f be decomposed by means of an $\mathbb{A}^1$-fibration over a surface (by restricting f onto a suitable open dense subset of X if necessary) ?

1.8. We need to mention a result due to Gurjar, Masuda and Miyanishi (cf. [GMM13]) concerning Problem 1.6. Before stating their result, we shall prepare the notion of type about $\mathbb{A}^1$-fibrations on affine surfaces.

Definition 1.7. Let Y be a normal affine surface with an $\mathbb{A}^1$-fibration, say $\pi : Y \to C$, which is surjective. We say that π is of *affine type* (resp. *complete type*) if the base curve C is affine (resp. projective).

It seems that the difference between being of affine type and complete type is geometrically tiny, however it is crucial in the sense that an $\mathbb{A}^1$-fibration on a normal affine surface Y is of affine type if and only if it is realized as a quotient map with respect to an effective $\mathbb{G}_a$-action on Y (cf. [GMM12]), which is in turn translated in terms of a purely algebraic object, so-called locally nilpotent derivation on the coordinate ring $\Gamma(\mathscr{O}_Y)$. By experience, $\mathbb{A}^1$-fibrations of affine type are more handy to deal with than those of complete type.

Example 1.8. (1) By an immediate consequence of the famous Abhyankar-Moh-Suzuki's theorem (cf. [AM75], [Su74], see also [Miy78, Chapter 2, §1]), every $\mathbb{A}^1$-fibration on the affine plane $\mathbb{A}^2$ is of affine type.

(2) Let X be the complement of $\mathbb{P}^1 \times \mathbb{P}^1$ with respect to a diagonal $\Delta \subseteq \mathbb{P}^1 \times \mathbb{P}^1$. Let $p_i : \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$ $(i = 1, 2)$ be the projection, and let l_i be a fiber of p_i such that l_1 and l_2 meet at a point on Δ. Then the linear pencil $\mathscr{L}$ on $\mathbb{P}^1 \times \mathbb{P}^1$ spanned by Δ and $l_1 + l_2$ gives rise to an $\mathbb{A}^1$-fibration $\Phi_{\mathscr{L}}|_X : X \to \mathbb{A}^1$ of affine type, where $\Phi_{\mathscr{L}}$ is the rational map determined by $\mathscr{L}$. Meanwhile, $p_i|_X : X \to \mathbb{P}^1$ is an $\mathbb{A}^1$-fibration of complete type. Thus the property of affine or complete type is not, in general, intrinsic on a given surface.

(3) Let $S \subseteq \mathbb{P}^3$ be a smooth cubic surface defined as follows:

$$S = \left\{ xy\,(x+y) + zu\,(z+u) = 0 \right\} \subseteq \mathbb{P}^3{}_{[x:y:z:u]}.$$

Then the hyperplane $H := \{u = 0\}$ cuts S in such a way that $H|_S = L_1 + L_2 + L_3$, where $L_1 = \{x = u = 0\}$, $L_2 = \{y = u = 0\}$ and $L_3 = \{x + y = u = 0\}$. Notice that these lines L_1, L_2, L_3 meet each other only at $P := [0 : 0 : 1 : 0]$, which is an Eckardt point of S, and $-K_S \sim H|_S = L_1 + L_2 + L_3$ by adjunction. Then, by a straightforward computation, it follows that $\overline{\kappa}(X) = -\infty$, where $X := S \backslash (L_1 \cup L_2 \cup L_3)$. Thus X admits an $\mathbb{A}^1$-fibration by [MS80]. More precisely, we know that any $\mathbb{A}^1$-fibration on X is defined over the projective line $\mathbb{P}^1$, i.e., it is of complete type (cf. [DuKi12]).

1.9. With the notation as in Problem 1.6, it is known that if general fibers of $f : X \to B$ are isomorphic to the affine plane $\mathbb{A}^2$, then it is a Zariski-locally trivial $\mathbb{A}^2$-bundle after shrinking the base, namely, there exists an open affine subset $W \subseteq B$ such that $U := f^{-1}(W)$ is the fiber product of W with $\mathbb{A}^2$, i.e., $U \cong W \times \mathbb{A}^2$ (cf. [KaZa01]). Thence for such a case Problem 1.6 is satisfactorily settled out. Note that as remarked in Example 1.8 (1), every $\mathbb{A}^1$-fibration on $\mathbb{A}^2$ is of affine type. Thus in some extent it may be reasonable to expect that a factorization property asked in Problem 1.6 holds true in the case where a general fiber admits an $\mathbb{A}^1$-fibration of affine type. However, it seems that the reality is far from this intuition. Indeed, the work due to Gurjar, Masuda and Miyanishi (cf. [GMM13, Example 2.6]) yields such an example in which a general fiber of $f : X \to B$ is isomorphic to $(\mathbb{P}^1 \times \mathbb{P}^1)\backslash\Delta$ as in Example 1.8 (2), and such an f can not be decomposed by means of an $\mathbb{A}^1$-fibration even if we restrict f onto an open dense subset of X however. But, instead of an f itself, if we perform a suitable étale finite covering of degree two, the resulting one from f, say $\widetilde{f} : \widetilde{X} \to \widetilde{B}$ is factored via an $\mathbb{A}^1$-fibration. In general, Gurjar, Masuda and Miyanishi (cf. [GMM13]) prove the following result:

