ON GAUSS MAPS IN POSITIVE CHARACTERISTICS

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Abstract. In [FI2], K. Furukawa and the author study Gauss maps in positive characteristics. In this note, we review [FI2].

1. Introduction

Let $X \subset \mathbb{P}^N$ be an $n$-dimensional projective variety over an algebraically closed field of characteristic $p \geq 0$. Then we can define the Gauss map $\gamma_X$ of $X$ to be the rational map

$$\gamma_X : X \dashrightarrow \mathbb{G}(n, \mathbb{P}^N)$$

which maps a smooth point $x \in X$ to the embedded tangent space $T_xX \subset \mathbb{P}^N$. Here $\mathbb{G}(n, \mathbb{P}^N)$ is the Grassmannian which parametrizes $n$-dimensional linear subvarieties of $\mathbb{P}^N$.

Example 1.1. (1) If $X \subset \mathbb{P}^N$ is an $n$-dimensional linear subvariety, $T_xX = X$ holds for any $x \in X$, i.e. $\gamma_X$ is a constant map. In fact, $\gamma_X$ is a constant map for a variety $X \subset \mathbb{P}^N$ if and only if $X$ is linear.

(2) Let $X \subset \mathbb{P}^N$ be the projective cone of $Y \subset \mathbb{P}^{N-1}$ with the vertex $v \in \mathbb{P}^N$. Then $\gamma_X$ is constant on the line $\overline{vy}$ for each $y \in Y$.

(3) If $C \subset \mathbb{P}^N$ is a nonlinear curve in $p = 0$, it is known that $\gamma_C$ is birational onto the image. Geometrically, this means that a general tangent line of $C$ is tangent to $C$ at only one point.

If we take a local parametrization of $X$, we can compute $\gamma_X$ explicitly as follows.

Example 1.2. Let $X := \mathbb{P}^1 \hookrightarrow \mathbb{P}^N$ be the twisted cubic curve, which is locally parametrized as $[1 : t] \mapsto [1 : t : t^2 : t^3] \in \mathbb{P}^3$. For $x = [1 : t]$, the embedded tangent space $T_xX \subset \mathbb{P}^3$ is the line spanned by $[1 : t : t^2 : t^3]$ and $[0 : 1 : 2t : 3t^2]$, where the latter point is obtained by differentiating $1, t, t^2, t^3$ by $t$. Then we have

$$\begin{bmatrix} 1 & t & t^2 & t^3 \\ 0 & 1 & 2t & 3t^2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -t^2 & -2t^3 \\ 0 & 1 & 2t & 3t^2 \end{bmatrix},$$

where $\sim$ means that the farmer matrix is transformed to the latter one by elementary transformations of rows. Hence the Gauss map $\gamma_X : X \rightarrow \mathbb{G}(3, \mathbb{P}^3)$.
\[ \mathbb{G}(1, \mathbb{P}^3) \] is locally described as
\[ [1 : t] \mapsto \begin{bmatrix} -t^2 & -2t^3 \\ 2t & 3t^2 \end{bmatrix} \in \mathbb{A}^4 \subset \mathbb{G}(1, \mathbb{P}^3). \]

By this description, \( \gamma_X \) is an embedding if \( p \neq 2 \), but is not birational if \( p = 2 \).

**Example 1.3.** Assume \( p > 0 \) and let \( c \) be a positive integer prime to \( p \). Let \( X := \mathbb{P}^1 \to \mathbb{P}^3 \), which is parametrized as \([1 : t] \mapsto [1 : t : t^{cp} : t^{cp+1}]\). Then \( \gamma_X : X \to \mathbb{G}(1, \mathbb{P}^3) \) is locally described as
\[ [1 : t] \mapsto \begin{bmatrix} t^{cp} \\ 0 \\ 0 \end{bmatrix} \in \mathbb{A}^4 \subset \mathbb{G}(1, \mathbb{P}^3). \]

Thus a general fiber of \( \gamma_X \) is (non-reduced) \( c \) points. Geometrically, this means that a general tangent line of \( X \) is tangent to \( X \) at \( c \) points.

