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On the concavity of the arithmetic volumes

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1 Introduction

We pursue the following analogy.

Convex geometry (Bonnesen, Diskant, ...)	Algebraic geometry (Boucksom-Favre -Jonsson, Cutkosky)	Arakelov geometry
convex bodies	nef & big divisors	nef & big —
Euclidean volumes	$\mathrm{vol}(P)$	$\widehat{\mathrm{vol}}(\overline{P})$
mixed volumes	$\deg(P^i \cdot Q^{\dim X - i})$	$\widehat{\deg}(\overline{P}^i \cdot \overline{Q}^{\dim X - i})$
P, Q: homothetic	$P \equiv_{\mathrm{num}} Q$	$\overline{P} \sim_{\mathbb{R}} \overline{Q}$
inradius $s(P,Q) = \sup\{t : P \supset tQ + c, \exists c\}$	$s(P,Q) = \sup\{t : P - tQ \text{ is psef}\}$	$s(\overline{P}, \overline{Q})$
$\vdots$	$\vdots$	$\vdots$

In [7], Yuan showed that the arithmetic volumes also fit into the Brunn-Minkowski inequality, that is, if X is a projective arithmetic variety and $\overline{P}, \overline{Q}$ are pseudo-effective arithmetic ($\mathbb{R}$-Cartier) $\mathbb{R}$-divisors on X, then

$$\widehat{\mathrm{vol}}(\overline{P} + \overline{Q})^{\frac{1}{\dim X}} \geqslant \widehat{\mathrm{vol}}(\overline{P})^{\frac{1}{\dim X}} + \widehat{\mathrm{vol}}(\overline{Q})^{\frac{1}{\dim X}}. \tag{1.1}$$

Our purpose is to obtain equality conditions for this inequality (Theorem 4.5). Let me illustrate the ideas with a toy example.

Toy case Let $A = \mathrm{diag}(a_1, \ldots, a_n)$, $B = \mathrm{diag}(b_1, \ldots, b_n)$ be diagonal positive-definite matrices. The mixed volumes of A, B are given by

$$V(A^{(k)} \cdot B^{(n-k)}) = \frac{1}{\binom{n}{k}} \sum_{\substack{I \subset \{1,\ldots,n\},\\ \sharp I = k}} \prod_{i \in I} a_i \cdot \prod_{j \notin I} b_j.$$

The AM-GM inequality says that $\forall k$

$$V(A^{(k)} \cdot B^{(n-k)}) \geqslant \left(\prod_{\substack{I \subset \{1,\ldots,n\}, \\ \sharp I = k}} \prod_{i \in I} a_i \cdot \prod_{j \notin I} b_j \right)^{\binom{n}{k}^{-1}} = \det(A)^{\frac{k}{n}} \det(B)^{\frac{n-k}{n}} \tag{1.2}$$

and

$$\begin{aligned} \det(A+B) &= \sum_{k=0}^{n} \binom{n}{k} V(A^{(k)} \cdot B^{(n-k)}) \\ &\geqslant \sum_{k=0}^{n} \binom{n}{k} \det(A)^{\frac{k}{n}} \det(B)^{\frac{n-k}{n}} = \left(\det(A)^{\frac{1}{n}} + \det(B)^{\frac{1}{n}}\right)^n . \end{aligned} \tag{1.3}$$

By the equality condition for the AM-GM inequality, we know that equalities in (1.2) $\forall k$ iff $a_1/b_1 = \cdots = a_n/b_n$. But we can also go by a very very roundabout way ...

Alexandrov inequality (Corollary 2.6)**.** *Let* $C = \operatorname{diag}(c_1, \ldots, c_n)$ *be another positive definite matrix. Then*

$$V\left((A+B)^{(n-1)} \cdot C\right)^{\frac{1}{n-1}} \geqslant V(A^{(n-1)} \cdot C)^{\frac{1}{n-1}} + V(B^{(n-1)} \cdot C)^{\frac{1}{n-1}}. \tag{1.4}$$

Diskant inequality (Theorem 4.4)**.** *Set* $s = s(A,B) = \min\{a_i/b_i\}$. *Then*

$$0 \leqslant \left(V(A^{(n-1)} \cdot B)^{\frac{1}{n-1}} - s \det(B)^{\frac{1}{n-1}}\right)^n \leqslant V(A^{(n-1)} \cdot B)^{\frac{n}{n-1}} - \det(A) \cdot \det(B)^{\frac{1}{n-1}}. \tag{1.5}$$

Proof. Since $s = \sup\{t \in \mathbb{R} : \det(A - tB) > 0\}$, we have

$$\begin{aligned} \det(A) &= n \int_{t=0}^{s} V\left((A-tB)^{(n-1)} \cdot B\right) dt \\ &\leqslant n \int_{t=0}^{s} \left(V(A^{(n-1)} \cdot B)^{\frac{1}{n-1}} - t \det(B)^{\frac{1}{n-1}}\right)^{n-1} dt \end{aligned}$$

by (1.4). We can calculate the last integral. □

If equality in (1.3), then, by (1.5), $s(A,B) = s(B,A)^{-1} = (\det(A)/\det(B))^{\frac{1}{n}}$.