Theorem 1.9. (cf. [GMM13, Theorem 2.8]) *Let $f : X \to B$ be a morphism from a smooth affine algebraic threefold X onto a smooth algebraic curve B such that a general fiber of f is a smooth affine surface with an $\mathbb{A}^1$-fibration of affine type. Then we have the following:*

(1) *After shrinking the base curve B if necessary, say $B_0 \subseteq B$, and after taking an étale finite morphism $\widetilde{B}_0 \to B_0$, the resulting morphism from f on the fiber product $\widetilde{X}_0 := f^{-1}(B_0) \times_{B_0} \widetilde{B}_0$, say $\widetilde{f} : \widetilde{X}_0 \to \widetilde{B}_0$ is factored in such a way that:*

$$\widetilde{f} = \widetilde{h} \circ \widetilde{g} : \widetilde{X}_0 \xrightarrow{\widetilde{g}} \widetilde{Y}_0 \xrightarrow{\widetilde{h}} \widetilde{B}_0,$$

where $\widetilde{g}$ is an $\mathbb{A}^1$-fibration over a surface $\widetilde{Y}_0$.

(2) *If we suppose additionally that there exists a relative completion of $f : X \to B$, say $\overline{f} : \overline{X} \to B$, which satisfies the following conditions: (Notation: Let $\Delta := \overline{X} \backslash X$, and for a point $b \in B$, let us put $\overline{X}_b := \overline{f}^*(b)$, $X_b := f^*(b)$ and $\Delta_b := \Delta \cdot \overline{X}_b$.)*

(i) *$(X, \overline{X}, \Delta, \overline{f}, 0)$ with a fixed point $0 \in B$ is a family of logarithmic deformations of the triple $(X_0, \overline{X}_0, \Delta_0)$,*

(ii) *A given $\mathbb{A}^1$-fibration of affine type on a general fiber X_b is extended to a $\mathbb{P}^1$-fibration on $\overline{X}_b$, say φ_b, and*

(iii) *A section of the $\mathbb{P}^1$-fibration φ_0 found in Δ_0 has no monodromy in $\overline{X}$,*

then after shrinking the base curve B if necessary, say $0 \in B_0 \subseteq B$, the restricted morphism $f|_{f^{-1}(B_0)}$ is factored by an $\mathbb{A}^1$-fibration (without the necessity to take an étale finite morphism as in the assertion (1)).

1.10. Summarizing a deformation $f : X \to B$ of smooth affine surfaces admitting $\mathbb{A}^1$-fibrations of affine type can be decomposed by means of an $\mathbb{A}^1$-fibration over a surface up to shrinking and taking an étale finite covering of the base curve B. On the other hand, if a general fiber of $f : X \to B$ is an affine surface that admits $\mathbb{A}^1$-fibrations of complete type only, then the structure of f about a possibility of a factorization by an $\mathbb{A}^1$-fibration becomes to be more subtle to grasp. Anyhow, it is useful to look for criteria to factor by an $\mathbb{A}^1$-fibration. In §2, we shall state main results which concern Problem 1.6, a criterion to factorize by an $\mathbb{A}^1$-fibration and several resulting corollaries something like this, by paying a special attention to the case of $X \cong \mathbb{A}^3$.

2. Main Results

2.1. If we observe a deformation of *irrational* affine ruled surfaces, then we obtain the following result about the possibility of factorizations:

Theorem 2.1. (cf. [DuKi13]) *Let X be a normal affine algebraic threefold with only $\mathbb{Q}$-factorial, terminal singularities, and let $f : X \to B$ be a morphism onto an algebraic curve such that a general fiber of f is irrational and affine ruled. Then there exists an open affine subset $U \subseteq X$ such that the restriction of f onto U is factored in such a way that:*

$$f|_U = h_0 \circ g_0 : U \xrightarrow{g_0} V \xrightarrow{h_0} W,$$

where W is an open subset of B, V is a normal affine surface such that $U \cong V \times \mathbb{A}^1$ and g_0 coincides with the projection to the first factor.