**Remark 1.4.** In [FI1], Furukawa and the author study Gauss maps of toric varieties.

About general fibers of Gauss maps, the following are known: If \( p = 0 \), (the closure of) a general fiber of \( \gamma_X \) is a linear subvariety for any \( X \) (see P. Griffiths and J. Harris [GH, (2.10)], F.L. Zak [Za, I, 2.3. Theorem (c)]). We will omit the words “the closure of” sometimes for simplicity. In particular, \( X \) is birational to \( \gamma_X(X) \times \mathbb{A}^\delta \) for \( \delta = \dim X - \dim \gamma_X(X) \), and the field extension \( K(X)/K(\gamma_X(X)) \) between function fields is purely transcendental. In \( p > 0 \), Furukawa [Fur] show that the same statement holds if \( \gamma_X \) is separable.

On the other hand, a general fiber of Gauss maps can be a union of points as we see in Example 1.3 (such examples were first found by H. Kaji [Ka1, Example 4.1] and J. Rathmann [Ra, Example 2.13]), and can be a nonlinear variety (S. Fukasawa [Fuk1, Section 7]) in \( p > 0 \). More generally, Fukasawa show that any projective variety appears as a general fiber of \( \gamma_X \) for some \( X \):

**Theorem 1.5** ([Fuk2]). Assume \( p > 0 \). Let \( F \subset \mathbb{P}^{N'} \) be a projective variety. If \( n \geq N' \), there exists an \( n \)-dimensional projective variety \( X \subset \mathbb{P}^N \) such that a general fiber of \( \gamma_X \) (with the reduced structure) is projectively equivalent to \( F \subset \mathbb{P}^{N'} \).

The following is an example of \( \gamma_X \) whose general fiber is a conic.

**Example 1.6.** Let \( \varphi : (k^\times)^2 \hookrightarrow \mathbb{P}^3 \) be an embedding defined by \([s : t] \mapsto [1 : s : s^2t : s^2t^{p+1}]\) for \( p > 0 \). Set \( X = \varphi((k^\times)^2) \subset \mathbb{P}^3 \). Then \( \gamma_X : X \to \mathbb{G}(2, \mathbb{P}^3) \simeq \mathbb{P}^3 \) is locally described as
\[ (s, t) \mapsto (0, 0, t^p) \in \mathbb{A}^3 \subset \mathbb{G}(2, \mathbb{P}^3). \]
Hence the fiber of $\gamma_X$ over $(0, 0, a) \in \mathbb{A}^3$ for $a \neq 0$ is
\[
\varphi(k^* \times \{a^{1/p}\}) = \{[1 : s : a^{1/p}s^2 : a^{p+1/p}s^2] \mid s \in k^*\} \subset \mathbb{P}^3,
\]
which is a conic.

By these results, we see that fibers of Gauss maps in $p > 0$ are quite different from those in $p = 0$. Hence we ask the following question.

**Question 1.7.** What kind of difference appears about Gauss maps in $p > 0$ and $p = 0$?

In [FI2], Furukawa and the author study Question 1.7 from the view point of images $\gamma_X(X)$, general fibers, and field extensions $K(X)/K(\gamma_X(X))$ induced by $\gamma_X$. In this article, we explain the results in [FI2].

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## 2. Main results

I.M. Landsberg and J. Piontkowski independently gave a necessary and sufficient condition for a subvariety $Y \subset G(n, \mathbb{P}^N)$ to be the image of some $n$-dimensional variety $X \subset \mathbb{P}^N$ in $p = 0$. In other word, they characterized subvarieties of $G(n, \mathbb{P}^N)$ which are written as the image of some Gauss map (see [FP, 2.4.7] and [IL, Theorem 3.4.8]). By their characterization, we know that in $p = 0$,

1. not all subvariety of $G(n, \mathbb{P}^N)$ is the image of some gauss maps.
   That is, there exists a subvariety $Y \subset G(n, \mathbb{P}^N)$ with $\dim Y \leq n$ such that $Y \neq \overline{\gamma_X(X)}$ for any $n$-dimensional variety $X \subset \mathbb{P}^N$.
2. If $Y = \overline{\gamma_X(X)}$ holds for some $X$, such $X$ is uniquely determined by $Y$.

