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# Multiplicity and invariants in birational geometry

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Definition 1

Let $X$ be a variety and let $x$ be a point of $X$. The multiplicity of $x$ on $X$, denoted $\mathrm{mult}_x X$ , means $\mathrm{mult} O_{X,x}$ .

For a normal variety $X$, a prime divisor $E$ over $X$ means that a prime divisor $E$ appears on a resolution $f: Y \to X$. Let $E$ be a prime divisor over $X$ and appear on a log resolution $f: Y \to X$ of $X$. The log discrepancy of $X$ with respect to E is

$$a_E(X) := \mathrm{ord}_E(K_Y - f^*K_X) + 1.$$

For a closed subset $Z$ of $X$, the minimal log discrepancy $\mathrm{mld}_Z(X)$ over $Z$ is the infimum of $a_E(X)$ for all prime divisor $E$ over $X$ with center in $Z$.

Let $I \subset O_X$ be a nonzero ideal. The log-canonical threshold of $I$ in $X$ is defined as follows:

$$\mathrm{lct}(I) = \sup\{ c \in \mathbb{Q} \mid (X, I^c) \text{ is log canonical}\}.$$

**Conjecture 1**
Let $X$ be an $n$ -dimensional locally a complete intersection variety. Then $\mathrm{mult}_x X \leqq 2^{n-\mathrm{mld}_x X}$ for a closed point $x$ of $X$ and equality holds if and only if emb$(X,x) = 2n - \mathrm{mld}_x X$.

**Conjecture 2**
Let $X$ be an $n$ -dimensional locally a complete intersection variety with log canonical singularities. Then $\mathrm{mult}_x X \leqq 2^{n-\ulcorner \mathrm{lct}(m_x) \urcorner}$ for a closed point $x$ of $X$ and equality holds if and only if emb$(X,x) = 2n - \ulcorner \mathrm{lct}(m_x) \urcorner$.

**Watanabe's Conjecture**
Let $X$ be an $n$ -dimensional locally a complete intersection variety with canonical singularities. Then $\mathrm{mult}_x X \leqq 2^{n-1}$ for a closed point $x$ of $X$ and equality holds if and only if emb$(X,x) = 2n - 1$.

Remark
- $\ulcorner \mathrm{lct}(m_x) \urcorner \leqq \mathrm{mld}_x X$
- Conjecture 1⇒Conjecture 2⇒Watanabe's Conjecture
- Watanabe's Conjecture is a generalization of Proposition 3.1 in [W].

Proposition 1

Let $X$ be an $n$ -dimensional Gorenstein variety with canonical singularities and let $x$ in $X$ be a closed point. Let emb$(X,x) = v$ and $\ulcorner \mathrm{lct}(m_x) \urcorner = c$.

(1) If $n + 1 - c = 2r + 1$, then

$$\mathrm{mult}_x X \leqq \binom{v-n+r}{r} + \binom{v-n+r-1}{r-1}.$$

(2) If $n + 1 - c = 2r$, then

$$\mathrm{mult}_x X \leqq 2\binom{v-n+r-1}{r-1}.$$

Remark
This Proposition 1 is a generalization of Theorem 3.1 in [HW] of characteristic zero. But our proof is just an imitation of the proof of Theorem 3.1 in [HW].

Proposition 2
Let $X$ be an $n$ -dimensional normal locally complete intersection variety with canonical singularities.
If $n \leqq 5$, $\mathrm{mult}_x X \leqq 2^{n-\mathrm{mld}_x X}$ for a closed point $x$ of $X$.
If $n \leqq 4$, $\mathrm{mult}_x X \leqq 2^{n-\mathrm{mld}_x X}$ for a closed point $x$ of $X$ and equality holds if and only if emb$(X,x) = 2n - \mathrm{mld}_x X$.

**Theorem 1**
Let $X$ be an $n$ -dimensional locally a complete intersection variety with canonical singularities. Let $\ulcorner \mathrm{lct}(m_x) \urcorner = c$. Let $e = \mathrm{emb}(X,x) - n$. Let $c + 2e - n > 0$.
(1) If $n - c + 1 = 2r$, then

$$\mathrm{mult}_x X \leqq 2\sum_{j=0}^{c+2e-n} \binom{c+2e-n}{j} \binom{n-c-e+r-j-1}{r-j-1}.$$

(2) If $n - c + 1 = 2r + 1$, then

$$\mathrm{mult}_x X \leqq \sum_{j=0}^{c+2e-n} \binom{c+2e-n}{j} \left\{ \binom{n-c-e+r-j-1}{r-j-1} + \binom{n-c-e+r-j}{r-j} \right\}.$$

Key idea
By the formula (13) in [IR], we obtain $\mathrm{mld}_x X = \inf_m \{(m+1)n - \dim \pi_m^{-1}(x)\}$, where $\pi_m : X_m \to X$ is a truncation map . By this equality, we get a regular sequence consisting the initial terms of some defining polynomials of $X$. Using this fact and Proposition 1, I proved Proposition 2 and Theorem 1.

**Corollary 1**
Let $X$ be an $n$ -dimensional locally a complete intersection variety with canonical singularities. If $n \leqq 32$, then $\mathrm{mult}_x X \leqq 2^{n-\ulcorner \mathrm{lct}(m_x) \urcorner}$ for a closed point $x$ of $X$ and equality holds if and only if emb$(X,x) = 2n - \ulcorner \mathrm{lct}(m_x) \urcorner$.

**Corollary 2**
Watanabe's Conjecture is true if $n \leqq 32$.

**Theorem 2**
Let $X$ be an $n$ -dimensional locally a complete intersection simplicial toric variety. Then $\mathrm{mld}_x X = \ulcorner \mathrm{lct}(m_x) \urcorner$ for a fixed closed point $x$ of $X$.

**Theorem 3**
Let $X$ be an $n$ -dimensional locally a complete intersection simplicial toric variety. Then $\mathrm{mult}_x X \leqq 2^{n-\mathrm{mld}_x X}$ for a fixed closed point $x$ of $X$ and equality holds if and only if emb$(X,x) = 2n - \mathrm{mld}_x X$.

Key idea
Nakajima classified cones corresponding to locally a complete intersection toric varieties in [N]. Using this classification, I proved Theorem 2 and Theorem 3.

Reference
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