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Induced automorphisms on Hyperkähler manifolds

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On Hyperkähler manifolds

Hyperkähler manifolds are simply connected complex analytic varieties endowed with a symplectic holomorphic 2-form generating $H^{2,0}$ or, equivalently, are Riemannian manifolds of real dimension $4m$ such that their holonomy group is $Sp(m)$.
The simplest example of such manifolds are K3 surfaces and it turns out that higher dimensional examples behave very much like K3's: the second integral cohomology forms a lattice which determines much of the variety itself due to the weaker global Torelli theorem of M. Verbitsky (see [3]).
There are few examples of such varieties, mainly 2 families in each even complex dimension due to Beauville [1] and two isolated examples in dimension 6 and 10 due to O'Grady. We are interested with the two families of Beauville, which consist of the deformations of Hilbert schemes of points of a K3 surface (in the following we will call them varieties of $K3^{[n]}$-type) and the Albanese fibre of the Hilbert scheme of points of an Abelian surface (varieties of Kummer type).
Another important set of examples (deformation equivalent to the above) are given by moduli spaces of stable sheaves on $K3$ or abelian surfaces.

Induced automorphisms

If we take a $K3$ (or abelian) surface with an automorphism φ, we can consider the induced action of φ on coherent sheaves. When the automorphism preserves also stable sheaves with given chern classes, it induces an automorphism on the moduli space of such sheaves. We call such automorphisms induced.
Together with Malte Wandel, we answered the following questions:

- When is a manifold of $K3^{[n]}$-type a moduli space of sheaves on a $K3$ surface?
- When is a manifold of Kummer type the albanese fibre of a moduli space of sheaves on an abelian surface?
- Which of its automorphisms are induced from the surface?

In the following we focus on the $K3$ case, as the abelian is analogous.

The tools

We use some theory of lattices and their discriminant forms (see [5]), as our result uses the action of an automorphism on the second integral cohomology.
For this purpose let us give at least a

Definition

Let $T_G(X) = H^2(X,\mathbb{Z})^G$ be the invariant lattice and let $S_G(X) = T_G(X)^\perp$ be the coinvariant lattice.
Let $T(X) = S(X)^\perp$ be the transcendental lattice.

To determine when a manifold is a moduli space of stable sheaves, we use the more general theory of stability conditions on derived categories. One can think of moduli spaces of stable objects in the derived category of a surface as a natural generalisation of moduli spaces of stable sheaves. Let S be a $K3$ surface and let $\Lambda := H^0(S,\mathbb{Z}) \oplus H^2(S,\mathbb{Z}) \oplus H^4(S,\mathbb{Z})$. There is a map $\nu\,:\, D^b(S) \to \Lambda$ with image in the $(1,1)$ part. With a suitable notion of stability, there is a smooth moduli space $M_v(S)$ of stable objects E with $\nu(E) = v \in \Lambda$ for all primitive v with square at least -2. It has dimension v^2+2. Moreover there is the following isometry:

$$H^2(M_v(S),\mathbb{Z}) \cong v^\perp. \qquad (1)$$

Which holds if $v^2 \geq 2$.
We also use Verbitsky's global Torelli theorem and Markman's determination of the fibres of the period map to recover a manifold from its second cohomology [3].

Preliminaries

There is a well defined non separated space of Hyperkähler manifolds together with a marking, i. e. an isometry between their second cohomology and a fixed lattice. The period map sends a marked Hyperkähler manifold to its Hodge structure. Let X and Y be two Hyperkähler manifolds. An isometry $f\,:\,H^2(X)\to H^2(Y)$ is of parallel transport if and only if it is obtained by parallel transport along a family of Hyperkähler manifolds containing X and Y. The following theorem of Huybrechts, Markman and Verbitsky tells us how to recover a manifold from its Hodge structure.

Theorem

[3] The period map is generically injective on each connected component of the moduli space of marked Hyperkähler manifolds. Let (X,f) and (Y,g) be two marked manifolds with the same period. Then the following are equivalent:

- *$f\circ g^{-1}$ is of parallel transport.*
- *$f\circ g^{-1}$ is the composition of a birational map and reflections on reduced irreducible divisors.*
- *(X,f) and (Y,g) lay in the same connected component of the moduli space of marked Hyperkähler manifolds.*

In the peculiar case of manifolds of $K3^{[n]}$-type , Markman also describes the group of parallel transport operators and Bayer and Macrì describe more in detail the birational geometry of moduli spaces of stable objects:

Theorem

[2] Let X be a manifold of $K3^{[n]}$-type that is birational to a moduli space of stable sheaves on a K3 surface S. Then X is a moduli space of stable objects in the derived category of S

For manifolds of Kummer type we use a similar result of Yoshioka [6].
A precise statement of our results are the following:

Theorem

[4] Let X be a manifold of $K3^{[n]}$-type and let $i\,:\,H^2(X,\mathbb{Z})\to\Lambda$ be any primitive embedding of integer Hodge structures. Then X is a moduli space of stable objects on the surface S if and only if $\Lambda^{1,1} = U \oplus NS(S)$.

Let G be a finite group of automorphisms of X and let $H^2(X)\to\Lambda$ be an embedding. We denote $T_G(\Lambda)$ the invariant part of the induced action of G on Λ.

Theorem

[4] Let X be a manifold of $K3^{[n]}$-type , $G\subset Aut(X)$ and let $i\,:\,H^2(X,\mathbb{Z})\to\Lambda$ be any primitive embedding of integer Hodge structures. Then the group G is induced from automorphisms of S if and only if $\Lambda^{1,1} = U\oplus NS(S)$, the previous copy of U is inside $T_G(\Lambda)$ and $T_G(\Lambda) = U \oplus T_G(S)$.

Sketch of the proof

First, let us remark that there exists always an embedding $H^2(X)\to\Lambda$ and let us endow Λ with the weight two Hodge structure induced from $H^2(X)$ (i. e. the $(2,0)$ part comes from $H^2(X)$). In particular, the splitting of a copy of U from the $(1,1)$ part of Λ does not depend on the choice of the embedding.
If X were a moduli space on S, one such embedding would be the "'inverse"' of (1), therefore a copy of $U = H^0(S)\oplus H^4(S)$ would split from the $(1,1)$ part of Λ.
On the other hand, if a copy of U splits, its orthogonal is the Hodge structure of a $K3$ surface S and we are able to construct a moduli space of stable objects on S having the same period of X (and laying in the same connected component of the moduli space of marked manifolds, with a suitable marking) hence X is birational to a moduli space of stable objects on S and, by [2], is itself a moduli space of objects on S.
The statement concerning automorphisms is similar, we proceed as above and construct a moduli space on S having the same period of X and with an action of G and we conclude using the injectivity of the map $Aut(X)\to O(H^2(X))$.

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