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Minimal stratification for line arrangements and Milnor fibers

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1 Introduction

One of the most basic tool for the study of topology of spaces is the cell decomposition. For example, the complex projective space has the following famous decomposition:

$$\mathbb{CP}^n = \mathbb{C}^n \sqcup \mathbb{C}^{n-1} \sqcup \cdots \sqcup \mathbb{C} \sqcup \{pt\}.$$

On the other hand, the case of affine varieties is more complicated. For example, $\mathbb{C}^\times = \mathbb{C} \setminus \{0\}$ is known to be homotopically equivalent to the circle

$$\mathbb{C}^\times \approx S^1.$$

The descent of the dimension may be notable. In general, Lefschetz proved the following.

Theorem 1.1. (Lefschetz, [10]) Let M be a complex smooth affine variety of $\dim_{\mathbb{C}} M = n$. Then M is homotopic to a finite CW-complex X of $\dim_{\mathbb{R}} X \leq n$.

(Sketch of the proof.) Assume that $M \subset \mathbb{C}^N$ is a closed subvariety. Choose a point $p \in \mathbb{C}^N \setminus M$ generically. Then the distance function

$$M \longrightarrow \mathbb{R}_{\leq 0},\ x \longmapsto ||x - p||$$

becomes a Morse function. By the Cauchy-Riemann equation, the Morse index of each critical point is $\leq n$. Using the gradient flow, one can construct

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a homotopy equivalence between M and a finite CW-complex of $\dim_{\mathbb{R}} \leq n$. (Q. E. D.)

Morse theoretic construction of the cell decomposition is a sort of "proof of existence". Except for the very special cases, e.g., bouquet of spheres, it is difficult to deduce the information of the attaching maps of the cells in general. Next one is an example of Morse function on the two-punctured plane.

Example 1.2. Let $M = \mathbb{C} \setminus \{0,1\}$ and $\varphi(z) := \frac{(z+1)^2}{\sqrt{z(z-1)}}$. We consider $|\varphi| : M \to \mathbb{R}$ as a Morse function which has three critical points $z = -1, \frac{5-\sqrt{17}}{4}, \frac{5+\sqrt{17}}{4}$ with index $0, 1, 1$ respectively. Note that all critical points are real and $0 < \frac{5-\sqrt{17}}{4} < 1 < \frac{5+\sqrt{17}}{4}$. The unstable manifolds present a one-dimensional CW complex which is homotopic to M. Since $|\varphi(z)| \to \infty$ as $|z| \to \infty$, the unstable cells are as in Figure 1. It is not easy to describe the unstable manifolds explicitly even for one-dimensional cases. Nevertheless, the stable manifolds can be explicitly described: two open segments $(0,1), (1,\infty)$ and the remainder $U = M \setminus ((0,1) \cup (1,\infty))$.

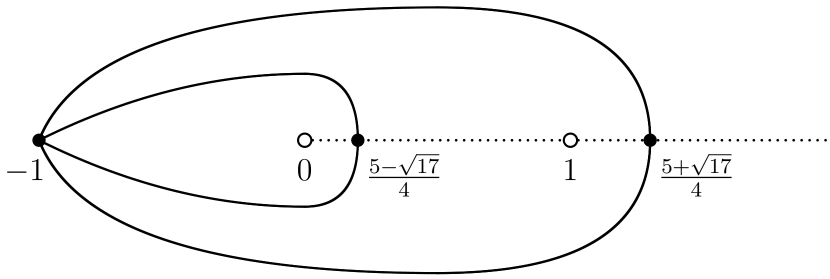

Figure 1: Unstable and stable manifolds (thick and dotted line, respectively).

