代数幾何学シンポジウム記録
2014年度 pp.80-87

GOOD REDUCTION CRITERION FOR K3 SURFACES

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Abstract. This proceeding article is based on the author's talk at the Kinosaki symposium on algebraic geometry on 2014/10/23.

We prove a Néron–Ogg–Shafarevich type criterion for good reduction of K3 surfaces: unramified Galois action on the second l-adic cohomology of a K3 surface implies good reduction after a finite unramified extension of the base field. The proof involves birational geometry of certain threefolds of mixed characteristic. This is a joint work with Christian Liedtke.

1. Introduction — complex case

We first consider the problem of good reduction in the classical complex case.

Let $\Delta \subset \mathbb{C}$ be a (small) disc containing the point 0, and let $\Delta^* = \Delta \setminus \{0\}$ the punctured disc. We consider the following problem: given a family $X \to \Delta^*$ of smooth proper varieties, does there exist an extension $\mathcal{X} \to \Delta$ (also smooth proper over 0)? If such an extension exists, we say that X has *good reduction*.

There is an obvious necessary condition. Fix a point $\eta \in \Delta^*$. Then the fundamental group $\pi_1(\Delta^*, \eta)$ (which is an infinite cyclic group generated by the loop around 0) acts on (Betti) cohomology groups $H^i(X_\eta, \mathbb{Z})$ (here X_η denotes the fiber above η). This is called the monodromy action.

Proposition 1.1. *If X has good reduction, then the monodromy action is trivial.*

Proof. Indeed, if X extends to a family $\mathcal{X} \to \Delta$, then the monodromy action factors the group $\pi_1(\Delta, \eta)$, which is trivial. □

One can ask the converse problem: whether trivial monodromy action (on all H^i) implies good reduction. In general this does not hold, even for (families of) curves of genus ≥ 2. But it does hold if X is a family of abelian varieties, see Theorem 2.3.

In the case of K3 surfaces, there is the following result of Kulikov and Persson–Pinkham.

Date: 2014/12/19.

Theorem 1.2 (Kulikov [Kul77, Theorems I and II], Persson–Pinkham [PP81])**.** *Let $X \to \Delta^*$ be a family of K3 surfaces.*

(1) After replacing Δ^* by a finite covering (which we also denote by Δ^*), there exists an extension $\mathcal{X} \to \Delta$ which is proper and semistable (that is, $\mathcal{X}_0$ is a normal crossing divisor of $\mathcal{X}$), and with relative canonical divisor $K_{\mathcal{X}/\Delta} = 0$.

(2) There are three possible types of special fiber $\mathcal{X}_0$ of such $\mathcal{X}$. Type I is smooth K3 surfaces, and types II and III have more than one irreducible components[1]*. If $\mathcal{X}_0$ is of type II or III then the monodromy action of $\pi_1(\Delta^*, \eta)$ on $H^2(X_\eta, \mathbb{Z})$ is non-trivial.*

As a consequence, trivial monodromy action implies potential good reduction[2]*.*

Remark 1.3. Even if $X \to \Delta^*$ is algebraic, their construction of $\mathcal{X}$ involves non-algebraic transformations, and $\mathcal{X}$ may not be algebraic.

2. Algebraic case

Now we consider the algebraic version of the problem.

First we explain the setting. In place of the disc Δ and the punctured disc Δ^*, we consider $\operatorname{Spec} \mathcal{O}_K$ and $\operatorname{Spec} K$, where K is a complete discrete valuation field and $\mathcal{O}_K \subset K$ is its valuation ring. The point $0 \in \Delta$ corresponds to $\operatorname{Spec} k$, where $k = \mathcal{O}_K/\mathfrak{m}_K$ is the residue field.

Example 2.1. A typical example is $K = \mathbb{Q}_p$ (p-adic numbers): then $\mathcal{O}_K = \mathbb{Z}_p$ (p-adic integers) and $k = \mathbb{F}_p$. We can also consider finite extensions of K, or the maximal unramified extensions of such fields.