Remark 2.2. Note that the assumption in Theorem 2.1 that a general fiber is *irrational* is crucial to obtain a desired decomposition. For instance, the discussion in 1.5 and Example 1.4 give rise to examples where we can not factor by an $\mathbb{A}^1$-fibration even if we look at the restriction onto an open dense subset however.

As a corollary of Theorem 2.1, we see:

Corollary 2.3. *With the notation and the assumption as in Theorem 2.1, suppose in addition that the Picard group* $\mathrm{Pic}(X)$ *of X is finite. Then f itself can be factored in such a way that:*

$$f = h \circ g : X \xrightarrow{g} Y \xrightarrow{h} B,$$

where g is a quotient map with respect to an effective $\mathbb{G}_a$-action on X. In particular, a general fiber of f possesses an $\mathbb{A}^1$-fibration of affine type.[3]

Remark 2.4. In Corollary 2.3, we do not impose a priori any condition about the type of an $\mathbb{A}^1$-fibration on a general fiber of f, notwithstanding, a posteriori it follows that it admits an $\mathbb{A}^1$-fibration of affine type.

2.2. In the case where X is isomorphic to the affine 3-space $\mathbb{A}^3$, we are able to say further about properties on a morphism $f : \mathbb{A}^3 \to B$ onto an algebraic curve whose general fibers are irrational and affine ruled as in the following fashion:

Theorem 2.5. *Let $f : \mathbb{A}^3 \to B$ be a morphism from the affine 3-space $\mathbb{A}^3$ onto a smooth algebraic curve B such that a general fiber of f is irrational and affine ruled. Then we have the following:*

(1) *The base curve B is isomorphic to either $\mathbb{P}^1$ or $\mathbb{A}^1$,*

[3] In case of $\mathrm{Pic}(X) = (0)$, we can see furthermore that *every* fiber of f admits an effective $\mathbb{G}_a$-action. See also Theorem 2.5 (3) below.

(2) *The morphism f is decomposed in such a way that:*

$$f = h \circ g : \mathbb{A}^3 \xrightarrow{g} \mathbb{A}^2 \xrightarrow{h} B,$$

where $g : \mathbb{A}^3 \to \mathbb{A}^2$ is an $\mathbb{A}^1$-fibration obtained as the quotient morphism of an effective $\mathbb{G}_a$-action on the affine 3-space $\mathbb{A}^3$.

(3) *Every fiber of f (taken with the reduced scheme structure) admits an effective $\mathbb{G}_a$-action, in particular, it possesses an $\mathbb{A}^1$-fibration of affine type.*

Example 2.6. Notice that the base curve B in Theorem 2.5 is not necessarily affine. For instance, let us consider the following linear pencil Λ on the affine 3-space $\mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x, y, z])$:

$$\Lambda = \left\{ S_\lambda = \mathbb{V}_{\mathbb{A}^3}\left((xz - y^2)z^2 + \lambda(xz - y^2)^3 + 1\right) \,\middle|\, \lambda \in \mathbb{P}^1 \right\}.$$

It is easy to confirm that Λ is free of base points, so that it defines a morphism onto $\mathbb{P}^1$:

$$f := \Phi_\Lambda : \mathbb{A}^3 \longrightarrow \mathbb{P}^1 = \mathbb{P}(\Lambda),$$

whose fiber $f^*(\lambda) =: S_\lambda$ over $\lambda \in \mathbb{P}^1$ is isomorphic to $S_\lambda \cong C_\lambda \times \mathbb{A}^1$, where C_λ is the affine curve in $\mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[xz - y^2, z])$ defined by the same polynomial as S_λ. For a general $\lambda \in \mathbb{P}^1$, it follows that S_λ is irrational and affine ruled. Moreover, every fiber of f (taken with the reduced scheme structure) admits an $\mathbb{A}^1$-fibration of affine type. The morphism f can be decomposed in such way that:

$$f = h \circ g : \mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x, y, z]) \xrightarrow{g} \mathbb{A}^2 = \mathrm{Spec}(\mathbb{C}[xz - y^2, z]) \xrightarrow{h} \mathbb{P}^1,$$

where g is associated to the inclusion $\mathbb{C}[xz - y^2, z] \hookrightarrow \mathbb{C}[x, y, z]$ and $h = \Phi_L$ is determined by the pencil $L = \{ C_\lambda \,|\, \lambda \in \mathbb{P}^1 \}$, which is free of base points, on $\mathbb{A}^2$.