2 Arithmetic $\mathbb{R}$-divisors

Let me explain some terminology. Let X be a normal projective arithmetic variety, that is, a normal and integral scheme projective and flat over $\mathrm{Spec}(\mathbb{Z})$. We set $d := \dim X - 1$ and denote the rational function field of X by $\mathrm{Rat}(X)$.

Definition 2.1 (Arith. $\mathbb{R}$-divisors)**.** An *arithmetic* $\mathbb{R}$*-divisor* is a pair $\overline{D} = (D, g)$ of an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor $D = a_1D_1 + \cdots + a_lD_l$ and a D*-Green function* $g : (X \setminus \bigcup \mathrm{Supp}(D_i))(\mathbb{C}) \to \mathbb{R}$, that is, g is continuous, invariant under the complex conjugation, and, $\forall p \in X(\mathbb{C})$,

$$u_p(x) := g(x) + \sum_{i=1}^{l} a_i \log |f_i(x)|^2 \tag{2.1}$$

extends to a C^0-function around p, where f_i is a local equation defining D_i around p. We denote the (∞-dimensional) $\mathbb{R}$-vector space of all the arith. $\mathbb{R}$-divisors on X by $\widehat{\mathrm{Div}}(X)$.

Example 2.1. Let $\overline{L} = (L, |\cdot|)$ be a continuous Hermitian line bundle on X, and let s be a non-zero rational section of L. Then $\widehat{\mathrm{div}}(s) := (\mathrm{div}(s), -\log|s|^2)$ is an arith. $\mathbb{R}$-divisor of C^0-type.

Example 2.2. A $\phi \in \mathrm{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R}$ is a formal product $\phi_1^{e_1} \cdots \phi_r^{e_r}$ with $\phi_i \in \mathrm{Rat}(X)^\times$ and $e_i \in \mathbb{R}$. Such ϕ defines an arith. $\mathbb{R}$-divisor by

$$\widehat{(\phi)} := e_1((\phi_1), -\log|\phi_1|^2) + \cdots + e_r((\phi_r), -\log|\phi_r|^2).$$

Given an arith. $\mathbb{R}$-divisor $\overline{D}$ on X, we set

$$H^0(D) := \left\{\phi \in \mathrm{Rat}(X)^\times \,:\, (\phi) + D \geqslant 0\right\} \cup \{0\}$$

and

$$\widehat{H}^0(\overline{D}) := \left\{\phi \in H^0(D) \,:\, \|\phi\|_{\sup}^g \leqslant 1\right\},$$

where $\|\cdot\|_{\sup}^g$ is the sup norm on $H^0(D) \otimes_{\mathbb{Z}} \mathbb{R}$ defined as

$$\|\phi\|_{\sup}^g := \operatorname*{ess.sup}_{x \in X(\mathbb{C})} |\phi(x)| \exp\left(\frac{g(x)}{2}\right).$$

An arith. $\mathbb{R}$-divisor $\overline{D}$ is said to be *effective* if $D \geqslant 0$ and $g \geqslant 0$. $\overline{D}$ is effective iff $1 \in \widehat{H}^0(\overline{D})$.

Definition 2.2 (Arith. volumes)**.** The *arith. volume* of $\overline{D}$ is defined as

$$\widehat{\mathrm{vol}}(\overline{D}) = \limsup_{m \to \infty} \frac{\log \sharp \widehat{H}^0(m\overline{D})}{m^{\dim X} / \dim X!}.$$

Remark 2.1. (1) The function $\overline{D} \to \widehat{\mathrm{vol}}(\overline{D})$ is positively homogeneous of degree $\dim X$ and continuous (Moriwaki [5]).

(2) $\overline{D}$ is called *big* if $\widehat{\mathrm{vol}}(\overline{D}) > 0$. The cone of all the big arith. $\mathbb{R}$-divisors is denoted by $\widehat{\mathrm{Big}}(X)$.