Another typical example is $\mathcal{O}_K = F[[t]]$ (formal power series) for a field F: then $K = F((t)) = F[[t]][t^{-1}]$ and $k = F$. If $F = \mathbb{C}$ then this is very close to the case considered in the previous section. We can also consider finite extensions of K, but such fields are always of the form $F'((u))$ for some finite extension F'/F.

In the following we restrict to the case where $\operatorname{char} K = 0$ and k is perfect (of characteristic $p \geq 0$).

As in the previous section we consider the following problem. Given a smooth proper variety X over K, does there exist an extension $\mathcal{X}$ which is smooth and proper over $\mathcal{O}_K$? If such $\mathcal{X}$ exists we say that X has *good reduction*.

Although this setting looks parallel to the previous section, the present situation differs from the complex case in the following points.

[1]They have (and use) further informations on the components and the configuration, but we omit the details.

[2]"potential" means "after changing the base by a finite extension".

First, when $p > 0$, the resolution of singularity and the potential semistable reduction is only known for special cases (e.g. $\dim X = 1$). We therefore need some technical assumption on our main theorems. Second, the residue field k is not necessarily algebraically closed. This becomes essential in some of our main theorems, as we will see later.

We now explain an (obvious) necessary condition concerning the monodromy action. For (any) variety X over K, and for any integer i and prime l, the i-th l-adic étale cohomology group $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_l)$ is defined. It is a $\mathbb{Z}_l$-module and it is equipped with an action of $G_K = \mathrm{Gal}(\overline{K}/K)$. Define the *inertia subgroup* I_K of G_K by $I_K = \mathrm{Ker}(G_K \to G_k)$. We say a representation of G_K is *unramified* if I_K acts trivially.

Proposition 2.2. *Assume $l \neq p$. If a smooth proper variety X has good reduction, then $H^i_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_l)$ is an unramified representation.*

This is indeed an analogue of Proposition 1.1 because it says that, if X extends to $\mathcal{X} \to \operatorname{Spec} \mathcal{O}_K$, then the action of $\pi_1^{\text{ét}}(\operatorname{Spec} K) = \mathrm{Gal}(\overline{K}/K)$ on $H^i_{\text{ét}}$ factors through $\pi_1^{\text{ét}}(\operatorname{Spec} \mathcal{O}_K) = \mathrm{Gal}(\overline{k}/k)$ (here $\pi_1^{\text{ét}}$ denotes the étale fundamental group).

In case of abelian varieties the converse is true (and it suffices to check H^1 for a single l):

Theorem 2.3 (Serre–Tate [ST68, Theorem 1])**.** *An abelian variety A over K has good reduction if and only if $H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Z}_l)$ is an unramified representation for some prime $l \neq p$. (And if these conditions are satisfied, then $H^i_{\text{ét}}(A_{\overline{K}}, \mathbb{Z}_l)$ is unramified for any i and any $l \neq p$.)*

Our main theorems, stated in the next section, are K3 versions of this Serre–Tate theorem and algebraic versions of Kulikov and Persson–Pinkham's result.

Remark 2.4. Although these results are limited to l-adic cohomology with $l \neq p$, there is also a theory of p-adic representations, in which crystalline representations are supposed to play the role of unramified ones in the l-adic theory. But we do not pursue this in this article.

3. MAIN THEOREMS

We state the theorems in this section, and give sketches of proofs in the later sections.

First we (the author) showed the following.

Theorem 3.1 (M. [Mat14, Theorem 1.1])**.** *Let X be a K3 surface over K and $l \neq p$ a prime. Assume $p \neq 2, 3$. Assume the following condition:*

($*$) *There exists a smooth proper surface X', birational to X, having potential semistable reduction (in the category of schemes)*[3].