As a corollary of Theorem 2.5, we can classify, in some sense, morphisms $f : \mathbb{A}^3 \to \mathbb{A}^1$ from the affine 3-space $\mathbb{A}^3$ fibered by irrational affine ruled surfaces in terms of locally nilpotent derivation on $\mathbb{C}[x, y, z]$. In order to state the result, we prepare:

Definition 2.7. Let (δ, h) be a pair composed of a locally nilpotent derivation on $\mathbb{C}[x, y, z]$ and a polynomial $h \in \mathrm{Ker}(\delta)$.[4] The pair (δ, h) is called *irrational* if general fibers of the polynomial map defined by h:

$$h : \mathrm{Spec}\,(\mathrm{Ker}(\delta)) \cong \mathbb{A}^2 \longrightarrow \mathbb{A}^1 \cong \mathrm{Spec}(\mathbb{C}[h])$$

are irrational. Two irrational pairs (δ_1, h_1) and (δ_2, h_2) are said to be *equivalent*, denoted by $(\delta_1, h_1) \sim (\delta_2, h_2)$, if $\mathrm{Ker}(\delta_1) = \mathrm{Ker}(\delta_2)$ and the polynomial maps:

$$h_i : \mathrm{Spec}\,(\mathrm{Ker}(\delta_i)) \cong \mathbb{A}^2 \longrightarrow \mathbb{A}^1 \cong \mathrm{Spec}(\mathbb{C}[h_i]) \quad (i = 1, 2)$$

coincide to each other up to automorphisms of the base, namely, there exists an automorphism $\alpha \in \mathrm{Aut}(\mathbb{A}^1)$ such that $h_2 = \alpha \circ h_1$.

With the notion of equivalence as above, we deduce the following result from Theorem 2.5:

Corollary 2.8. *There is a one to one correspondence between the set of morphisms $f : \mathbb{A}^3 \to \mathbb{A}^1$ whose general fibers are irrational and affine ruled and the set of all irrational pairs modulo equivalence (see Definition 2.7).*

Proof. Let $f : \mathbb{A}^3 \to \mathbb{A}^1$ be a morphism such that a general fiber of f is irrational and affine ruled. Then by virtue of Theorem 2.5, this morphism f can be factorized in such a way that:

$$f = h \circ g : \mathbb{A}^3 = \mathrm{Spec}(\mathbb{C}[x, y, z]) \xrightarrow{g} \mathbb{A}^2 \xrightarrow{h} \mathbb{A}^1,$$

where g is a quotient map with respect to an effective $\mathbb{G}_a$-action on $\mathbb{A}^3$, and letting δ be a locally nilpotent derivation corresponding to this action, g is obtained associated to the inclusion $\mathrm{Ker}(\delta) \subseteq \mathbb{C}[x, y, z]$. This implies that the polynomial $f(x, y, z) \in \mathbb{C}[x, y, z]$ in question is contained in $\mathrm{Ker}(\delta)$. For a general point $\alpha \in \mathbb{A}^1$ of the base curve, we have $f^*(\alpha) = g^*(h^*(\alpha))$. As $f^*(\alpha)$ is irrational, it follows that $h^*(\alpha)$ is an irrational curve since an $\mathbb{A}^1$-fibration does contain an $\mathbb{A}^1$-cylinder. Thus we deduce that (δ, f) is an irrational pair. The converse direction is also easy to construct, hence we shall omit the detail. □

[4] It is known that $\mathrm{Ker}(\delta)$ is in fact a polynomial ring in two variables by virtue of [Mi84b].

As an immediate corollary of Theorem 2.5, we see the following result:

Corollary 2.9. *There does not exist any polynomial $f(x, y, z) \in \mathbb{C}[x, y, z]$ which defines a polynomial morphism $f : \mathbb{A}^3 \to \mathbb{A}^1$ such that a general fiber of f is irrational and equipped with an $\mathbb{A}^1$-fibration of complete type only.*

Remark 2.10. Corollary 2.9 is remarkable in consideration of the fact that there do exist polynomials $f(x, y, z) \in \mathbb{C}[x, y, z]$ defining morphisms $f : \mathbb{A}^3 \to \mathbb{A}^1 = \mathrm{Spec}(\mathbb{C}[f])$ whose general fibers are *rational* and are equipped with $\mathbb{A}^1$-fibrations of complete type only (see Example 1.4, for instance).

3. Sketch of the proof of Theorem 2.1

3.1. We shall say about a very rough sketch of the proof of Theorem 2.1. Let $f : X \to B$ be a morphism from a normal affine algebraic threefold X onto a smooth algebraic curve B such that a general fiber of f is irrational and affine ruled.

