On the concavity of the arithmetic volumes

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1 Introduction

We pursue the following analogy.

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In [7], Yuan showed that the arithmetic volumes also fit into the Brunn-Minkowski inequality, that is, if $X$ is a projective arithmetic variety and $P, Q$ are pseudo-effective arithmetic ($\mathbb{R}$-Cartier) $\mathbb{R}$-divisors on $X$, then

$$\tilde{\text{vol}}(P + Q)^{\frac{1}{\dim X}} \geq \tilde{\text{vol}}(P)^{\frac{1}{\dim X}} + \tilde{\text{vol}}(Q)^{\frac{1}{\dim X}}. \quad (1.1)$$

Our purpose is to obtain equality conditions for this inequality (Theorem 4.5). Let me illustrate the ideas with a toy example.

**Toy case** Let $A = \text{diag}(a_1, \ldots, a_n)$, $B = \text{diag}(b_1, \ldots, b_n)$ be diagonal positive-definite matrices. The mixed volumes of $A, B$ are given by

$$V(A^{(k)} \cdot B^{(n-k)}) = \frac{1}{\binom{n}{k}} \sum_{I \subseteq \{1, \ldots, n\}, |I| = k} \prod_{i \in I} a_i \cdot \prod_{j \notin I} b_j.$$
The AM-GM inequality says that \( \forall k \)

\[
V(A^{(k)} \cdot B^{(n-k)}) \geq \left( \prod_{i \in I \cup \{1, \ldots, n\}} \prod_{i \in I, j \notin I} a_i \cdot b_j \right)^{(n)^{-1}} = \det(A)^{\frac{k}{n}} \det(B)^{\frac{n-k}{n}}
\]

and

\[
\det(A + B) = \sum_{k=0}^{n} \binom{n}{k} V(A^{(k)} \cdot B^{(n-k)})
\]

\[
\geq \sum_{k=0}^{n} \binom{n}{k} \det(A)^{\frac{k}{n}} \det(B)^{\frac{n-k}{n}} = \left( \det(A)^{\frac{1}{n}} + \det(B)^{\frac{1}{n}} \right)^n. \tag{1.3}
\]

By the equality condition for the AM-GM inequality, we know that equalities in (1.2) \( \forall k \) iff \( a_1/b_1 = \cdots = a_n/b_n \). But we can also go by a very very roundabout way ...

**Alexandrov inequality** (Corollary 2.6). Let \( C = \text{diag}(c_1, \ldots, c_n) \) be another positive definite matrix. Then

\[
V\left((A + B)^{(n-1)} \cdot C\right)^{\frac{1}{n-1}} \geq V(A^{(n-1)} \cdot C)^{\frac{1}{n-1}} + V(B^{(n-1)} \cdot C)^{\frac{1}{n-1}}. \tag{1.4}
\]

**Diskant inequality** (Theorem 4.4). Set \( s = s(A, B) = \min\{a_i/b_i\} \). Then

\[
0 \leq \left( V(A^{(n-1)} \cdot B) V^{-\frac{1}{n-1}} - s \det(B)V^{-\frac{1}{n-1}} \right)^n \leq V(A^{(n-1)} \cdot B)^{\frac{n}{n-1}} - \det(A) \cdot \det(B)^{\frac{1}{n-1}}. \tag{1.5}
\]

**Proof.** Since \( s = \sup\{t \in \mathbb{R} : \det(A - tB) > 0\} \), we have

\[
\det(A) = n \int_{t=0}^{s} V((A - tB)^{(n-1)} \cdot B) \, dt
\]

\[
\leq n \int_{t=0}^{s} \left(V(A^{(n-1)} \cdot B)^{\frac{1}{n-1}} - t \det(B)^{\frac{1}{n-1}} \right)^{n-1} dt
\]

by (1.4). We can calculate the last integral. \( \Box \)

If equality in (1.3), then, by (1.5), \( s(A, B) = s(B, A)^{-1} = (\det(A)/\det(B))^{\frac{1}{n}} \).
2 Arithmetic $\mathbb{R}$-divisors

Let me explain some terminology. Let $X$ be a normal projective arithmetic variety, that is, a normal and integral scheme projective and flat over $\text{Spec}(\mathbb{Z})$. We set $d := \dim X - 1$ and denote the rational function field of $X$ by $\text{Rat}(X)$.