If $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_l)$ is an unramified representation, then X has potential good reduction, that is, there exists a finite extension K'/K and an algebraic space $\mathcal{X}$ smooth proper over $\operatorname{Spec} \mathcal{O}_{K'}$ *satisfying $\mathcal{X} \otimes_{\mathcal{O}_{K'}} K' \cong X_{K'}$.*

Remark 3.2. It is essential to consider algebraic spaces: the theorem fails if we replace "algebraic space" with "scheme". We gave a counterexample in [Mat14, Section 5.2].

On the other hand, if we allow the model $\mathcal{X}$ to have some "mild" singularities, then we have a positive result in the category of schemes. We omit the precise statement since it (which required field extensions) is now superseded by Theorem 3.4 (c) below.

Remark 3.3. If $p = 0$, assumption ($*$) (in fact, the stronger statement that X has potential semistable reduction) is known to be true for any variety X. But it is open even for (K3) surfaces if $p > 0$. One known case is the following: if (a K3 surface) X admits an ample line bundle L such that $L^2 + 4 < p$ then ($*$) is true (see [Mau14, Section 4] and [Mat14, Section 3]).

Recently we (the author with Christian Liedtke) made this theorem more precise:

Theorem 3.4 (Liedtke–M. [LM14, Theorems 5.1 and 6.2])**.** (a) *In Theorem 3.1, we can take K'/K to be an unramified extension*[4].

(b) *But we cannot take $K' = K$ in general.*

(c) *Under the assumptions of Theorem 3.1, there exists a* scheme $\mathcal{Z}$ *proper flat over* $\operatorname{Spec} \mathcal{O}_K$ *(without extending K) such that $\mathcal{Z} \otimes_{\mathcal{O}_K} K$ is isomorphic to X and $(\mathcal{Z}_0)_{\overline{k}}$ has only rational double point (RDP) singularities.*

Summarizing, we have the following equivalence.

Corollary 3.5. *A K3 surface (satisfying ($*$)) has good reduction over a finite unramified extension if and only if its l-adic H^2 is unramified. We cannot drop the phrase "over a finite unramified extension".*

Remark 3.6. A natural question is the uniqueness (up to changing K') of $\mathcal{X}$ and $\mathcal{Z}$ in the theorem, after fixing a polarization L of X

[3]That is, there exists a finite extension K'/K and a scheme $\mathcal{X}'$ proper flat over $\mathcal{O}_{K'}$ such that $\mathcal{X}' \otimes_{\mathcal{O}_{K'}} K'$ is isomorphic to X' and $\mathcal{X}'_0$ is a strict normal crossing divisor of $\mathcal{X}'$.

[4]K'/K is said to be *unramified* if $\mathfrak{m}_K$ generates $\mathfrak{m}_{K'}$ as an $\mathcal{O}_{K'}$-module, or equivalently, if the inclusion $I_{K'} \subset I_K$ of the inertia subgroups is an equality.

and assuming a further condition on extension of L to $\mathcal{X}$ or $\mathcal{Z}$. For (c), there exists a unique $\mathcal{Z}$ on which L extends as a polarization. For (a), if we require L to extend to a polarization on $\mathcal{X}$, then such $\mathcal{X}$ do not exist in general. Instead, it is natural to require L to extend to a quasi-polarization on $\mathcal{X}$ (i.e., to be a big and nef line bundle on $\mathcal{X}_0$), and then such $\mathcal{X}$ exists. But in this case uniqueness does not hold in general. There may be more than one such models, and they are connected by flops (in the sense of [LM14, Section 3.3]).

4. Proof of Theorem 3.1

(Details: [Mat14, Sections 2, 3])

In this section we freely replace the base field K by finite extensions.

Start from a proper semistable model $\mathcal{X}'$ of a surface X' birational to X (possibly after extending K) whose existence is assured by $(*)$.

First we will construct a model as in Theorem 1.2 (1) (that is, semistable and relative $K = 0$). To do this, we apply minimal model program (MMP) to get a "minimal" model, and then resolve the singularities (which are mild and well-known). This method of obtaining a semistable model is taken from Maulik's paper [Mau14, Section 4].

