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UNIRATIONALITY AND PURELY INSEPARABLE ISOGENIES OF SUPERSINGULAR K3 SURFACES

CHRISTIAN LIEDTKE

Rationality and Unirationality – the Lüroth problem

In 1876, Lüroth proved that if $k \subset L \subset k(t)$ is a field extension such that L is of transcendence degree 1 over k, then L itself is a purely transcendental extension of k, that is, $L = k(u)$ for some $u \in L$. Then, he asked, whether it is true in general that for every field extension

$$k \subset L \subset k(t_1, ..., t_n)$$

such that L is of transcendence degree n over k, it is true that L itself is of the form $k(u_1, ..., u_n)$ – this question is known as the *Lüroth problem*. The answer is "yes" in the following cases

- If $n = 1$ (Lüroth).
- If $n = 2$ and $k = \mathbb{C}$ (Castelnuovo).
- If $n = 2$, k is algebraically closed of positive characteristic, and the extension $k(t_1, t_2)/L$ is separable (Zariski).

On the other hand, the answer is in general "no" in the following cases

- If $n = 2$ and k is algebraically closed of positive characteristic (Zariski).
- If $n = 3$ and $k = \mathbb{C}$ (Iskovskikh–Manin, Clemens–Griffiths, Artin–Mumford).

Now, let us recall the following notions from algebraic geometry, which translate Lüroth's problem into birational geometry of algebraic varieties.

Definition. A n-dimensional variety X over a field k is called *rational* (resp. *unirational*) if there exists a dominant rational (resp. birational) map $\mathbb{P}^n_k \dashrightarrow X$.

We note that X is unirational if and only if there exists an inclusion of function fields $k \subset k(X) \subseteq k(\mathbb{P}^n)$, and that $k(\mathbb{P}^n) = k(t_1, ..., t_n)$. Thus, Lüroth's problem is equivalent to asking whether unirational varieties are rational. By the above, unirational curves are rational, unirational surfaces over $\mathbb{C}$ are rational, and separably unirational surfaces over algebraically closed fields of positive characteristic are rational.

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Supersingular surfaces

In 1958, Zariski [Za58] gave the first examples of unirational surfaces over algebraically closed fields of positive characteristic that are not rational. Interestingly, it is still a difficult question to decide whether a given surface is unirational or not. Let us first give a necessary condition. To state it, we denote for a smooth and proper variety by ρ the rank of its Néron–Severi group, and by b_i the rank of its i.th (ℓ-adic, crystalline) cohomlogy group. By a classical result of Igusa [Ig60], we have $\rho \leq b_2$.

Theorem (Shioda [Sh74])**.** *If X is a smooth, proper, and unirational variety, then $\rho(X) = b_2(X)$.*

Thus, unirational varieties have no transcendental cycles in their second étale and crystalline cohomology. Moreover, if X is a smooth and proper over an algebraically closed field k of positive characteristic, then the image of the crystalline first Chern class map c_1 from $\mathrm{Pic}(X)$ to $H^2_{\mathrm{cris}}(X/W)$ gives rise to an F-subcrystal of $H^2_{\mathrm{cris}}(X/W)$ that is of slope 1. In particular, if X satisfies $\rho(X) = b_2(X)$, then the whole F-isocrystal $H^2_{\mathrm{cris}}(X/W) \otimes K$ is of slope 1. These two results give rise to the following definitions.

Definition. Let X be a smooth and proper surface over an algebraically closed field of positive characteristic. Then, X is called

- *Shioda-supersingular* if $\rho(X) = b_2(X)$, and
- *Artin-supersingular* if $H^2_{\mathrm{cris}}(X/W)$ is of slope 1.

By the above discussion, these notions are related as follows

$$\begin{array}{rl} & X \text{ is unirational} \\ \Rightarrow & X \text{ is Shioda-supersingular} \\ \Rightarrow & X \text{ is Artin-supersingular.} \end{array}$$

Quite generally, Tate [Ta65] conjectured that for a smooth and proper variety X over $\overline{\mathbb{F}}_p$ the F-sub-isocrystal of $H^2_{\mathrm{cris}}(X/W) \otimes K$ arising from $c_1(\mathrm{Pic}(X))$ is not only contained in, but actually equal to the slope 1 sub-isocrystal of $H^2_{\mathrm{cris}}(X/W) \otimes K$. This conjecture thus characterizes classes in $H^2_{\mathrm{cris}}(X/W)$ arising from Chern classes of line bundles, and should be thought of as a characteristic-p analog of the Lefschetz theorem on $(1,1)$-classes. This conjecture is still wide open. However, if true, it would imply the equivalence of the notions of Shioda-supersingularity and Artin-supersingularity.

On the other hand, there do exist Godeaux surfaces (minimal surfaces of general type with $p_g = q = 0$ and $K^2 = 5$) in every characteristic $p \equiv 1$ mod 5 that are Shioda-supersingular but *not* unirational [Sh77]. Thus, a Shioda-supersingular surface need not be unirational in general.