3.2. We embed X into a normal projective threefold V with at worst $\mathbb{Q}$-factorial, terminal singularities and we denote by Δ the boundary divisor. Further, let us denote by $\pi : V \dashrightarrow C$ a rational map induced by $f : X \to B$, where C is a smooth projective curve containing B as an open subset. If π is not a morphism, then the indeterminacy of π is located in Δ, hence by performing a succession of blowing-ups whose centers are outside X, we may assume that π is a morphism from the beginning. Then we apply an usual minimal model program (mmp) for V (not for (V, Δ)) with respect to the morphism $\pi : V \to C$ to obtain a diagram:

$$(*) \quad V = V_0 \overset{\varphi_1}{\dashrightarrow} V_1 \overset{\varphi_2}{\dashrightarrow} V_2 \dashrightarrow \cdots \dashrightarrow V_{t-1} \overset{\varphi_t}{\dashrightarrow} V_t = V',$$

where $\varphi_i : V_i \dashrightarrow V_{i+1}$ is a birational map, i.e., it is either a divisorial contraction or a flip, and the right terminal V' is an output of an (mmp). Since the process $(*)$ is done with respect to $\pi : V \to C$, each V_i possesses a morphism $\pi_i : V_i \to C$ induced from π. For the simplicity of the notation, we use $\pi' : V' \to C$ instead of π_t.

3.3. We can check that π' is factored in such a way that $\pi' = \eta' \circ \xi'$, where $\xi' : V' \to W'$ is a Mori conic bundle structure (MCB) associated to an extremal ray in $\overline{\mathrm{NE}}(V'/C)$. In this step, we make use of the assumption that a general fiber of f is irrational.

3.4. Usually, an application of birational geometry for studies of affine algebraic varieties has several crucial obstacles. By experience, the most crucial one consists in how to understand the change of the inside affine parts via the process of (mmp). In our present situation neither, we are not able to describe in an explicit manner how the initial affine algebraic threefold $X = V \backslash \Delta$ varies via $(*)$. More precisely to say, letting Δ' be the proper transform on V' of Δ, we do not know how to look into a difference between X and $X' := V' \backslash \Delta'$. (The same difficulties occur even if we perform $(K_V + \Delta)$-mmp in a relative setting instead.) For instance, we do not know whether or not X' is still affine once a flip emerges in the process of $(*)$. However, for our purpose, we have only to observe how a general fiber of $f : X \to B$ varies via $(*)$ taking an influence of (MCB) $\xi' : V' \to W'$ into account.

3.5. For a point $b \in B$, let us denote by $S_b := f^*(b)$ the fiber over b, meanwhile, we put $S'_b := \pi'^*(b) \backslash (\Delta' \cap \pi'^*(b))$. Notice that if a fiber S_b is contracted in $(*)$, then $S'_b = \emptyset$. Anyway, if $b \in B$ is general, then S_b has no effect from flips appearing in $(*)$. Moreover, we may assume that S_b is disjoint from all exceptional divisors emerging in $(*)$ whose proper transforms on V are mapped onto points with respect to π by choosing $b \in B$ generally. Whereas, the most troublesome matter for our purpose lies in an effect arising from horizontal exceptional divisors, i.e., exceptional divisors appearing in the process $(*)$ such that their proper transforms on V dominates C via π. Namely, such exceptional divisors may violate the structure of an $\mathbb{A}^1$-fibration found on a general fiber S_b. But, we can guarantee that effects coming from these exceptional divisors on S_b with $b \in B$ general is to delete several fiber components of an $\mathbb{A}^1$-fibration by noticing that S_b is irrational, which means that S'_b possesses still an $\mathbb{A}^1$-fibration.

3.6. Letting E the union of proper transforms on V of exceptional divisors appearing in $(*)$, and let us denote by F the union of fibers of π, which have effects arising from flips in $(*)$. Then $U' := X\backslash(X\cap(E\cup F))$ is an open dense subset of V and V' simultaneously. Let us investigate the restriction $\xi'|_{U'}$ of (MCB) ξ' onto U'. In fact, the observation in 3.5 above combined with the assumption that general fibers of f are irrational implies that $\xi'|_{U'}$ yields an $\mathbb{A}^1$-fibration. Then [KaMi78] ascertains that there exists an open affine subset $V \subseteq W'$ such that the inverse image $U := (\xi'|_{U'})^{-1}(V)$ is isomorphic to the fiber product with the affine line $\mathbb{A}^1$, that is, $U \cong V \times \mathbb{A}^1$, which is what we want to show.

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