(3) $\overline{D}$ is called *pseudo-effective* if $\widehat{\mathrm{vol}}(\overline{A}) > 0$ implies $\widehat{\mathrm{vol}}(\overline{D} + \overline{A}) > 0$.

Let $\overline{D} = (a_1 D_1 + \cdots + a_l D_l, g)$ be an arith. $\mathbb{R}$-divisor on X. Assume that D_i are all effective and Cartier.

Definition 2.3 (Heights). Given a rational point $x \in X(\overline{\mathbb{Q}})$, we denote the minimal field of definition for x by $K(x)$ and the normalization of $\overline{\{x\}}$ by C_x.

If $(*)$ $x \notin \mathrm{Supp}(D_i)$, $\forall i$, then we define the *height* of x as

$$h_{\overline{D}}(x) := \frac{1}{[K(x) : \mathbb{Q}]} \left(\sum_{i=1}^{l} a_i \log \sharp \mathcal{O}_{C_x}(D_i)/\mathcal{O}_{C_x} + \frac{1}{2} \sum_{\sigma : K(x) \to \mathbb{C}} g(x^\sigma) \right).$$

In general, we can choose a suitable $\phi \in \mathrm{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R}$ s.t. $\overline{D} + \widehat{(\phi)}$ satisfies the condition $(*)$.

(1) $\overline{D}$ is said to be *nef* if D is relatively nef, u_p (2.1) is continuous PSH $\forall p$, and $h_{\overline{D}}(x) \geqslant 0$ $\forall x \in X(\overline{\mathbb{Q}})$. The cone of all the nef arith. $\mathbb{R}$-divisors on X is denoted by $\widehat{\mathrm{Nef}}(X)$.

(2) $\overline{D}$ is said to be *integrable* if $\overline{D}$ can be written as (nef arith. div.) $-$ (nef arith. div.). The (∞-dimensional) $\mathbb{R}$-vector space of all the integrable arith. $\mathbb{R}$-divisors on X is denoted by $\widehat{\mathrm{Int}}(X)$.

Example 2.3. Let $\mathbb{P}^d_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[X_0, \dots, X_d])$ be the projective space. Let $H := \{X_0 = 0\}$ and let

$$g_{\mathrm{FS}} := \log\left(1 + |X_1/X_0|^2 + \cdots + |X_d/X_0|^2\right).$$

Then $\overline{H} = (H, g_{\mathrm{FS}})$ is nef and big (but not arithmetically ample). If we add some $\lambda > 0$, then $(H, g_{\mathrm{FS}} + \lambda)$ is arithmetically ample.

Define the naive height of a rational point $x := (x_0 : \cdots : x_d) \in \mathbb{P}^d_{\mathbb{Z}}(\overline{\mathbb{Q}})$ as

$$h_{\mathrm{naive}}(x) := \frac{1}{[K(x) : \mathbb{Q}]} \sum_{v \in M_{K(x)}} \log\left(\max_i \{|x_i|_v\}\right),$$

which is invariant under the multiplication by $\alpha \in K(x)^\times$ by the product formula. Then we can prove $h_{\mathrm{naive}}(x) = h_{\overline{H}}(x) + O(1)$. (In other words, $h_{\overline{D}} + O(1)$ gives the Weil height associated to D.)

Proposition-Definition 2.2. *There exists a unique, symmetric (in $\overline{D}_0, \ldots, \overline{D}_{d-1}$), multilinear, and continuous map*

$$\widehat{\deg} : \overbrace{\widehat{\mathrm{Int}}(X) \times \cdots \times \widehat{\mathrm{Int}}(X)}^{d\text{-times}} \times \widehat{\mathrm{Div}}(X) \to \mathbb{R},$$
$$(\overline{D}_0, \ldots, \overline{D}_{d-1}; \overline{D}_d) \mapsto \widehat{\deg}(\overline{D}_0 \cdots \overline{D}_d)$$

having the following properties.

(1) *For every nef arith. $\mathbb{R}$-divisor $\overline{N}$, $\widehat{\deg}(\overline{N}^{\cdot d+1}) = \widehat{\mathrm{vol}}(\overline{N})$.*

(2) *If $\overline{D}_0, \ldots, \overline{D}_{d-1}$ are nef and $\overline{D}_d$ is pseudo-effective, then $\widehat{\deg}(\overline{D}_0 \cdots \overline{D}_d) \geqslant 0$.*

Remark 2.3. (1) The above map extends the usual arith. intersection numbers of C^∞-Hermitian line bundles (that is defined by the $*$-products).