**Definition 2.1** (Arith. $\mathbb{R}$-divisors). An arithmetic $\mathbb{R}$-divisor is a pair $\mathcal{D} = (D, g)$ of an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $D = a_1 D_1 + \cdots + a_l D_l$ and a $D$-Green function $g : (X \setminus \bigcup \text{Supp}(D_i))(\mathbb{C}) \to \mathbb{R}$, that is, $g$ is continuous, invariant under the complex conjugation, and, $\forall p \in X(\mathbb{C})$,

$$
u_p(x) := g(x) + \sum_{i=1}^{l} a_i \log |f_i(x)|^2$$

extends to a $C^0$-function around $p$, where $f_i$ is a local equation defining $D_i$ around $p$. We denote the ($\infty$-dimensional) $\mathbb{R}$-vector space of all the arith. $\mathbb{R}$-divisors on $X$ by $\widehat{\text{Div}}(X)$.

**Example 2.1.** Let $\mathcal{L} = (L, \cdot | \cdot)$ be a continuous Hermitian line bundle on $X$, and let $s$ be a non-zero rational section of $L$. Then $\text{div}(s) := (\text{div}(s), -\log |s|^2)$ is an arith. $\mathbb{R}$-divisor of $C^0$-type.

**Example 2.2.** A $\phi \in \text{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R}$ is a formal product $\phi_1^{e_1} \cdots \phi_r^{e_r}$ with $\phi_i \in \text{Rat}(X)^\times$ and $e_i \in \mathbb{R}$. Such $\phi$ defines an arith. $\mathbb{R}$-divisor by

$$\widehat{\phi} := e_1((\phi_1), -\log |\phi_1|^2) + \cdots + e_r((\phi_r), -\log |\phi_r|^2).$$

Given an arith. $\mathbb{R}$-divisor $\mathcal{D}$ on $X$, we set

$$H^0(D) := \{ \phi \in \text{Rat}(X)^\times : (\phi) + D \geq 0 \} \cup \{0\}$$

and

$$\widehat{H}^0(\mathcal{D}) := \{ \phi \in H^0(D) : \| \phi \|_{\text{sup}}^2 \leq 1 \},$$

where $\| \cdot \|_{\text{sup}}$ is the sup norm on $H^0(D) \otimes_{\mathbb{Z}} \mathbb{R}$ defined as

$$\| \phi \|_{\text{sup}}^2 := \text{ess.sup}_{x \in X(\mathbb{C})} |\phi(x)| \exp \left( \frac{g(x)}{2} \right).$$

An arith. $\mathbb{R}$-divisor $\mathcal{D}$ is said to be effective if $D \geq 0$ and $g \geq 0$. $\mathcal{D}$ is effective iff $1 \in \widehat{H}^0(\mathcal{D})$.

**Definition 2.2** (Arith. volumes). The arith. volume of $\mathcal{D}$ is defined as

$$\widehat{\text{vol}}(\mathcal{D}) = \limsup_{m \to \infty} \frac{\log \| H^0(m\mathcal{D}) \|_{\text{sup}}}{m \dim X / \dim X!}.$$
Remark 2.1. (1) The function $\overline{D} \to \overline{\text{vol}(D)}$ is positively homogeneous of degree $\dim X$ and continuous (Moriwaki [5]).

(2) $\overline{D}$ is called big if $\overline{\text{vol}(D)} > 0$. The cone of all the big arith. $\mathbb{R}$-divisors is denoted by $\text{Big}(X)$.

(3) $\overline{D}$ is called pseudo-effective if $\overline{\text{vol}(A)} > 0$ implies $\overline{\text{vol}(D + A)} > 0$.