Furukawa [Fur] generalized their characterization to separable Gauss maps in any characteristics. In particular, (1), (2) hold under the assumption that $\gamma_X$ is separable in $p \geq 0$.

On the other hand, Kaji show the following proposition about images of Gauss maps of curves in $p > 0$:

**Proposition 2.1 ([Ka2, Appendix]).** Assume $p > 0$. For any curve $Y \subset G(1, \mathbb{P}^N)$, there exists a curve $X \subset \mathbb{P}^N$ such that $\overline{\gamma_X(X)} = Y$.

More precisely, for any curve $Y \subset G(1, \mathbb{P}^N)$ and any finite and inseparable field extension $L/K(Y)$, there exists a curve $X \subset \mathbb{P}^N$ such that $\overline{\gamma_X(X)} = Y$, $K(X) = L$, and the field extension $K(X)/K(Y)$ induced by $\gamma_X$ coincides with the given extension $L/K(Y)$.
Remark 2.2. For a fixed curve \( Y \subset G(1, \mathbb{P}^N) \), there exist infinitely many field extensions \( L/K(Y) \) which are finite and inseparable. Thus there are infinitely many curves \( X \subset \mathbb{P}^N \) such that \( \gamma_X(X) = Y \) by Proposition 2.1. Hence the above (1), (2) does not hold in \( p > 0 \).

We generalize Proposition 2.1 to higher dimensional \( X, Y \). In the rest of this article, a field means a finitely generated field over \( k \). For a field extension \( L/K \), we denote by \( \delta_{L/K} \) the natural \( L \)-linear map \( \Omega_{K/k} \otimes_K L \to \Omega_{L/k} \). The following is our main result.

**Theorem 2.3.** Assume \( p > 0 \). For any variety \( Y \subset G(1, \mathbb{P}^N) \) with \( 1 \leq \dim Y \leq n \), there exists an \( n \)-dimensional variety \( X \subset \mathbb{P}^N \) such that \( \gamma_X(X) = Y \).

More precisely, for any variety \( Y \subset G(1, \mathbb{P}^N) \) with \( 1 \leq \dim Y \leq n \) and any field extension \( L/K(Y) \) such that \( \text{tr.deg}_k L = n \) and the linear map \( \delta_{L/K(Y)} \) is zero, there exists an \( n \)-dimensional variety \( X \subset \mathbb{P}^N \) such that \( \gamma_X(X) = Y \), \( K(X) = L \), and the field extension \( K(X)/K(Y) \) induced by \( \gamma_X \) coincides with the given extension \( L/K(Y) \).

**Remark 2.4.** We give some remarks about Theorem 2.3.

(a) For a field extension \( L/K \), \( \delta_{L/K} \) is injective if and only if \( L/K \) is separable. Hence the condition \( \delta_{L/K} = 0 \) is the “opposite” of the separability.

(b) If \( \dim Y = 1 \), \( \delta_{L/K} : \Omega_{K/k} \otimes_K L \to \Omega_{L/k} \) is zero if and only if \( \delta_{L/K} \) is not injective since \( \dim L \Omega_{K/k} \otimes_K L = 1 \). Thus \( \delta_{L/K} = 0 \) if and only if \( L/K \) is inseparable in this case by (a). Hence Theorem 2.3 generalizes Proposition 2.1.

(c) If we replace the assumption \( \delta_{L/K(Y)} = 0 \) with “\( L/K(Y) \) is inseparable” as in Proposition 2.1, Theorem 2.3 does not hold in general for \( \dim Y \geq 2 \), even if we assume that \( L/K(Y) \) is finite. See [FI2, Example 3.4].

In Theorem 2.3, we consider images and field extensions. As a geometric version of Theorem 2.3, and as a generalization of Theorem 1.5 about general fibers of Gauss maps, we have the following theorem.