(2) As in the algebraic case, $\overline{D}$ is pseudo-effective iff, for any normalized blow-up $\varphi : X' \to X$ and for any nef arith. $\mathbb{R}$-divisor $\overline{H}$ on X',

$$\widehat{\deg}(\overline{H}^{\cdot d} \cdot \varphi^* \overline{D}) \geqslant 0$$

([4, Theorem 6.4]).

Theorem 2.4 (Faltings, Hriljac, Moriwaki, Yuan-Zhang, ...). *Let $\overline{D}$ be an integrable arith. $\mathbb{R}$-divisor. Let $\overline{H}_1, \ldots, \overline{H}_d$ be nef arith. $\mathbb{R}$-divisors s.t. $H_{1,\mathbb{Q}}, \ldots, H_{d,\mathbb{Q}}$ are all big.*

(1) *If $\deg(D_\mathbb{Q} \cdot H_{2,\mathbb{Q}} \cdots H_{d,\mathbb{Q}}) = 0$, then $\widehat{\deg}(\overline{D}^{\cdot 2} \cdot \overline{H}_2 \cdots \overline{H}_d) \leqslant 0$.*

(2) *If $\widehat{\deg}(\overline{D} \cdot \overline{H}_1 \cdots \overline{H}_d) = 0$, then $\widehat{\deg}(\overline{D}^{\cdot 2} \cdot \overline{H}_2 \cdots \overline{H}_d) \leqslant 0$.*

Sketch of proof. (1) By using an arith. Bertini theorem, we can reduce the result to Faltings-Hriljac's theorem (on arith. surfaces).

(2) Set $t = \deg(D_\mathbb{Q} \cdot H_{2,\mathbb{Q}} \cdots H_{d,\mathbb{Q}}) / \deg(H_{1,\mathbb{Q}} \cdot H_{2,\mathbb{Q}} \cdots H_{d,\mathbb{Q}})$ and apply (1) to $\overline{D} - t\overline{H}_1, \overline{H}_2, \ldots, \overline{H}_d$.

□

Remark 2.5. Yuan and Zhang [8] have proved that (under suitable conditions) the equality holds in (1) iff $\overline{D}$ comes from $\mathrm{Spec}(H^0(\mathcal{O}_X))$.

Corollary 2.6. *Let $\overline{D}, \overline{E}, \overline{H}_1, \ldots, \overline{H}_d$ be nef arith. $\mathbb{R}$-divisors on X.*

(1) *(Teissier-Khovanskii-type) For any i with $1 \leqslant i \leqslant d$,*

$$\widehat{\deg}(\overline{D}^{\cdot i} \cdot \overline{E}^{\cdot(d-i+1)})^2 \geqslant \widehat{\deg}(\overline{D}^{\cdot(i-1)} \cdot \overline{E}^{\cdot(d-i+2)}) \cdot \widehat{\deg}(\overline{D}^{\cdot(i+1)} \cdot \overline{E}^{\cdot(d-i)}).$$

(2) *For any k with $1 \leqslant k \leqslant d+1$ and for any i with $0 \leqslant i \leqslant k$,*

$$\widehat{\deg}(\overline{D}^{\cdot i} \cdot \overline{E}^{\cdot(k-i)} \cdot \overline{H}_k \cdots \overline{H}_d)^k \geqslant \widehat{\deg}(\overline{D}^{\cdot k} \cdot \overline{H}_k \cdots \overline{H}_d)^i \cdot \widehat{\deg}(\overline{E}^{\cdot k} \cdot \overline{H}_k \cdots \overline{H}_d)^{k-i}.$$

(3) *(Alexandrov-type) For any k with $1 \leqslant k \leqslant d+1$,*

$$\widehat{\deg}((\overline{D}+\overline{E})^{\cdot k} \cdot \overline{H}_k \cdots \overline{H}_d)^{\frac{1}{k}} \geqslant \widehat{\deg}(\overline{D}^{\cdot k} \cdot \overline{H}_k \cdots \overline{H}_d)^{\frac{1}{k}} + \widehat{\deg}(\overline{E}^{\cdot k} \cdot \overline{H}_k \cdots \overline{H}_d)^{\frac{1}{k}}.$$

3 Arithmetic positive intersection numbers

An *approximation* of $\overline{D}$ is a pair $(\varphi : X' \to X, \overline{M})$ having the following properties.

(1) φ is a projective birational morphism s.t. X' is normal and $X'_{\mathbb{Q}}$ is smooth.