Let $\overline{D} = (a_1 D_1 + \cdots + a_l D_l, g)$ be an arith. $\mathbb{R}$-divisor on $X$. Assume that $D_i$ are all effective and Cartier.

Definition 2.3 (Heights). Given a rational point $x \in X(\overline{\mathbb{Q}})$, we denote the minimal field of definition for $x$ by $K(x)$ and the normalization of $\{x\}$ by $C_x$.

If $(*)$ $x \notin \text{Supp}(D_i), \forall i$, then we define the height of $x$ as

$$h_{\overline{D}}(x) := \frac{1}{[K(x) : \mathbb{Q}]} \left( \sum_{i=1}^{l} a_i \log \#O_{C_x}(D_i)/O_{C_x} + \frac{1}{2} \sum_{\sigma : K(x) \to \mathbb{C}} g(x^\sigma) \right).$$

In general, we can choose a suitable $\phi \in \text{Rat}(X)^{\times} \otimes_{\mathbb{Z}} \mathbb{R}$ s.t. $\overline{D} + (\overline{\phi})$ satisfies the condition $(*)$.

(1) $\overline{D}$ is said to be nef if $D$ is relatively nef, $u_p (2.1)$ is continuous PSH $\forall p$, and $h_{\overline{D}}(x) > 0 \forall x \in X(\overline{\mathbb{Q}})$. The cone of all the nef arith. $\mathbb{R}$-divisors on $X$ is denoted by $\text{Nef}(X)$.

(2) $\overline{D}$ is said to be integrable if $\overline{D}$ can be written as (nef arith. div.) $-$ (nef arith. div.). The ($\infty$-dimensional) $\mathbb{R}$-vector space of all the integrable arith. $\mathbb{R}$-divisors on $X$ is denoted by $\text{Int}(X)$.

Example 2.3. Let $\mathbb{P}_2^d = \text{Proj}(\mathbb{Z}[X_0, \ldots, X_d])$ be the projective space. Let $H := \{X_0 = 0\}$ and let

$$g_{FS} := \log \left( 1 + |X_1/X_0|^2 + \cdots + |X_d/X_0|^2 \right).$$

Then $\overline{H} = (H, g_{FS})$ is nef and big (but not arithmetically ample). If we add some $\lambda > 0$, then $(H, g_{FS} + \lambda)$ is arithmetically ample.

Define the naive height of a rational point $x := (x_0 : \cdots : x_d) \in \mathbb{P}_2^d(\overline{\mathbb{Q}})$ as

$$h_{\text{naive}}(x) := \frac{1}{[K(x) : \mathbb{Q}]} \sum_{v \in M_{K(x)}} \log \left( \max_i |x_i|_v \right),$$

which is invariant under the multiplication by $\alpha \in K(x)^{\times}$ by the product formula. Then we can prove $h_{\text{naive}}(x) = h_{\overline{H}}(x) + O(1)$. (In other words, $h_{\overline{H}} + O(1)$ gives the Weil height associated to $D$.)
**Proposition-Definition 2.2.** There exists a unique, symmetric (in $D_0, \ldots, D_{d-1}$), multilinear, and continuous map

$$
\widehat{\deg} : \text{Int}(X) \times \cdots \times \text{Int}(X) \times \text{Div}(X) \to \mathbb{R},
$$

$$
(D_0, \ldots, D_{d-1}; D_d) \mapsto \widehat{\deg}(D_0 \cdots D_d)
$$

having the following properties.

1. For every nef arith. $\mathbb{R}$-divisor $N$, $\widehat{\deg}(N^{d+1}) = \text{vol}(N)$.

2. If $D_0, \ldots, D_{d-1}$ are nef and $D_d$ is pseudo-effective, then $\widehat{\deg}(D_0 \cdots D_d) \geq 0$.

**Remark 2.3.** (1) The above map extends the usual arith. intersection numbers of $C^\infty$-Hermitian line bundles (that is defined by the $*$-products).

