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Multiplicity and invariants in birational geometry

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Definition 1
Let $X$ be a variety and let $x$ be a point of $X$. The multiplicity of $x$ on $X$, denoted $\text{mult}_x X$, means $\text{mult}_x \mathcal{O}_{X,x}$. For a normal variety $X$, a prime divisor $E$ over $X$ means that a prime divisor $E$ appears on a resolution $f: Y \to X$. Let $E$ be a prime divisor over $X$ and appear on a log resolution $f: Y \to X$. The log discrepancy of $X$ with respect to $E$ is

$$a_E(X):=\text{ord}_x(K_Y-f^*K_X)+1.$$ 

For a closed subset $Z$ of $X$, the minimal log discrepancy $\text{mld}_Z(X)$ over $Z$ is the infimum of $a_E(X)$ for all prime divisors $E$ over $X$ with center in $Z$.

Let $I \subset \mathcal{O}_X$ be a nonzero ideal. The log-canonical threshold of $I$ in $X$ is defined as follows:

$$\text{lct}(I)=\sup\{c \in \mathbb{Q} | \text{mult}_x \mathcal{O}_X(x) \leq c \}.$$ 

Conjecture 1
Let $X$ be an $n$-dimensional locally a complete intersection variety. Then $\text{mult}_x X \leq 2^n - \text{mld}_x X$ for a closed point $x$ of $X$ and equality holds if and only if $\text{mult}(x)=2^n - \text{lct}(m_x)$.

Conjecture 2
Let $X$ be an $n$-dimensional locally a complete intersection variety with log canonical singularities. Then $\text{mult}_x X \leq 2^{n-1} - \text{lct}(m_x)$ for a closed point $x$ of $X$ and equality holds if and only if $\text{mult}(x)=2^n - \text{lct}(m_x)$.

Watanabe’s Conjecture
Let $X$ be an $n$-dimensional locally a complete intersection variety with canonical singularities. Then $\text{mult}_x X \leq 2^{n-1}$ for a closed point $x$ of $X$ and equality holds if and only if $\text{mult}(x)=2^n - \text{lct}(m_x)$.

Remark

- $\text{lct}(m_x) \leq \text{mld}_x X$
- Conjecture 1 $\Rightarrow$ Conjecture 2 $\Rightarrow$ Watanabe’s Conjecture
- Watanabe’s Conjecture is a generalization of Proposition 3.1 in [W].

Proposition 1
Let $X$ be an $n$-dimensional Gorenstein variety with canonical singularities and let $x$ in $X$ be a closed point. Let $\text{mult}(x)=c$ and $\text{mult}_x \mathcal{O}_X(x)=v$. Then

- If $n+1-c=2r+1$, then $\text{mult}_x X \leq (\frac{v-r-1}{r})^r$.
- If $n+1-c=2r$, then $\text{mult}_x X \leq 2^{n-r-1}$.

Remark

This Proposition 1 is a generalization of Theorem 3.1 in [HW] of characteristic zero. But our proof is just an imitation of the proof of Theorem 3.1 in [HW].

Proposition 2
Let $X$ be an $n$-dimensional normal locally complete intersection variety with canonical singularities. If $n \leq 5$, $\text{mult}_x X \leq 2^n - \text{mld}_x X$ for a closed point $x$ of $X$. If $n \geq 4$, $\text{mult}_x X \leq 2^n - \text{mld}_x X$ for a closed point $x$ of $X$ and equality holds if and only if $\text{mult}(x)=2n - \text{mld}_x X$.

Theorem 1
Let $X$ be an $n$-dimensional locally a complete intersection variety with canonical singularities. Let $\text{lct}(m_x) = c$. Let $e = \text{mult}(x) - n$. Then

- If $n - c + 1 = 2r$, then $\text{mult}_x X \leq \sum_{j=0}^c (\frac{c^{2e+n} - (c+2e-n)}{r-1})$.
- If $n - c + 1 = 2r + 1$, then $\text{mult}_x X \leq \sum_{j=0}^c (\frac{c^{2e+n} - (c+2e-n)}{r-1}) + (\frac{n-c-e+r}{r-j})$.

Key idea

By the formula (13) in [IR], we obtain $\text{mld}_x X = \inf \{ (m+1)n - \text{dim}_{\mathbb{C}}(x) \}$, where $\mu_{m-x}$ is a truncation map. By this equality, we get a regular sequence consisting the initial terms of some defining polynomials of $X$. Using this fact and Proposition 1, I proved Proposition 2 and Theorem 1.

Corollary 1
Let $X$ be an $n$-dimensional locally a complete intersection variety with canonical singularities. If $n \leq 32$, then $\text{mult}_x X \leq 2^n - \text{lct}(m_x)$ for a closed point $x$ of $X$ and equality holds if and only if $\text{mult}(x)=2n - \text{lct}(m_x)$.

Corollary 2
Watanabe’s Conjecture is true if $n \leq 32$.

Theorem 2
Let $X$ be an $n$-dimensional locally a complete intersection simplicial toric variety. Then $\text{mld}_x X = \text{lct}(m_x)$ for a fixed closed point $x$ of $X$.

Theorem 3
Let $X$ be an $n$-dimensional locally a complete intersection simplicial toric variety. Then $\text{mult}_x X \leq 2^{n-\text{mld}_x X}$ for a fixed closed point $x$ of $X$ and equality holds if and only if $\text{mult}(x)=2n - \text{mld}_x X$.

Key idea

Nakajima classified cones corresponding to locally a complete intersection toric varieties in [N]. Using this classification, I proved Theorem 2 and Theorem 3.

Reference