(2) $\overline{M}$ is a nef arith. $\mathbb{R}$-divisor on X' s.t. $\varphi^*\overline{D} - \overline{M}$ is pseudo-effective.

We denote the set of all the approximations of $\overline{D}$ by $\widehat{\Theta}(\overline{D})$. If $\overline{D}$ is pseudo-effective, then $\widehat{\Theta}(\overline{D}) \neq \emptyset$.

Definition 3.1. Let $0 \leqslant n \leqslant d$. Suppose that $\overline{D}_0, \dots, \overline{D}_n$ are all big and that $\overline{D}_{n+1}, \dots, \overline{D}_d$ are all nef and big. The *arithmetic positive intersection number* of $(\overline{D}_0, \dots, \overline{D}_n; \overline{D}_{n+1}, \dots, \overline{D}_d)$ is defined as

$$\langle \overline{D}_0 \cdots \overline{D}_n \rangle \overline{D}_{n+1} \cdots \overline{D}_d := \sup_{(\varphi, \overline{M}_i) \in \widehat{\Theta}(\overline{D}_i)} \widehat{\deg}(\overline{M}_0 \cdots \overline{M}_n \cdot \varphi^*\overline{D}_{n+1} \cdots \varphi^*\overline{D}_d).$$

Proposition 3.1. *(1) The map*

$$\widehat{\mathrm{Big}}(X)^{\times(n+1)} \times (\widehat{\mathrm{Nef}}(X) \cap \widehat{\mathrm{Big}}(X))^{\times(d-n)} \to \mathbb{R},$$
$$(\overline{D}_0, \dots, \overline{D}_n; \overline{D}_{n+1}, \dots, \overline{D}_d) \mapsto \langle \overline{D}_0 \cdots \overline{D}_n \rangle \overline{D}_{n+1} \cdots \overline{D}_d,$$

is multi-additive in $\overline{D}_{n+1}, \dots, \overline{D}_d$ and uniquely extends to

$$\widehat{\mathrm{Big}}(X)^{\times(n+1)} \times \widehat{\mathrm{Int}}(X)^{\times(d-n)} \to \mathbb{R},$$
$$(\overline{D}_0, \dots, \overline{D}_n; \overline{D}_{n+1}, \dots, \overline{D}_d) \mapsto \langle \overline{D}_0 \cdots \overline{D}_n \rangle \overline{D}_{n+1} \cdots \overline{D}_d.$$

(2) If $n = d-1$, then we can further extend the map to

$$\widehat{\mathrm{Big}}(X)^{\times d} \times \widehat{\mathrm{Div}}(X) \to \mathbb{R},$$
$$(\overline{D}_0, \ldots, \overline{D}_{d-1}; \overline{D}_d) \mapsto \langle \overline{D}_0 \cdots \overline{D}_{d-1} \rangle \overline{D}_d.$$

Theorem 3.2 (Arithmetic Fujita approximation: Yuan [7], Chen [2]). *If $\overline{D}$ is big, then $\widehat{\mathrm{vol}}(\overline{D}) = \langle \overline{D}^{\cdot (d+1)} \rangle$.*

By Corollary 2.6 + Theorem 3.2, we have

Proposition 3.3. *Let $\overline{D}, \overline{E}$ be big arith. $\mathbb{R}$-divisors. For any i with $1 \leqslant i \leqslant d-1$,*

$$\langle \overline{D}^{\cdot i} \cdot \overline{E}^{\cdot (d-i+1)} \rangle \geqslant \widehat{\mathrm{vol}}(\overline{D})^{\frac{i}{d+1}} \cdot \widehat{\mathrm{vol}}(\overline{E})^{\frac{d-i+1}{d+1}}$$

and

$$\langle \overline{D}^{\cdot d} \rangle \overline{E} \geqslant \langle \overline{D}^{\cdot d} \cdot \overline{E} \rangle \geqslant \widehat{\mathrm{vol}}(\overline{D})^{\frac{d}{d+1}} \cdot \widehat{\mathrm{vol}}(\overline{E})^{\frac{1}{d+1}}.$$

In particular,

$$\widehat{\mathrm{vol}}(\overline{D} + \overline{E}) \geqslant \sum_{i=0}^{d+1} \binom{d+1}{i} \langle \overline{D}^{\cdot i} \cdot \overline{E}^{d-i+1} \rangle \geqslant \left(\widehat{\mathrm{vol}}(\overline{D})^{\frac{1}{d+1}} + \widehat{\mathrm{vol}}(\overline{E})^{\frac{1}{d+1}} \right)^{d+1}.$$