(2) As in the algebraic case, $\overline{D}$ is pseudo-effective iff, for any normalized blow-up $\varphi : X' \to X$ and for any nef arith. $\mathbb{R}$-divisor $\overline{H}$ on $X'$,

$$
\widehat{\deg}(\overline{H}^d \cdot \varphi^* \overline{D}) \geq 0
$$

([4, Theorem 6.4]).

**Theorem 2.4** (Faltings, Hriljac, Moriwaki, Yuan-Zhang, ...). Let $\overline{D}$ be an integrable arith. $\mathbb{R}$-divisor. Let $\overline{H}_1, \ldots, \overline{H}_d$ be nef arith. $\mathbb{R}$-divisors s.t. $H_{1,\mathbb{Q}}, \ldots, H_{d,\mathbb{Q}}$ are all big.

1. If $\deg(D_{\mathbb{Q}} \cdot H_{2,\mathbb{Q}} \cdots H_{d,\mathbb{Q}}) = 0$, then $\widehat{\deg}(D^2 \cdot \overline{H}_2 \cdots \overline{H}_d) \leq 0$.

2. If $\widehat{\deg}(D \cdot \overline{H}_1 \cdots \overline{H}_d) = 0$, then $\widehat{\deg}(D^2 \cdot \overline{H}_2 \cdots \overline{H}_d) \leq 0$.

**Sketch of proof.** (1) By using an arith. Bertini theorem, we can reduce the result to Faltings-Hriljac’s theorem (on arith. surfaces).

(2) Set $t = \deg(D_{\mathbb{Q}} \cdot H_{2,\mathbb{Q}} \cdots H_{d,\mathbb{Q}})/ \deg(H_{1,\mathbb{Q}} \cdot H_{2,\mathbb{Q}} \cdots H_{d,\mathbb{Q}})$ and apply (1) to $D - t\overline{H}_1, \overline{H}_2, \ldots, \overline{H}_d$.

**Remark 2.5.** Yuan and Zhang [8] have proved that (under suitable conditions) the equality holds in (1) iff $\overline{D}$ comes from Spec($H^0(\mathcal{O}_X)$).

**Corollary 2.6.** Let $D, E, \overline{H}_1, \ldots, \overline{H}_d$ be nef arith. $\mathbb{R}$-divisors on $X$.
(1) (Teissier-Khovanskii-type) For any $i$ with $1 \leq i \leq d$,
\[
\deg(D^i \cdot E^{(d-i+1)})^2 \geq \deg(D^{(i-1)} \cdot E^{(d-i+2)}) \cdot \deg(D^{(i+1)} \cdot E^{(d-i)}).
\]

(2) For any $k$ with $1 \leq k \leq d+1$ and for any $i$ with $0 \leq i \leq k$,
\[
\deg(D^i \cdot E^{(k-i)} \cdot H_k \cdots H_d)^k \geq \deg(D^k \cdot H_k \cdots H_d)^i \cdot \deg(E^k \cdot H_k \cdots H_d)^{k-i}.
\]

(3) (Alexandrov-type) For any $k$ with $1 \leq k \leq d+1$,
\[
\deg((D + E)^k \cdot H_k \cdots H_d)^\frac{1}{k} \geq \deg(D^k \cdot H_k \cdots H_d)^\frac{1}{k} + \deg(E^k \cdot H_k \cdots H_d)^\frac{1}{k}.
\]

3 Arithmetic positive intersection numbers

An approximation of $\overline{D}$ is a pair $(\varphi : X' \to X, M)$ having the following properties.

(1) $\varphi$ is a projective birational morphism s.t. $X'$ is normal and $X'_Q$ is smooth.

(2) $M$ is a nef arith. $\mathbb{R}$-divisor on $X'$ s.t. $\varphi^* - M$ is pseudo-effective.

We denote the set of all the approximations of $\overline{D}$ by $\hat{\Theta}(\overline{D})$. If $\overline{D}$ is pseudo-effective, then $\hat{\Theta}(\overline{D}) \neq \emptyset$.