We apply on $\mathcal{X}$ the semistable MMP for 3-dimensional mixed characteristic schemes, which is accomplished by Kawamata [Kaw94]. The program terminates (possibly after extending K) at $\mathcal{X}''$ which is either a minimal model ($K_{\mathcal{X}''/\mathcal{O}_K}$ nef) or a Mori fiber space (whose description we omit). In our case (of a K3 surface X), always the former occur, the generic fiber of $\mathcal{X}''$ is isomorphic (not only birational) to X, and we have $K_{\mathcal{X}''/\mathcal{O}_K} = 0$.

By the classification of log terminal singularities (Kawamata [Kaw94, Section 4]) and the fact that $K_{\mathcal{X}''/\mathcal{O}_K}$ is Cartier (not only $\mathbb{Q}$-Cartier) in our case, the possible non-smooth points of $\mathcal{X}''$ are of the following two types: either semistable, or $\mathcal{X}''_0$ has rational double point singularity (over $\overline{k}$) at that point (this latter may be viewed as an arithmetic analogue of compound Du Val singularities).

To get a semistable model, we want to resolve the latter type of singularities (without changing the generic fiber). Unfortunately, such resolution does not always exist in the category of schemes. Fortunately, Artin's result [Art74] assures that such resolution $\mathcal{X}''' \to \mathcal{X}''$ exists in the category of *algebraic spaces* (possibly after extending K). Thus we obtain a semistable model $\mathcal{X}'''$ of X.

The possible special fibers of such $\mathcal{X}'''$ is classified by Nakkajima [Nakk00, Proposition 3.4] in the positive characteristic case. The conclusion is the same as Kulikov's list.

Now we want to compute the Galois action on the cohomology $H^2_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_l)$ $(\otimes_{\mathbb{Z}_l}\mathbb{Q}_l)$ in terms of cohomology of (intersections of) irreducible components of $\mathcal{X}_0'''$. If the semistable model $\mathcal{X}'''$ is a scheme, then we can do this using the Rapoport–Zink spectral sequence (parallel to the Steenbrink spectral sequence in the complex case). We showed [Mat14, Section 2] the existence and the needed properties of this spectral sequence for the case $\mathcal{X}'''$ is an algebraic space (with some conditions). Using it we can compute the Galois action and obtain the same conclusion as in Kulikov's complex case: I_K acts non-trivially if $\mathcal{X}'''$ is not smooth.

5. Proof of Theorem 3.4(b)

(Details: [LM14, Sections 3, 6])

In essence, the non-existence of the resolution of the following type is the obstruction for having good reduction without extending K.

Lemma 5.1. *Let $p \geq 5$ and $d \in \mathbb{Z}_p^*$ be such that $d \notin \mathbb{Z}_p^{*2}$. Let $A = \mathbb{Z}_p[x,y,z]/(xy + z^2 - dp^2)$. Then A does not admit a simultaneous resolution over $\mathbb{Z}_p$: that is, there does not exist an algebraic space $\mathcal{Y}$ smooth over $\mathbb{Z}_p$ with a morphism $\psi\colon \mathcal{Y} \to \operatorname{Spec} A$ which is isomorphic on the generic fiber and is the resolution of the (RDP) singularities on the special fiber.*

The reader might recall the singularity $(xy - zw = 0) \subset \mathbb{A}^4$ and the (Atiyah) flop between the two resolutions.

$A \otimes_{\mathbb{Z}_p} \mathbb{Z}_p[\sqrt{d}]$ does admit a simultaneous resolution over $\mathbb{Z}_p[\sqrt{d}]$: blowing up either $(x, z-\sqrt{d}\cdot p)$ or $(x, z+\sqrt{d}\cdot p)$ gives a desired resolution (each of these ideals are principal except exactly at the singular point). But these ideals are not defined over $\mathbb{Z}_p$. This is not a proof, but this explains the idea of the example.

Now we construct a K3 surface which would be a counterexample.