4 Concavity of the arithmetic volumes

Theorem 4.1 (Yuan [6]). *If $\overline{D}, \overline{E}$ are nef arith. $\mathbb{R}$-divisors, then*

$$\widehat{\mathrm{vol}}(\overline{D} - \overline{E}) \geqslant \widehat{\mathrm{vol}}(\overline{D}) - (\dim X) \widehat{\deg}(\overline{D}^{\cdot d} \cdot \overline{E}).$$

Corollary 4.2. *The function $\overline{D} \mapsto \widehat{\mathrm{vol}}(\overline{D})$ is differentiable at big arithmetic $\mathbb{R}$-divisors. If $\overline{D}$ is big and $\overline{E}$ is arbitrary, then*

$$\lim_{t \to 0} \frac{\widehat{\mathrm{vol}}(\overline{D} + t\overline{E}) - \widehat{\mathrm{vol}}(\overline{D})}{t} = (\dim X) \langle \overline{D}^{\cdot d} \rangle \overline{E}.$$

Suppose that $\overline{D}$ is big. The *(positive) height* of X is defines as

$$h^+_{\overline{D}}(X) := \frac{\widehat{\mathrm{vol}}(\overline{D})}{(\dim X) \mathrm{vol}(D_{\mathbb{Q}})}. \tag{4.1}$$

A sequence (x_n) of rational points on X is said to be *generic* if every subsequence is Zariski dense in X. If (x_n) is generic, then

$$\liminf_{n \to \infty} h_{\overline{D}}(x_n) \geqslant h^+_{\overline{D}}(X). \tag{4.2}$$

Moreover, if $h_{\overline{D}}(x_n)$ converges to $h^+_{\overline{D}}(X)$ and we move $\overline{D}$ along $\overline{D} + t(0, 2f)$, then the both functions in (4.2) have the same slope at $\overline{D}$. So we can extend the equidistribution theorem (Yuan [6], Berman-Boucksom [1], Chen [3], ...) to the case of big arith. $\mathbb{R}$-div'rs.

Corollary 4.3. *Let $f : X(\mathbb{C}) \to \mathbb{R}$ be a continuous function that is invariant under the complex conjugation, and let (x_n) be a generic sequence of rational points. If $h_{\overline{D}}(x_n)$ converges to $h^+_{\overline{D}}(X)$, then*

$$\lim_{n\to\infty} \frac{1}{[K(x_n):\mathbb{Q}]} \sum_{\sigma : K(x_n)\to\mathbb{C}} f(x_n^\sigma) = \frac{\langle \overline{D}^{\cdot d}\rangle(0, 2f)}{\mathrm{vol}(D_{\mathbb{Q}})}.$$

Theorem 4.4 (Diskant inequality)**.** *If $\overline{D}$ is big and $\overline{P}$ is nef and big, then*

$$0 \leqslant \left((\langle \overline{D}^{\cdot d}\rangle \overline{P})^{\frac{1}{d}} - s\widehat{\mathrm{vol}}(\overline{P})^{\frac{1}{d}} \right)^{d+1} \leqslant (\langle \overline{D}^{\cdot d}\rangle \overline{P})^{1+\frac{1}{d}} - \widehat{\mathrm{vol}}(\overline{D}) \cdot \widehat{\mathrm{vol}}(\overline{P})^{\frac{1}{d}},$$

where $s = s(\overline{D}, \overline{P}) = \sup\{t \in \mathbb{R} : \overline{D} - t\overline{P}$ is pseudo-effective$\}$.

Theorem 4.5 ([4])**.** *Let $\overline{D}, \overline{E}$ be nef and big arith. $\mathbb{R}$-divisors. TFAE.*

(1) $\widehat{\mathrm{vol}}(\overline{D} + \overline{E})^{\frac{1}{d+1}} = \widehat{\mathrm{vol}}(\overline{D})^{\frac{1}{d+1}} + \widehat{\mathrm{vol}}(\overline{E})^{\frac{1}{d+1}}$.

(2) For i with $1 \leqslant i \leqslant d$, $\widehat{\deg}(\overline{D}^{\cdot i} \cdot \overline{E}^{\cdot (d-i+1)}) = \widehat{\mathrm{vol}}(\overline{D})^{\frac{i}{d+1}} \cdot \widehat{\mathrm{vol}}(\overline{E})^{\frac{d-i+1}{d+1}}$.