**Definition 3.1.** Let $0 \leq n \leq d$. Suppose that $\overline{D}_0, \ldots, \overline{D}_n$ are all big and that $\overline{D}_{n+1}, \ldots, \overline{D}_d$ are all nef and big. The arithmetic positive intersection number of $(\overline{D}_0, \ldots, \overline{D}_n; \overline{D}_{n+1}, \ldots, \overline{D}_d)$ is defined as
\[
(\overline{D}_0 \cdots \overline{D}_n)\overline{D}_{n+1} \cdots \overline{D}_d := \sup_{(\varphi, M) \in \hat{\Theta}(\overline{D})} \deg(M_0 \cdots M_n \cdot \varphi^* D_{n+1} \cdots \varphi^* D_d).
\]

**Proposition 3.1.** (1) The map
\[
\hat{\text{Big}}(X)^{(n+1)} \times (\text{Nef}(X) \cap \hat{\text{Big}}(X))^{(d-n)} \to \mathbb{R},
\]
\[
(\overline{D}_0, \ldots, \overline{D}_n; \overline{D}_{n+1}, \ldots, \overline{D}_d) \mapsto (\overline{D}_0 \cdots \overline{D}_n)\overline{D}_{n+1} \cdots \overline{D}_d,
\]
is multi-additive in $\overline{D}_{n+1}, \ldots, \overline{D}_d$ and uniquely extends to
\[
\hat{\text{Big}}(X)^{(n+1)} \times \text{Int}(X)^{(d-n)} \to \mathbb{R},
\]
\[
(\overline{D}_0, \ldots, \overline{D}_n; \overline{D}_{n+1}, \ldots, \overline{D}_d) \mapsto (\overline{D}_0 \cdots \overline{D}_n)\overline{D}_{n+1} \cdots \overline{D}_d.
\]
(2) If $n = d - 1$, then we can further extend the map to
\[
\widehat{\text{Big}}(X)^{x_d} \times \widehat{\text{Div}}(X) \to \mathbb{R},
\]
\[
(D_0, \ldots, D_{d-1}; D_d) \mapsto (D_0 \cdots D_{d-1})D_d.
\]

**Theorem 3.2** (Arithmetic Fujita approximation: Yuan [7], Chen [2]). If $D$ is big, then $\widehat{\text{vol}}(D) = (D^{(d+1)})$.

By Corollary 2.6 + Theorem 3.2, we have

**Proposition 3.3.** Let $D, E$ be big arith. $\mathbb{R}$-divisors. For any $i$ with $1 \leq i \leq d - 1$,
\[
\langle D^i \cdot E^{(d-i+1)} \rangle \geq \widehat{\text{vol}}(D)^{\frac{i}{d+1}} \cdot \widehat{\text{vol}}(E)^{\frac{d-i+1}{d+1}}
\]
and
\[
\langle D^d \cdot E \rangle \geq \langle D^d \cdot E \rangle \geq \widehat{\text{vol}}(D)^{\frac{d}{d+1}} \cdot \widehat{\text{vol}}(E)^{\frac{1}{d+1}}.
\]
In particular,
\[
\widehat{\text{vol}}(D + E) \geq \sum_{i=0}^{d+1} \binom{d+1}{i} \langle D^i \cdot E^{d-i+1} \rangle \geq \left( \frac{\widehat{\text{vol}}(D)^{\frac{i}{d+1}} + \widehat{\text{vol}}(E)^{\frac{1}{d+1}}}{d+1} \right)^{d+1}.
\]

4 Concavity of the arithmetic volumes

**Theorem 4.1** (Yuan [6]). If $D, E$ are nef arith. $\mathbb{R}$-divisors, then
\[
\widehat{\text{vol}}(D - E) \geq \widehat{\text{vol}}(D) - (\dim X)\deg(D^d \cdot E).
\]