**Theorem 2.5.** Assume \( p > 0 \). Let \( Y \subset G(n, \mathbb{P}^N) \) be a projective variety with \( \dim Y \geq 1 \) and let \( F \subset Y \times \mathbb{P}^{N'} \) be an \( n \)-dimensional projective variety such that the first projection \( f : F \to Y \) is surjective. Then there exist an \( n \)-dimensional projective variety \( X \subset \mathbb{P}^N \) and a generically bijective rational map \( h : X \to F \) such that the Gauss map \( \gamma_X \) of \( X \) is equal to \( f \circ h \). In particular, \( \gamma_X(X) = Y \) holds.

Furthermore, if we assume \( n \geq N' \), we can take \( X \subset \mathbb{P}^N \) such that the fiber \( \gamma_X^{-1}(y)_{\text{red}} \subset \mathbb{P}^N \) of \( \gamma_X \) over general \( y \in Y \) is projectively equivalent to
$F_y := f^{-1}(y)_{\text{red}} \subset \{y\} \times \mathbb{P}^{N'}.$

\[ X \xrightarrow{\text{gen. bij.}} \gamma_X \xrightarrow{\gamma_X^{-1}} Y \subset \mathbb{G}(n, \mathbb{P}^N). \]

Roughly, Theorem 2.5 states that any surjective morphism $F \to Y$ appears as a Gauss map up to generically bijective rational maps. We note that the assumption $n \geq N'$ is necessary in the last statement of Theorem 2.5 since for $y = [L] \in Y$, the general fiber $\overline{\gamma_X^{-1}(y)}_{\text{red}}$ must be contained in $L \simeq \mathbb{P}^n$.

3. Idea of proof

Let $X \subset \mathbb{P}^N$ be an $n$-dimensional projective variety. Then we have a rational map

$$(\gamma_X, \text{id}_X) : X \dashrightarrow \mathbb{G}(n, \mathbb{P}^N) \times \mathbb{P}^N,$$

where $\text{id}_X$ is the inclusion of $X$ into $\mathbb{P}^N$.

**Definition 3.1.** We define a subvariety $\Gamma_X \subset \mathbb{G}(n, \mathbb{P}^N) \times \mathbb{P}^N$ to be the closure of the image of $(\gamma_X, \text{id}_X)$. In other word, $\Gamma_X$ is the graph of the rational map $\gamma_X$.

Landsberg, Piontkowski, and Furukawa characterize images of separable Gauss maps. However, it seems that such characterization does not work well for inseparable Gauss maps.

On the other hand, $(\gamma_X, \text{id}_X) : X \dashrightarrow \Gamma_X$ is birational since $\Gamma_X$ is the graph of $\gamma_X$. In particular, $(\gamma_X, \text{id}_X)$ is separable. Under this observation, we have the following idea:

**Idea.** We characterize not images but graphs of Gauss maps to investigate not necessarily separable Gauss maps.

To state a key theorem, we prepare some notation.

Since $\mathbb{G}(n, \mathbb{P}^N)$ is a Grassmannian, there exist locally free sheaves $S$ and $Q$ on $\mathbb{G}(n, \mathbb{P}^N)$ of ranks $N - n$ and $n + 1$ respectively with the exact sequence

$$0 \to S \to H^0(\mathbb{P}^N, \mathcal{O}(1)) \otimes \mathcal{O}_{\mathbb{G}(n, \mathbb{P}^N)} \to Q \to 0.$$

Hence each point $y \in \mathbb{G}(n, \mathbb{P}^N)$ corresponds to the linear variety $\mathbb{P}(Q \otimes k(y)) \subset \mathbb{P}^N$.

By the surjection $H^0(\mathbb{P}^N, \mathcal{O}(1)) \otimes \mathcal{O}_{\mathbb{G}(n, \mathbb{P}^N)} \to Q$, we have a projective bundle

$$U := \mathbb{P}(Q) \subset \mathbb{P}(H^0(\mathbb{P}^N, \mathcal{O}(1)) \otimes \mathcal{O}_{\mathbb{G}(n, \mathbb{P}^N)}) = \mathbb{G}(n, \mathbb{P}^N) \times \mathbb{P}^N.$$
over \( \mathbb{G}(n, \mathbb{P}^N) \). Hence we have the tautological invertible sheaf \( \mathcal{O}_U(1) \) on \( U \) with the surjection \( \text{pr}_1^* \mathcal{Q} \to \mathcal{O}_U(1) \). By definition, it holds that
\[
U = \{([L], x) \in \mathbb{G}(n, \mathbb{P}^N) \times \mathbb{P}^N \mid x \in L\},
\]
that is, \( U \) is the incidence variety.