We have a partition $U \sqcup (0,1) \sqcup (1,\infty)$ of M by contractible pieces, and note that the number of codimension zero piece is equal to $b_0(M) = 1$ and that of codimension one is $b_1(M) = 2$. Also note that codimension one pieces $(0,1)$ and $(1,\infty)$ are nothing but chambers of the real hyperplane arrangement $\{0,1\}$. These pieces are expressed in terms of defining linear forms as follows,

$$\begin{aligned} (0,1) &= \left\{ z \in M \,\middle|\, \frac{z-1}{z} \in \mathbb{R}_{<0} \right\}, \\ (1,\infty) &= \left\{ z \in M \,\middle|\, \frac{-1}{z-1} \in \mathbb{R}_{<0} \right\}, \end{aligned} \tag{1}$$

where $\mathbb{R}_{<0}$ is the set of negative real numbers.

The homotopy types of the unstable cells for higher dimensional cases are discussed in [14]. The unstable cell itself is highly transcendental. On the other hand, the stable manifolds are semi-algebraic sets (open segments and their complements). The purpose of this paper is to explain that similar description works for the complement of line arrangements (defined over $\mathbb{R}$) in $\mathbb{C}^2$. Then we will apply our semi-algebraic stratification to the computation of the monodromy of Milnor fibers.

2 Minimal stratification

A real arrangement $\mathcal{A} = \{H_1, \dots, H_n\}$ is a finite set of affine lines in the affine plane $\mathbb{R}^2$. Each line is defined by some affine linear form

$$\alpha_H(x_1, x_2) = ax_1 + bx_2 + c = 0, \tag{2}$$

with $a, b, c \in \mathbb{R}$ and $(a, b) \neq (0, 0)$. A connected component of $\mathbb{R}^2 \setminus \bigcup_{H \in \mathcal{A}} H$ is called a chamber. The set of all chambers is denoted by $\mathsf{ch}(\mathcal{A})$. The affine linear equation (2) defines a complex line $\{(z_1, z_2) \in \mathbb{C}^2 \mid az_1 + bz_2 + c = 0\}$ in $\mathbb{C}^2$. We denote the set of complexified lines by $\mathcal{A}_{\mathbb{C}} = \{H_{\mathbb{C}} = H \otimes \mathbb{C} \mid H \in \mathcal{A}\}$. The object of our interest is the complexified complement $M(\mathcal{A}) = \mathbb{C}^2 \setminus \bigcup_{H \in \mathcal{A}} H_{\mathbb{C}}$.

To describe the semi-algebraic stratification, we need a good numbering of lines associated with a fixed generic line. Let $\mathcal{F}$ be an oriented generic line in $\mathbb{R}^2$. We assume that the numbering $H_1, \dots, H_n$ satisfies

$$H_1 \cap \mathcal{F} < H_2 \cap \mathcal{F} < \cdots < H_n \cap \mathcal{F},$$

and for each H the positive half-space $\{\alpha_H > 0\} \subset \mathbb{R}^2$ covers positive direction of $\mathcal{F}$.

Definition 2.1. $\mathsf{ch}^{\mathcal{F}}(\mathcal{A}) := \{C : \text{ chamber} \mid C \cap \mathcal{F} = \emptyset\}$.

Definition 2.2. Set $\alpha_{n+1} = -1$. For $i = 1, 2, \dots, n$, define $S_i \subset M(\mathcal{A})$ as follows.

$$S_i = \left\{ z = (z_1, z_2) \in M(\mathcal{A}) \,\middle|\, \frac{\alpha_{i+1}(z)}{\alpha_i(z)} \in \mathbb{R}_{<0} \right\}.$$

The following is the main result.

Theorem 2.3. *The closed submanifolds $S_1, \dots, S_n \subset M(\mathcal{A})$ satisfy the following.*

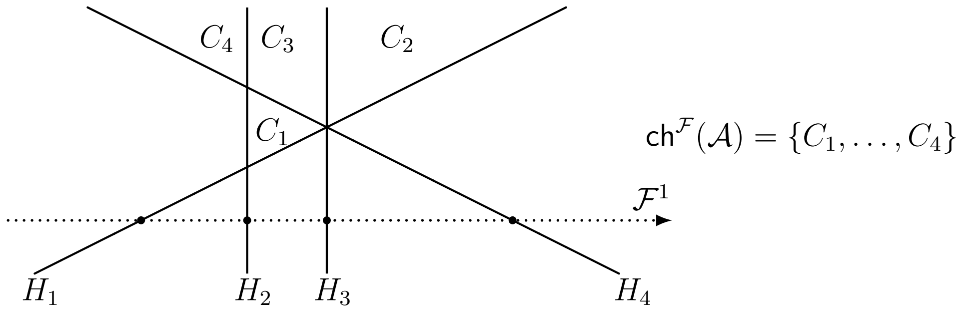

Figure 2: Numbering of lines and chambers.