(3) $\widehat{\deg}(\overline{D}^{\cdot d} \cdot \overline{E}) = \widehat{\mathrm{vol}}(\overline{D})^{\frac{d}{d+1}} \cdot \widehat{\mathrm{vol}}(\overline{E})^{\frac{1}{d+1}}$.

(4) $\exists \phi \in \mathrm{Rat}(X)^\times$,

$$\frac{\overline{D}}{\widehat{\mathrm{vol}}(\overline{D})^{\frac{1}{d+1}}} - \frac{\overline{E}}{\widehat{\mathrm{vol}}(\overline{E})^{\frac{1}{d+1}}} = \widehat{(\phi)}.$$

Proof of Theorem 4.5. (4) $\Rightarrow$ (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious (by the arith. Teissier-Khovanskii inequalities). The key is (3) $\Rightarrow$ (4).

By the arith. Diskant inequality, we have

$$s = s(\overline{D}, \overline{E}) = \left(\frac{\widehat{\mathrm{vol}}(\overline{D})}{\widehat{\mathrm{vol}}(\overline{E})} \right)^{\frac{1}{d+1}} \quad \text{and} \quad s(\overline{E}, \overline{D}) = s^{-1}.$$

Thus $\overline{D} - s\overline{E}$ and $s\overline{E} - \overline{D}$ are both pseudo-effective. By Moriwaki's Dirichlet theorem, we have (4). □

5 Computation formula

Suppose that $X_{\mathbb{Q}}$ is smooth and fix a volume form ω with $\int_{X(\mathbb{C})} \omega = 1$. Given a big arith. divisor $\overline{D}$, blow-up X along

$$\mathfrak{b}(m\overline{D}) := \operatorname{Image}\left(\left\langle \widehat{H}^0(m\overline{D})\right\rangle_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_X(-mD) \to \mathcal{O}_X\right).$$

We obtain $\mu_m : X_m \to X$ s.t. X_m is normal, the generic fibre $X_{m,\mathbb{Q}}$ is smooth, and $\mathfrak{b}(m\overline{D})\mathcal{O}_{X_m}$ is Cartier. Set

$$F(m\overline{D}) := \mathfrak{b}(m\overline{D})\mathcal{O}_{X_m} \quad \text{and} \quad M(m\overline{D}) := \mu_m^*(m\overline{D}) - F(m\overline{D}).$$

We can endow these divisors with Green functions as follows:

Take an L^2-ONB $e_1, \ldots, e_{r_m}$ for $\left\langle \widehat{H}^0(m\overline{D})\right\rangle_{\mathbb{C}}$ and let

$$\operatorname{Berg}(m\overline{D})(x) := |e_1(x)|^2 + \cdots + |e_{r_m}(x)|^2, \quad x \in X(\mathbb{C}),$$

be the Bergman function.

We can define a continuous Hermitian metric on $\mathcal{O}_{X_m}(F(m\overline{D}))$ by

$$|1_{F(m\overline{D})}|(x) = \sqrt{\operatorname{Berg}(m\overline{D})(\mu_m(x))}, \quad x \in X_m(\mathbb{C}).$$

Then $\overline{F}(m\overline{D}) := (F(m\overline{D}), -\mu_m^* \log \operatorname{Berg}(m\overline{D}))$ is effective and $\overline{M}(m\overline{D}) := \mu_m^*(m\overline{D}) - \overline{F}(m\overline{D})$ is nef.

Suppose that $X_{\mathbb{Q}}$ is smooth. Let $\overline{D}$ be a big arith. divisor.

Theorem 5.1. *Let k be an integer with $1 \leqslant k \leqslant d+1$, let $\overline{D}_k, \ldots, \overline{D}_n$ be big arith. $\mathbb{R}$-divisors, and let $\overline{D}_{n+1}, \ldots, \overline{D}_d$ be integrable arith. $\mathbb{R}$-divisors. Then*

$$\langle \overline{D}^{\cdot k} \cdot \overline{D}_k \cdots \overline{D}_n \rangle \overline{D}_{n+1} \cdots \overline{D}_d = \lim_{m \to \infty} \frac{\langle \overline{D}_k \cdots \overline{D}_n \rangle \overline{M}(m\overline{D})^{\cdot k} \cdot \overline{D}_{n+1} \cdots \overline{D}_d}{m^k}.$$

Corollary 5.2 (Asymptotic orthogonality)**.**

$$\lim_{m \to \infty} \frac{\widehat{\deg}(\overline{M}(m\overline{D})^{\cdot d} \cdot \overline{F}(m\overline{D}))}{m^{d+1}} = 0.$$

6 Applications

Definition 6.1. An arith. Zariski decomposition of a big arith. $\mathbb{R}$-divisor $\overline{D}$ is a sum $\overline{D} = \overline{P} + \overline{N}$ s.t. $\overline{P}$ is a nef arith. $\mathbb{R}$-divisor, $\overline{N}$ is an effective arith. $\mathbb{R}$-divisor, and $\widehat{\operatorname{vol}}(\overline{P}) = \widehat{\operatorname{vol}}(\overline{D})$.