For a variety \( X' \subset \mathbb{G}(n, \mathbb{P}^N) \times \mathbb{P}^N \), let \( \text{pr}_1 : X' \to \mathbb{G}(n, \mathbb{P}^N) \) and \( \text{pr}_2 : X' \to \mathbb{P}^N \) be the first and second projections respectively. Since \( \Omega_{\mathbb{G}(n, \mathbb{P}^N)} = \mathcal{Q}^\vee \otimes S \) holds, there exists a natural homomorphism
\[
\text{pr}_1^*(\mathcal{Q}^\vee \otimes S) = \text{pr}_1^* \Omega_{\mathbb{G}(n, \mathbb{P}^N)} \to \Omega_{X'}
\]
for a variety \( X' \subset \mathbb{G}(n, \mathbb{P}^N) \times \mathbb{P}^N \). This homomorphism induces
\[
\Phi : \text{pr}_1^* \mathcal{Q}^\vee \to \Omega_{X'} \otimes \text{pr}_1^* S^\vee.
\]

For an \( n \)-dimensional projective variety \( X \subset \mathbb{P}^N \), the graph \( \Gamma_X \) is contained in \( U \) since \( x \in \gamma_X(x) = T_xX \) for a smooth point \( x \in X \). Hence we have the following commutative diagram:

\[
\begin{array}{ccc}
\Gamma_X & \subset & U \\
& \searrow^{(\gamma_X, \text{id}_X)} & \swarrow_{\text{pr}_1} \\
X & \longrightarrow & \mathbb{G}(n, \mathbb{P}^N).
\end{array}
\]

The following theorem gives a characterization of graphs of Gauss maps. We note that this theorem holds for any characteristic.

**Theorem 3.2.** Let \( X' \subset U \) be a subvariety of dimension \( n \) such that \( \text{pr}_2 : X' \to \mathbb{P}^N \) is generically finite and separable. Then the following are equivalent:

(i) \( X' = \Gamma_X \) holds for some \( n \)-dimensional variety \( X \subset \mathbb{P}^N \).

(ii) \( X' = \Gamma_{X_0} \) holds for \( X_0 = \text{pr}_2(X') \).

(iii) The composite homomorphism \( \mathcal{O}_U(-1)|_{X'} \hookrightarrow \text{pr}_1^* \mathcal{Q}^\vee \xrightarrow{\Phi} \Omega_{X'} \otimes \text{pr}_1^* S^\vee \) is the zero map at the generic point of \( X' \).

**Proof.** See [FI2]. \( \Box \)

**Remark 3.3.** By definition, \( \Phi \) is the zero map at the generic point of \( X' \) if and only if so is \( \text{pr}_1^*(\mathcal{Q}^\vee \otimes S) = \text{pr}_1^* \Omega_{\mathbb{G}(n, \mathbb{P}^N)} \to \Omega_{X'} \), i.e. the \( K(X) \)-linear map \( \delta_{K(X')/K(\text{pr}_1(X'))} \) is zero. Hence (iii) in Theorem 3.2 holds automatically if \( \delta_{K(X')/K(\text{pr}_1(X'))} = 0 \).

By using Theorem 3.2, we can show Theorem 2.3 as follows.
Sketch of proof of Theorem 2.3. Assume that there exists an \( n \)-dimensional variety \( X' \subset U \) which satisfies

- \( \text{pr}_2 : X' \to \mathbb{P}^N \) is generically finite and separable,
- \( Y = \text{pr}_1(X') \),
- the field extension \( K(X')/K(Y) \) induced by \( \text{pr}_1 \) coincides with the given extension \( L/K(Y) \).