(i) S_i *and* S_j *($i \neq j$) intersect transversely, and* $S_i \cap S_j = \bigsqcup C$*, where* C *runs all chambers satisfying* $\alpha_i(C)\alpha_{i-1}(C) < 0$ *and* $\alpha_j(C)\alpha_{j-1}(C) < 0$.

(ii) $S_i^\circ := S_i \setminus \bigcup_{C \in \mathsf{ch}^{\mathcal{F}}(\mathcal{A})} C$ *is a contractible* 3*-manifold.*

(iii) $U := M(\mathcal{A}) \setminus \bigcup_{i=1}^n S_i$ *is a contractible* 4*-manifold.*

For the proof, see [15]. We call the above decomposition the *minimal stratification* of $M(\mathcal{A})$.

Remark 2.4. It is conjectured that similar construction provides a good minimal stratification for higher dimensional cases.

3 Milnor fibers

Let $\mathcal{A} = \{H_1, \ldots, H_n\}$ be an affine line arrangement in $\mathbb{R}^2$ with the defining equation $Q_{\mathcal{A}}(x, y) = \prod_{i=1}^n \alpha_i$, where α_i is a defining linear equation for H_i. The coning $c\mathcal{A}$ of $\mathcal{A}$ is an arrangement of $n + 1$ planes in $\mathbb{R}^3$ defined by the equation $Q_{c\mathcal{A}}(x, y, z) = z^{n+1}Q(\frac{x}{z}, \frac{y}{z})$. The line $\{z = 0\} \in c\mathcal{A}$ is called the line at infinity and is denoted by H_∞. The complexified complement $M(\mathcal{A}) = \mathbb{C}^2 \setminus \{Q_{\mathcal{A}} = 0\}$ can be identified with $\mathbb{P}^2_{\mathbb{C}} \setminus \{Q_{c\mathcal{A}} = 0\}$. We call $p \in \mathbb{RP}^2$ a *multiple point* if the multiplicity of $c\mathcal{A}$ at p (that is, the number of lines passing through p) is greater than or equal to 3.

Definition 3.1. $F_{\mathcal{A}} = \{(x, y, z) \in \mathbb{C}^3 \mid Q_{c\mathcal{A}}(x, y, z) = 1)\}$ is called the Milnor fiber of $\mathcal{A}$. The automorphism $\rho : F_{\mathcal{A}} \longrightarrow F_{\mathcal{A}},\ (x, y, z) \longmapsto (\zeta x, \zeta y, \zeta z)$, with $\zeta = \exp(2\pi i/(n + 1))$, is called the monodromy action.

The automorphism ρ has order $n + 1$. It generates the cyclic group $\langle\rho\rangle \simeq \mathbb{Z}/(n+1)\mathbb{Z}$. The monodromy ρ induces a linear map $\rho^* : H^1(F_{\mathcal{A}}, \mathbb{C}) \longrightarrow H^1(F_{\mathcal{A}}, \mathbb{C})$. Since $(\rho^*)^{n+1}$ is the identity, we have the eigenspace decomposition $H^1(F_{\mathcal{A}}, \mathbb{C}) = \bigoplus_{\lambda^{n+1}=1} H^1(F_{\mathcal{A}}, \mathbb{C})_\lambda$, where $H^1(F_{\mathcal{A}}, \mathbb{C})_\lambda$ is the the set of