Remark 6.1. (1) If $\dim X = 2$, then an arith. Zariski decomposition of a big $\overline{D}$ always exists and unique (Moriwaki [5]).

(2) If $\dim X \geqslant 3$, there exists no arith. Zariski decomposition in general even after any blow-up of X (Moriwaki '11).

Example 6.1. Let $\mathbb{P}^2_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[X_0, X_1, X_2])$ and let $z_i := X_i/X_0$ be the affine coordinate. Let $H := \{X_0 = 0\}$ and let

$$g := \max\left\{-2, \log|X_1/X_0|^2 + 2, \log|X_2/X_0|^2 + 2\right\},$$

which is an H-Green function of PSH-type. Moreover, we can add a "bump" $\rho : \mathbb{P}^2(\mathbb{C}) \to \mathbb{R}_{\geqslant 0}$ such that

$$\mathrm{Supp}(\rho) \Subset \{|z_1| < \exp(-2)\} \times \{|z_2| < \exp(-2)\}.$$

Then $\overline{H} = (H, g + \rho)$ are big and non-nef ($h_{\overline{H}}(1 : 0 : 0) < 0$ or $g + \rho$ is not of PSH-type).

Blow up $\mathbb{P}^2_{\mathbb{Z}}$ with center $(1 : 0 : 0)$, viz. over $\{X_0 \neq 0\}$,

$$\varphi : \mathrm{Proj}(\mathbb{Z}[z_1, z_2][Y_1, Y_2]/(z_2Y_1 - z_1Y_2)) \to \{X_0 \neq 0\}.$$

Then $\varphi^*\overline{H}$ admits an arith. Zariski decomposition. Let E be the exceptional divisor and let $w_{ij} := Y_j/Y_i$. Then the positive part is given by

$$\overline{P} = \left(\varphi^*H - \frac{1}{2}E, \max\left\{\log|z_iw_{i1}|, \log|z_iw_{i2}|, \log|z_iw_{i1}|^2 + 2, \log|z_iw_{i2}|^2 + 2\right\}\right),$$

the negative part is

$$\overline{N} = \left(\frac{1}{2}E, \max\left\{0, -2 - \max\left\{\log|z_iw_{i1}|, \log|z_iw_{i2}|\right\}\right\} + \varphi^*\rho\right) \geqslant 0,$$

and $\widehat{\mathrm{vol}}(\overline{H}) = \widehat{\mathrm{vol}}(\overline{P}) = 5/4$.

Corollary 6.2. *Let $\overline{P}, \overline{Q}$ be nef and big arith. $\mathbb{R}$-divisors. If $\widehat{\mathrm{vol}}(\overline{P}) = \widehat{\mathrm{vol}}(\overline{Q})$ and $\overline{P} \geqslant \overline{Q}$, then $\overline{P} = \overline{Q}$.*

Proof.

$$2\widehat{\mathrm{vol}}(\overline{P})^{\frac{1}{d+1}} = \widehat{\mathrm{vol}}(\overline{P})^{\frac{1}{d+1}} + \widehat{\mathrm{vol}}(\overline{Q})^{\frac{1}{d+1}} \leqslant \widehat{\mathrm{vol}}(\overline{P} + \overline{Q})^{\frac{1}{d+1}} \leqslant \widehat{\mathrm{vol}}(2\overline{P})^{\frac{1}{d+1}}.$$

Thus, by Theorem 4.5, $\exists \phi \in \mathrm{Rat}(X)^\times \otimes_{\mathbb{Z}} \mathbb{R}$ s.t. $\overline{P} - \overline{Q} = \widehat{(\phi)} \geqslant 0$.

$$\widehat{(\phi)} \geqslant 0 \quad \Leftrightarrow \quad \widehat{(\phi)} = 0 \quad (\Leftrightarrow \quad \phi \in H^0(\mathcal{O}_X^*) \otimes_{\mathbb{Z}} \mathbb{R}).$$

□

Corollary 6.3. *An arith. Zariski decomposition of a big arith. $\mathbb{R}$-divisor is (if it exists) unique.*

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