Then \( X' \) is the graph of \( \gamma_X \) for \( X = \text{pr}_2(X') \subset \mathbb{P}^N \) by Theorem 3.2 and the assumption \( \delta_{L/K(Y)} = 0 \) (see Remark 3.3), and this \( X \) satisfies the conditions in Theorem 2.3.

Hence it suffices to find such \( X' \). To find such \( X' \), it is enough to find \( x_1, \ldots, x_n \in L \) such that

- \( L/k(x_1, \ldots, x_n) \) is finite and separable,
- \( L = K(Y)(x_1, \ldots, x_n) \).

In fact, we take \( X' \) to be the image of

\[ \text{Spec } L \xrightarrow{(f,g)} Y \times \mathbb{P}^n \subset G(n, \mathbb{P}^N) \times \mathbb{P}^n \cong \mathbb{P}(\Omega) = U, \]

where \( f : \text{Spec } L \to Y \) is induced by \( L/K(Y) \) and \( g : \text{Spec } L \to \mathbb{P}^n \) is the morphism which is parametrized as \([1 : x_1 : \cdots : x_n]\).

By a suitable argument about the generators of field extensions, we can find such \( x_i \) and Theorem 2.3 is proved. \( \square \)

In Theorem 2.3, we consider only field extensions \( L/K(Y) \) with \( \delta_{L/K(Y)} = 0 \). How about field extensions with \( \delta_{L/K(Y)} \neq 0 \)? By using Theorem 3.2, we can show the following theorem about inseparable field extensions \( L/K \), not necessarily \( \delta_{L/K} = 0 \).

**Theorem 3.4.** Let \( L/K \) be an inseparable field extension. If \( p \geq 3 \) or rank \( \delta_{L/K} \) is even, there exists a hypersurface \( X \subset \mathbb{P}^{n+1} \) for \( n = \text{tr.deg}_k L \) such that the extension \( K(X)/K(\gamma_X(X)) \) induced by \( \gamma_X \) coincides with the given extension \( L/K \).

The difference of Theorems 2.3 and 3.4 are as follows: In Theorem 2.3, we fix an embedding \( \text{Spec } K \hookrightarrow G(n, \mathbb{P}^N) \) as \( Y \subset G(n, \mathbb{P}^N) \), and consider field extensions \( L/K \) such that \( \delta_{L/K} \) is zero. In Theorem 3.4, we do not fix an embedding \( \text{Spec } K \hookrightarrow G(n, \mathbb{P}^N) \), and consider \( L/K \) such that \( \delta_{L/K} \) is not necessarily zero.

In the proof of Theorem 2.3, the condition (iii) in Theorem 3.2 automatically holds since \( \Phi = 0 \), which follows from the assumption \( \delta_{L/K} = 0 \). On the other hand, \( \Phi \) is not zero in the setting of Theorem 3.4. Hence we need to take an embedding \( \text{Spec } K \hookrightarrow G(n, \mathbb{P}^{n+1}) \) and \( x_1, \ldots, x_n \in L \) carefully so that the condition (iii) in Theorem 3.2 holds.
Remark 3.5. If $p = 2$, the behavior of Gauss maps is sometimes different from that in other characteristics. For example,

- For any hypersurface $X \subset \mathbb{P}^{n+1}$, $\text{rank}_{K(X)} \delta_{K(X)/K(\gamma(X))}$ is even.
  Hence the statement of Theorem 3.4 does not hold if $p = 2$ and $\text{rank}_L \delta_{L/K}$ is odd.

- For any variety $X \subset \mathbb{P}^N$ (which is not necessarily a hypersurface), $\text{rank}_{K(X)} \delta_{K(X)/K(\gamma(X))}$ cannot be equal to one.

By Theorem 3.4 and Remark 3.5, we have the following question:

**Question 3.6.** Assume $p = 2$. Let $L/K$ be an inseparable field extension with $n = \text{tr.deg}_k L$ and $\text{rank}_L \delta_{L/K}$ is odd. Then is there an $n$-dimensional variety $X \subset \mathbb{P}^N$ with $N \geq n + 2$ such that $K(X)/K(\gamma(X)) = L/K$?

Unfortunately, we do not know the answer of this question.

**References**


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