**Corollary 4.2.** The function $D \mapsto \widehat{\text{vol}}(D)$ is differentiable at big arithmetic $\mathbb{R}$-divisors. If $D$ is big and $E$ is arbitrary, then
\[
\lim_{t \to 0} \frac{\widehat{\text{vol}}(D + tE) - \widehat{\text{vol}}(D)}{t} = (\dim X)\langle D^d \cdot E \rangle.
\]

Suppose that $D$ is big. The (positive) height of $X$ is defined as
\[
h^+(D)(X) := \frac{\widehat{\text{vol}}(D)}{(\dim X) \text{vol}(D_Q)}.
\]
(4.1)

A sequence $(x_n)$ of rational points on $X$ is said to be generic if every subsequence is Zariski dense in $X$. If $(x_n)$ is generic, then
\[
\liminf_{n \to \infty} h^+_D(x_n) \geq h^+_D(X).
\]
(4.2)
Moreover, if $h_D(x_n)$ converges to $h^+_D(X)$ and we move $D$ along $D + t(0, 2f)$, then the both functions in (4.2) have the same slope at $D$. So we can extend the equidistribution theorem (Yuan [6], Berman-Boucksom [1], Chen [3], ...) to the case of big arith. $\mathbb{R}$-divisors.

**Corollary 4.3.** Let $f : X(\mathbb{C}) \to \mathbb{R}$ be a continuous function that is invariant under the complex conjugation, and let $(x_n)$ be a generic sequence of rational points. If $h_D(x_n)$ converges to $h^+_D(X)$, then

\[
\lim_{n \to \infty} \frac{1}{[K(x_n) : \mathbb{Q}]} \sum_{\sigma : K(x_n) \to \mathbb{C}} f(x_n^\sigma) = \frac{\langle D^d \rangle(0, 2f)}{\text{vol}(D_Q)}.
\]

**Theorem 4.4 (Diskant inequality).** If $\bar{D}$ is big and $\overline{P}$ is nef and big, then

\[
0 \leq \left( (\langle D^d \rangle \mathcal{P})^{\frac{1}{2}} - s \text{vol}(\mathcal{P})^{\frac{1}{2}} \right)^{d+1} \leq (\langle D^d \rangle \mathcal{P})^{1 + \frac{1}{2}} - \text{vol}(\mathcal{D}) \cdot \text{vol}(\mathcal{P})^{\frac{1}{2}},
\]

where $s = s(\bar{D}, \overline{P}) = \sup \{ t \in \mathbb{R} : \bar{D} - t \overline{P} \text{ is pseudo-effective} \}$.

**Theorem 4.5 ([4]).** Let $\bar{D}, \overline{E}$ be nef and big arith. $\mathbb{R}$-divisors. TFAE.

1. $\text{vol}(\bar{D} + \overline{E})^{\frac{1}{d+1}} = \text{vol}(\mathcal{D})^{\frac{1}{d+1}} + \text{vol}(\mathcal{E})^{\frac{1}{d+1}}$.

2. For $i$ with $1 \leq i \leq d$, $\deg(\bar{D}^i \cdot \overline{E}^{d-i+1}) = \text{vol}(\mathcal{D})^{\frac{i}{d+1}} \cdot \text{vol}(\mathcal{E})^{\frac{d-i+1}{d+1}}$.

3. $\deg(\bar{D}^d \cdot \overline{E}) = \text{vol}(\mathcal{D})^{\frac{d}{d+1}} \cdot \text{vol}(\mathcal{E})^{\frac{1}{d+1}}$.

4. $\exists \phi \in \text{Rat}(X)^\times$, 

\[
\frac{\bar{D}}{\text{vol}(\mathcal{D})^{\frac{1}{d+1}}} - \frac{\overline{E}}{\text{vol}(\mathcal{E})^{\frac{1}{d+1}}} = (\phi).
\]

**Proof of Theorem 4.5.** (4) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious (by the arith. Teissier-Khovanskii inequalities). The key is (3) $\Rightarrow$ (4).

