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<th>Multiplicity and invariants in birational geometry</th>
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Definition 1
Let $X$ be a variety and let $x$ be a point of $X$. The multiplicity of $x$ on $X$, denoted $\text{mult}_x X$, means $\text{mult}_x O_X$.

For a normal variety $X$, a prime divisor $E$ over $X$ means that a prime divisor $E$ appears on a resolution $f: Y \to X$. Let $E$ be a prime divisor over $X$ and appear on a log resolution $f: Y \to X$. The log discrepancy of $X$ with respect to $E$ is

$$\alpha_E(x) := \text{ord}_E(K_Y - f^*K_X) + 1.$$ 

For a closed subset $Z$ of $X$, the minimal log discrepancy $\text{mld}_E(Z)$ over $Z$ is the infimum of $\alpha_E(x)$ for all prime divisors $E$ over $X$ with center in $Z$.

Let $f \subseteq O_X$ be a nonzero ideal. The log-canonical threshold of $f$ in $X$ is defined as follows:

$$\text{lct}(f) = \sup \{ c \in \mathbb{Q} | (X^f)^c \text{ is log canonical} \}.$$ 

Conjecture 1
Let $X$ be an $n$-dimensional locally a complete intersection variety. Then $\text{mult}_x X \leq 2^{n-\text{mld}_x X}$ for a closed point $x$ of $X$ and equality holds if and only if $\text{emb}(X,x) = 2n - \text{mld}_x X$.

Conjecture 2
Let $X$ be an $n$-dimensional locally a complete intersection variety with log canonical singularities. Then $\text{mult}_x X \leq 2^{n-\text{lct}(m_x)}$ for a closed point $x$ of $X$ and equality holds if and only if $\text{emb}(X,x) = 2n - \text{lct}(m_x)$.

Watanabe’s Conjecture
Let $X$ be an $n$-dimensionally locally a complete intersection variety with canonical singularities. Then $\text{mult}_x X \leq 2^{n-1}$ for a closed point $x$ of $X$ and equality holds if and only if $\text{emb}(X,x) = 2n - 1$.

Remark
- $\text{lct}(m_x) \leq \text{mld}_x X$
- Conjecture 1 $\Rightarrow$ Conjecture 2 =⇒ Watanabe’s Conjecture
- Watanabe’s Conjecture is a generalization of Proposition 3.1 in [W].

Proposition 1
Let $X$ be an $n$-dimensional Gorenstein variety with canonical singularities and let $x$ in $X$ be a closed point. Let $\text{emb}(X,x) = v$ and $\text{lct}(m_x) = c$.

1. If $n + 1 - c = 2r + 1$, then $\text{mult}_x X \leq 2^{n-r+1}$. 
2. If $n + 1 - c = 2r$, then $\text{mult}_x X \leq 2^{n-r+1}$.

Remark
This Proposition 1 is a generalization of Theorem 3.1 in [HW] of characteristic zero. But our proof is just an imitation of the proof of Theorem 3.1 in [HW].

Proposition 2
Let $X$ be an $n$-dimensional normal locally complete intersection variety with canonical singularities. If $n \leq 5$, $\text{mult}_x X \leq 2^{n-\text{mld}_x X}$ for a closed point $x$ of $X$. If $n \leq 4$, $\text{mult}_x X \leq 2^{n-\text{mld}_x X}$ for a closed point $x$ of $X$ and equality holds if and only if $\text{emb}(X,x) = 2n - \text{mld}_x X$.

Theorem 1
Let $X$ be an $n$-dimensional locally a complete intersection variety with canonical singularities. Let $\text{lct}(m_x) = c$. Let $e = \text{emb}(X,x) - n$. Let $n + 2e - n > 0$.

1. If $n + 1 - c = 2r$, then $\text{mult}_x X \leq \sum_{j=0}^{c+2e-n}(c+2e-n)_j \left( \binom{n-c-e+r-j-1}{r-j} \right)$.
2. If $n + 1 - c = 2r + 1$, then $\text{mult}_x X \leq \sum_{j=0}^{c+2e-n}(c+2e-n)_j \left( \binom{n-c-e+r-j}{r-j} \right)$.

Key idea
By the formula (13) in [IR], we obtain $\text{mld}_x X = \inf \{ (n+1)n - \dim_{\mathbb{Q}}(\mathbb{Q}(x)) \}$, where $\pi_{m_x}: X \to X$ is a truncation map. By this equality, we get a regular sequence consisting the initial terms of some defining polynomials of $X$. Using this fact and Proposition 1, I proved Proposition 2 and Theorem 1.

Corollary 2
Watanabe’s Conjecture is true if $n \leq 32$.

Theorem 2
Let $X$ be an $n$-dimensionally locally a complete intersection simplicial toric variety. Then $\text{mld}_x X = \text{lct}(m_x) - 1$ for a fixed closed point $x$ of $X$.

Key idea
Nakajima classified cones corresponding to locally a complete intersection toric varieties in [N]. Using this classification, I proved Theorem 2 and Theorem 3.

Reference