λ-eigenvectors with eigenvalue $\lambda \in \mathbb{C}^*$. When $\lambda = 1$, $H^1(F_{\mathcal{A}})_1 = H^1(F_{\mathcal{A}})^{\rho^*}$ is the subspace of elements fixed by ρ^*, which is isomorphic to $H^1(F_{\mathcal{A}}/\langle\rho\rangle)$. It is easily seen that the quotient by the monodromy action is $F_{\mathcal{A}}/\langle\rho\rangle \simeq \mathsf{M}(\mathcal{A})$. Therefore, the 1-eigenspace of the first cohomology is combinatorially determined, $H^1(F_{\mathcal{A}})_1 \simeq H^1(\mathsf{M}(\mathcal{A})) \simeq \mathbb{C}^n$. In general, let $\mathcal{L}_\lambda$ be a complex rank one local system associated with a representation

$$\pi_1(\mathsf{M}(\mathcal{A})) \longrightarrow \mathbb{C}^*, \ \gamma_H \longmapsto \lambda,$$

where γ_H is a meridian loop of the line H. Then it is known that

$$H^1(F_{\mathcal{A}})_\lambda \simeq H^1(\mathsf{M}(\mathcal{A}), \mathcal{L}_\lambda). \tag{3}$$

(See [1] for details.)

One can compute the dimension of the monodromy eigen space by using the minimal stratification.

Theorem 3.2. *We can formulate an algorithm computing* $\dim H^1(F_{\mathcal{A}})_\lambda$ *in terms of the "real picture". (See [16] for details.)*

As we mentioned, $\dim H^1(F_{\mathcal{A}})_1 = n$. Therefore the non-trivial part of the first cohomology group is

$$H^1(F_{\mathcal{A}})_{\neq 1} = \bigoplus_{\lambda \neq 1} H^1(F_{\mathcal{A}})_\lambda.$$

One of the main problem is whether the non-trivial part $H^1(F_{\mathcal{A}})_{\neq 1}$ is combinatorially determined or not. Indeed the nontrivial part is conjectured to be determined by the following combinatorial structure.

Definition 3.3. A *k-multinet* on $c\mathcal{A}$ is a pair $(\mathcal{N}, \mathcal{X})$, where $\mathcal{N}$ is a partition of $c\mathcal{A}$ into $k \geq 3$ classes $\mathcal{A}_1, \dots, \mathcal{A}_k$ and $\mathcal{X}$ is a set of multiple points such that

(i) $|\mathcal{A}_1| = \cdots = |\mathcal{A}_k|$;

(ii) $H \in \mathcal{A}_i$ and $H' \in \mathcal{A}_j$ $(i \neq j)$ imply that $H \cap H' \in \mathcal{X}$;

(iii) for all $p \in \mathcal{X}$, $|\{H \in \mathcal{A}_i \mid H \ni p\}|$ is constant and independent of i;

(iv) for any $H, H' \in \mathcal{A}_i$ $(i = 1, \dots, k)$, there is a sequence $H = H_0, H_1, \dots, H_r = H'$ in $\mathcal{A}_i$ such that $H_{j-1} \cap H_j \notin \mathcal{X}$ for $1 \leq j \leq r$.

Example 3.4. There are infinitely many 3-multinets. See [4] and [16] for examples.