By the arith. Diskant inequality, we have

\[
s = s(\bar{D}, \overline{E}) = \left( \frac{\text{vol}(\mathcal{D})}{\text{vol}(\mathcal{E})} \right)^{\frac{1}{d+1}} \quad \text{and} \quad s(\mathcal{E}, \bar{D}) = s^{-1}.
\]

Thus $\bar{D} - s \overline{E}$ and $s \overline{E} - \bar{D}$ are both pseudo-effective. By Moriwaki’s Dirichlet theorem, we have (4).
5 Computation formula

Suppose that $X_{\mathbb{Q}}$ is smooth and fix a volume form $\omega$ with $\int_{X(\mathbb{C})} \omega = 1$. Given a big arith. divisor $\overline{D}$, blow-up $X$ along

$$b(m\overline{D}) := \text{Image} \left( \left( \mathcal{H}^0(m\overline{D}) \right)_{\mathbb{Z}} \otimes_{ \mathcal{O}_X } \mathcal{O}_X(-m\overline{D}) \to \mathcal{O}_X \right).$$

We obtain $\mu_m : X_m \to X$ s.t. $X_m$ is normal, the generic fibre $X_{m,\mathbb{Q}}$ is smooth, and $b(m\overline{D})\mathcal{O}_{X_m}$ is Cartier. Set

$$F(m\overline{D}) := b(m\overline{D})\mathcal{O}_{X_m} \text{ and } M(m\overline{D}) := \mu_m^*(m\overline{D}) - F(m\overline{D}).$$

We can endow these divisors with Green functions as follows:

Take an $L^2$-ONB $e_1, \ldots, e_m$ for $\left( \mathcal{H}^0(m\overline{D}) \right)_{\mathbb{C}}$ and let

$$\text{Berg}(m\overline{D})(x) := |e_1(x)|^2 + \cdots + |e_m(x)|^2, \quad x \in X(\mathbb{C}),$$

be the Bergman function.

We can define a continuous Hermitian metric on $\mathcal{O}_{X_m}(F(m\overline{D}))$ by

$$|1_{F(m\overline{D})}|(x) = \sqrt{\text{Berg}(m\overline{D})(\mu_m(x))}, \quad x \in X_m(\mathbb{C}).$$

Then $\overline{F}(m\overline{D}) := (F(m\overline{D}), -\mu_m^* \log \text{Berg}(m\overline{D}))$ is effective and $\overline{M}(m\overline{D}) := \mu_m^*(m\overline{D}) - \overline{F}(m\overline{D})$ is nef.

Suppose that $X_{\mathbb{Q}}$ is smooth. Let $\overline{D}$ be a big arith. divisor.

**Theorem 5.1.** Let $k$ be an integer with $1 \leq k \leq d + 1$, let $\overline{D}_k, \ldots, \overline{D}_n$ be big arith. $\mathbb{R}$-divisors, and let $\overline{D}_{n+1}, \ldots, \overline{D}_d$ be integrable arith. $\mathbb{R}$-divisors. Then

$$(\overline{D}^k \cdot \overline{D}_k \cdots \overline{D}_n)\overline{D}_{n+1} \cdots \overline{D}_d = \lim_{m \to \infty} \frac{\langle \overline{D}_k \cdots \overline{D}_n \rangle M(m\overline{D})^k \cdot \overline{D}_{n+1} \cdots \overline{D}_d}{m^k}.$$

**Corollary 5.2** (Asymptotic orthogonality).

$$\lim_{m \to \infty} \frac{\deg(\overline{M}(m\overline{D})^d \cdot \overline{F}(m\overline{D}))}{m^{d+1}} = 0.$$

6 Applications

**Definition 6.1.** An arith. Zariski decomposition of a big arith. $\mathbb{R}$-divisor $\overline{D}$ is a sum $\overline{D} = \overline{P} + \overline{N}$ s.t. $\overline{P}$ is a nef arith. $\mathbb{R}$-divisor, $\overline{N}$ is an effective arith. $\mathbb{R}$-divisor, and $\text{vol}(\overline{P}) = \text{vol}(\overline{D})$.  