Let $\mathcal{X}' \subset \mathbb{P}^3_{\mathbb{Z}_p}$ be a "quartic surface" satisfying the following conditions: $X = \mathcal{X}'_{\mathbb{Q}_p}$ is a smooth K3 surface; $\mathcal{X}' \to \operatorname{Spec} \mathbb{Z}_p$ is smooth outside a finite set $\Sigma \subset \mathcal{X}'_0$; each $q \in \Sigma$ is a RDP of $\mathcal{X}'_0$, and the ring $\mathcal{O}_{\mathcal{X}',q}$ is "as in the previous lemma".

Proposition 5.2. *This X does not have good reduction over $\mathbb{Q}_p$. (But it does have good reduction over $\mathbb{Q}_p(\sqrt{d})$.)*

Since X has good reduction over $\mathbb{Q}_p(\sqrt{d})$, we see that H^2 of X is unramified as a representation of $G_{\mathbb{Q}_p(\sqrt{d})}$. But since $\mathbb{Q}_p(\sqrt{d})/\mathbb{Q}_p$ is unramified, this is equivalent to saying that it is unramified as a representation of $G_{\mathbb{Q}_p}$. Thus this X gives a desired counterexample.

Assuming the lemma, it is easy to show the non-existence of a smooth proper model admitting a morphism to $\mathcal{X}'$. But what we have to show is the non-existence without assuming such a morphism. To show this, we prove appropriate versions of existence and termination of flop.

Proof. Assume there exists a smooth proper model $\mathcal{Y}$ of X over $\mathbb{Z}_p$.

Take an ample line bundle L on $\mathcal{X}'$ (for example $\mathcal{O}(1)$), and denote by $L_{\mathcal{Y}_0}$ the restriction of the transform of L.

If $L_{\mathcal{Y}_0}$ is (big and) nef on $\mathcal{Y}_0$, then the linear system $|L_{\mathcal{Y}}^{\otimes m}|$ defines a morphism $\mathcal{Y} \to \mathcal{X}'$, a simultaneous resolution of $\mathcal{X}'$, which contradicts the previous lemma.

Now assume $L_{\mathcal{Y}_0}$ is not nef. We want to replace $\mathcal{Y}$ with another smooth proper model on which L becomes nef. We prove:

(Existence of flop): Take a $L_{\mathcal{Y}_0}$-negative curve $C \subset \mathcal{Y}_0$. There exists another smooth proper model $\mathcal{Y}^+$ of X and a rational map

$$\mathcal{Y} \dashrightarrow \mathcal{Y}^+,$$

inducing an isomorphism outside C:

$$\mathcal{Y} \setminus C \xrightarrow{\sim} \mathcal{Y}^+ \setminus C^+,$$

with the ("flopped") curve $C^+ \subset \mathcal{Y}^+$ being $L_{\mathcal{Y}_0^+}$-positive.

(Termination of flop): Starting from $(\mathcal{Y}, L)$ and repeatedly applying the flop on the previous paragraph, we eventually obtain $\mathcal{Y} \dashrightarrow \mathcal{Y}^{++}$, $\mathcal{Y}^{++}$ another smooth proper model such that $L_{\mathcal{Y}_0^{++}}$ is big and nef.

Thus we reduce to the nef case and get a contradiction. □

6. Proof of Theorem 3.4(a)(c)

(Details: [LM14, Sections 4, 5])

By Theorem 3.1, we have $\mathcal{X}$ smooth proper over $\mathcal{O}_{K'}$, K'/K a (possibly ramified) finite extension. We may assume K'/K is Galois. Let M the maximal unramified extension of K inside K'. The idea is to equip $\mathcal{X}$ with a $\mathrm{Gal}(K'/M)$-action and take the quotient. However, the Galois action on the generic fiber $X_{K'}$ does not always extend to an action on $\mathcal{X}$, even under the assumption that H^2 is unramified. In order to get a Galois action we have to replace $\mathcal{X}$ using flops.