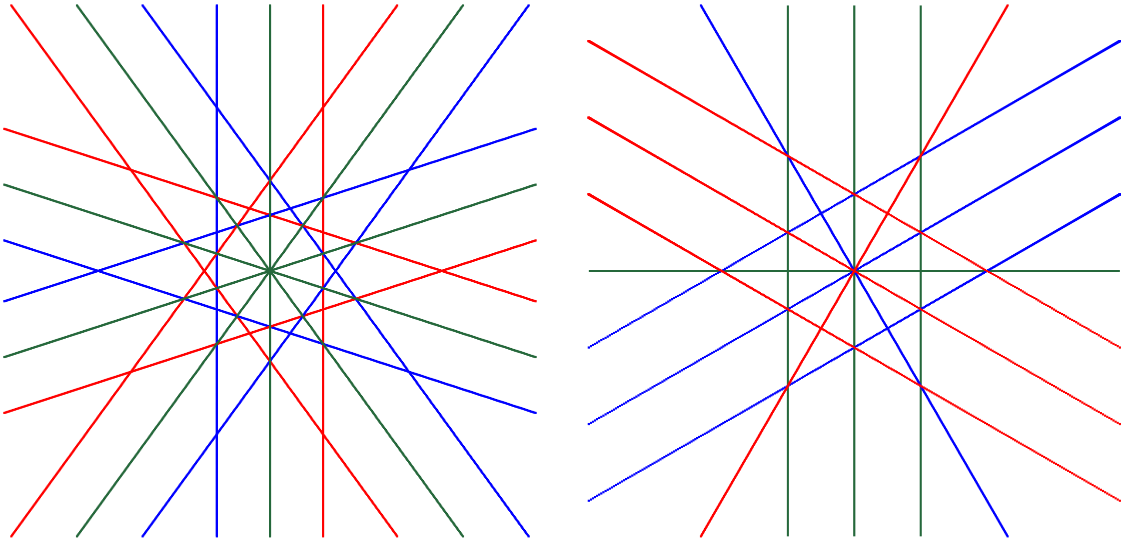

Figure 3: Examples of 3-multinets

Example 3.5. (Hessian arrangement) Let $f_\mu(z_1, z_2, z_3) = 3z_1z_2z_3 - \mu(z_1^3 + z_2^3 + z_3^3)$, with $\mu \in \mathbb{C}$. It is well-known that f_μ factors into linear forms if and only if $\mu = 0, 1, \omega, \omega^2$, where $\omega = e^{2\pi i/3}$. For $\mu = 0, 1, \omega, \omega^2$, let us define $\mathcal{A}_\mu = \{f_\mu = 0\} \subset \mathbb{P}^2$, which is a union of three lines. Then

$$\mathcal{A} = \mathcal{A}_0 \sqcup \mathcal{A}_1 \sqcup \mathcal{A}_\omega \sqcup \mathcal{A}_{\omega^2}$$

determines a 4-multinet structure. This arrangement is called the *Hessian arrangement.*

The following is a consequence of [9], [6, Theorem 3.11] and [5, Theorem 3.1 (i)]

Theorem 3.6. *Under the notation as above.*

(1) Suppose there exists a k-multinet on $c\mathcal{A}$ for some $k \geq 3$ and set $\lambda = e^{2\pi i/k}$. Then
$$\dim H^1(F_\mathcal{A})_\lambda \geq k - 2.$$

(2) If $c\mathcal{A}$ has at most triple points (i.e., each intersection is either double or triple point), then $H^1(F_\mathcal{A})_{\neq 1} \neq 0$ if and only if $c\mathcal{A}$ supports a 3-multinet structure. (And then non-trivial eigenvalue is $\lambda = e^{2\pi i/3}$.)

These results indicates the following.

Conjecture 3.7. *Suppose $\lambda \in \mathbb{C}^\times$ has order k. Then $H^1(F_\mathcal{A})_\lambda \neq 0$ if and only if $c\mathcal{A}$ supports a k-multinet structure.*

Thus, the nontrivial eigenspace of the first cohomology group of the Milnor fiber is conjecturally described combinatorially. Furthermore, Yuzvinsky [17] proved that there does not exists k-multinet for $k \geq 5$. We also note that the Hessian arrangement (Example 3.5) is the only known 4-multinet, (hence only known k-multinet with $k \neq 3$).

Another problem concerning the topology of the Milnor fiber is the torsion freeness of the homology groups. We recently proved the following ([2, 12, 13]).

Theorem 3.8. *Let us assume that $\mathcal{A}$ is defined over $\mathbb{R}$. Then,*

(1) $c\mathcal{A}$ does not admit 4-multinet structure.

(2) $H_1(F_{\mathcal{A}}, \mathbb{Z})$ does not have torsion.

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