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Remark 6.1. (1) If $\dim X = 2$, then an arith. Zariski decomposition of a big $D$ always exists and unique (Moriwaki [5]).

(2) If $\dim X \geq 3$, there exists no arith. Zariski decomposition in general even after any blow-up of $X$ (Moriwaki ’11).

Example 6.1. Let $\mathbb{P}^2 = \text{Proj}(\mathbb{Z}[X_0, X_1, X_2])$ and let $z_i := X_i/X_0$ be the affine coordinate. Let $H := \{X_0 = 0\}$ and let
\[
g := \max \left\{ -2, \log |X_1/X_0|^2 + 2, \log |X_2/X_0|^2 + 2 \right\},
\]
which is an $H$-Green function of PSH-type. Moreover, we can add a “bump” $\rho : \mathbb{P}^2(\mathbb{C}) \to \mathbb{R}_\geq 0$ such that
\[
\text{Supp}(\rho) \subseteq \{|z_1| < \exp(-2)\} \times \{|z_2| < \exp(-2)\}.
\]
Then $\overline{H} = (H, g + \rho)$ are big and non-nef ($h\overline{H}(1 : 0 : 0) < 0$ or $g + \rho$ is not of PSH-type).

Blow up $\mathbb{P}^2$ with center $(1 : 0 : 0)$, viz. over $\{X_0 \neq 0\}$,
\[
\varphi : \text{Proj}(\mathbb{Z}[z_1, z_2][Y_1, Y_2]/(z_2Y_1 - z_1Y_2)) \to \{X_0 \neq 0\}.
\]
Then $\varphi^*\overline{H}$ admits an arith. Zariski decomposition. Let $E$ be the exceptional divisor and let $w_{ij} := Y_j/Y_i$. Then the positive part is given by
\[
\overline{P} = \left( \varphi^*H - \frac{1}{2}E, \max \left\{ \log |z_1w_{i1}|, \log |z_1w_{i2}|, \log |z_1w_{i1}|^2 + 2, \log |z_1w_{i2}|^2 + 2 \right\} \right),
\]
the negative part is
\[
\overline{N} = \left( \frac{1}{2}E, \max \left\{ 0, -2 - \max \left\{ \log |z_1w_{i1}|, \log |z_1w_{i2}| \right\} \right\} + \varphi^*\rho \right) \geq 0,
\]
and $\overline{\text{vol}(\overline{H})} = \overline{\text{vol}(\overline{P})} = 5/4$.

Corollary 6.2. Let $\overline{P}, \overline{Q}$ be nef and big arith. $\mathbb{R}$-divisors. If $\overline{\text{vol}(\overline{P})} = \overline{\text{vol}(\overline{Q})}$ and $\overline{P} \geq \overline{Q}$, then $\overline{P} = \overline{Q}$.

Proof.
\[
2\overline{\text{vol}(\overline{P})}^{\frac{1}{\dim X}} = \overline{\text{vol}(\overline{P})}^{\frac{1}{\dim X}} + \overline{\text{vol}(\overline{Q})}^{\frac{1}{\dim X}} \leq \overline{\text{vol}(\overline{P} + \overline{Q})}^{\frac{1}{\dim X}} \leq \overline{\text{vol}(2\overline{P})}^{\frac{1}{\dim X}}.
\]
Thus, by Theorem 4.5, $\exists \phi \in \text{Rat}(X)^\times$ s.t. $\overline{P} - \overline{Q} = (\phi) \geq 0$.
\[
(\phi) \geq 0 \iff (\overline{\phi}) = 0 \iff \phi \in H^0(\mathcal{O}_{\mathbb{X}}^\times) \otimes \mathbb{R}.
\]

Corollary 6.3. An arith. Zariski decomposition of a big arith. $\mathbb{R}$-divisor is (if it exists) unique.
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References


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