Take an ample line bundle L on X. By the flop technique (as in the previous section), we replace $\mathcal{X}$ with another model such that $L_{\mathcal{X}_0}$ is big and nef. Then $|L_{\mathcal{X}}^{\otimes m}|$ defines a morphism $\pi \colon \mathcal{X} \to \mathcal{Z} \subset \mathbb{P}^N_{\mathcal{O}_{K'}}$.

Clearly $\mathcal{Z}$ is a projective scheme with $\mathrm{Gal}(K'/K)$ action. We observe that π is birational, π is isomorphic on the generic fiber, and the (possible) singularity of $\mathcal{Z}_0$ are rational double points.

Take a $g \in \mathrm{Gal}(K'/M)$, and assume that the action of g on $\mathcal{Z}$ does not extend to $\mathcal{X}$, that is, $g^*\mathcal{X} \ncong \mathcal{X}$. Then this gives rise to a cycle in H^2 on which g acts non-trivially, and contradicts the assumption that H^2 is unramified. Therefore $\mathrm{Gal}(K'/M)$ acts on $\mathcal{X}$ (but the whole $\mathrm{Gal}(K'/K)$ does not act in general).

Next we show that $\mathcal{X}/\mathrm{Gal}(K'/M)$ is smooth. Since K'/M is totally ramified (in other words, the corresponding extension of residue fields is trivial), the action of $\mathrm{Gal}(K'/M)$ on H^2 of the special fiber of $\mathcal{X}_0$ is trivial. This implies that its action on $\mathcal{X}_0$ itself is trivial. If $[K' : M]$ is prime to p, then this easily implies that $\mathcal{X}/\mathrm{Gal}(K'/M)$ is smooth. If $[K' : M]$ is divisible by p, the situation is more complicated due to (possible) existence of infinitesimal actions, but in our case this does not happen because a K3 surface does not admit a (non-trivial) global derivation. It follows that $\mathcal{X}/\mathrm{Gal}(K'/M)$ is smooth in any case. This proves (a).

It remains to show (c). Form $\pi\colon \mathcal{X}/(\mathrm{Gal}(K'/M)) \to \mathcal{Z}'$ as above. Then $\mathrm{Gal}(M/K)$ acts on $\mathcal{Z}'$ and the quotient $\mathcal{Z}'/\mathrm{Gal}(M/K)$ gives a model satisfying the conditions of (c).

References

[Art74] M. Artin, *Algebraic construction of Brieskorn's resolutions*, J. Algebra **29** (1974), 330–348.

[Kaw94] Yujiro Kawamata, *Semistable minimal models of threefolds in positive or mixed characteristic*, J. Algebraic Geom. **3** (1994), no. 3, 463–491.

[Kul77] Viktor S. Kulikov, *Degenerations of $K3$ surfaces and Enriques surfaces*, Izv. Akad. Nauk SSSR Ser. Mat. **41** (1977), no. 5, 1008–1042, 1199 (Russian).

[LM14] Christian Liedtke and Yuya Matsumoto, *Good reduction of K3 surfaces* (2014), available at `http://arxiv.org/abs/1411.4797v1`.

[Mat14] Yuya Matsumoto, *Good reduction criterion for K3 surfaces*, Math. Z. (2014), to appear, available at `http://arxiv.org/abs/1401.1261v1`.

[Mau14] Davesh Maulik, *Supersingular K3 surfaces for large primes*, Duke Math. J. **163** (2014), no. 13, 2357–2425. With an appendix by Andrew Snowden.

[Nakk00] Yukiyoshi Nakkajima, *Liftings of simple normal crossing log $K3$ and log Enriques surfaces in mixed characteristics*, J. Algebraic Geom. **9** (2000), no. 2, 355–393.

[PP81] Ulf Persson and Henry Pinkham, *Degeneration of surfaces with trivial canonical bundle*, Ann. of Math. (2) **113** (1981), no. 1, 45–66.

[ST68] Jean-Pierre Serre and John Tate, *Good reduction of abelian varieties*, Ann. of Math. (2) **88** (1968), 492–517.

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