Supersingular K3 surfaces

Since the question of unirationality of surfaces in positive characteristic is still far from settled, we now consider a particular interesting special class that is a good testing ground.

Definition. A *K3 surface* is a smooth and proper surface X over a field k such that $\omega_X \cong \mathcal{O}_X$ and $h^1(\mathcal{O}_X) = 0$.

Classical examples of K3 surfaces are smooth quartic hypersurfaces in $\mathbb{P}^3$, as well as Kummer surfaces. By definition, the latter are the unique minimal resolution of singularities of $A/\pm\mathrm{id}$, where A is an Abelian variety of dimension 2 in characteristic $\neq 2$. By recent progress on the Tate-conjecture for K3 surfaces, we now have the following theorem.

Theorem (Charles [Ch13], Madapusi-Pera [M13], Maulik [Mau14])**.** *In odd characteristic, a K3 surface is Shioda-supersingular if and only if it is Artin-supersingular.*

Thus, in odd characteristic, we may and will simply talk about *supersingular K3 surfaces.* Their study was initiated around 1973 by Artin [Ar74] and Shioda [Sh72], [Sh73], [Sh74], [Sh75]. Then, it was continued by Rudakov and Shafarevich (see [RS81], for example), and Ogus [Og79], [Og83], and is still an active area of research. For a survey, we refer the interested reader to [Sh79] or [RS81].

Conjecture (Artin, Rudakov, Shafarevich, Shioda)**.** A K3 surface is unirational if and only if it is supersingular.

Until 2013, this conjecture was established in the following cases:

- For Shioda-supersingular K3 surfaces in characteristic 2 (Rudakov–Shafarevich [RS78]).
- For $\sigma_0 \leq 6$ in characteristic 3 (Rudakov–Shafarevich [RS78]).
- For $\sigma_0 \leq 3$ in characteristic 5 (Pho–Shimada [PS06]).
- For supersingular Kummer surfaces in every characteristic $p \geq 3$ (Shioda [Sh77]).

Here, σ_0 denotes the Artin invariant of the supersingular K3 surface, to which we will come back below. In particular, thanks to Shioda's result, we have at least examples of supersingular K3 surfaces that are unirational in *every* odd characteristic. This result is achieved by explicitly finding dominant and rational maps from $\mathbb{P}^2$ to certain Fermat surfaces, which was established in [Sh74].

For a supersingular K3 surface in odd characteristic, the Néron–Severi group is of rank 22, and its discriminant satisfies

$$\operatorname{disc}\mathrm{NS}(X) \;=\; -p^{2\sigma_0} \quad \text{for some integer} \quad 1 \leq \sigma_0 \leq 10.$$

This integer is called the *Artin-invariant* of the supersingular K3 surface and was introduced in [Ar74]. By [RS78], this integer σ_0 already determines the Néron–Severi lattice of the supersingular K3 surface up to isometry and such lattices are called *supersingular K3 lattices*. For example, Ogus [Og79] showed that a supersingular K3 surface satisfies $\sigma_0 \leq 2$ if and only if it is a Kummer surface associated to a supersingular Abelian surface (for an Abelian variety A, being supersingular means that the p-torsion subgroup scheme $A[p]$ is an infinitesimal group scheme).

The crystalline Torelli theorem

For supersingular K3 surfaces, there exists a period domain in terms of crystals, as well as a Torelli theorem, both of which are due to Ogus [Og79], [Og83]. More precisely, if X is a supersingular K3 surface of Artin invariant σ_0, then the second crystalline cohomology group $H := H^2_{\mathrm{cris}}(X/W)$ has the following properties:

- H is a free W-module of rank 22, where W denotes the ring of Witt vectors of k.
- It carries a Frobenius-linear operator Φ, as well as an intersection form $\langle -, - \rangle$.
- The *Tate-module*

$$T_H := \{ x \in H \mid \Phi(x) = px \} \subset H$$

  is a free $\mathbb{Z}_p$-module of rank 22 and the intersection form restricted to T_H is $\mathbb{Z}_p$-valued of discriminant $-p^{2\sigma_0}$.

The collection of this data (plus some compatibility conditions) is called a *supersingular K3 crystal*, and we refer to [Og79] for details and precise definitions. Given a supersingular K3 lattice N of Artin invariant σ_0, a supersingular K3 crystal $(H, T_H, \Phi, \langle -, - \rangle)$ together with an isometric embedding $N \to T_H$ is called an *N-rigidified supersingular K3 crystal.*

Theorem (Ogus [Og79])**.** *Given a supersingular K3 lattice N of Artin-invariant σ_0, there exists a moduli space*

$$\mathcal{M}_N \to \operatorname{Spec} \mathbb{F}_p$$

of N-rigidified supersingular K3 crystals. It is smooth, projective, and of dimension $\sigma_0 - 1$ over $\mathbb{F}_p$. Over $\overline{\mathbb{F}}_p$, it has two components.

This moduli space $\mathcal{M}_N$ will serve as the period domain for N-marked supersingular K3 surfaces. More precisely, there exists a moduli space $\mathcal{S}_N \to \operatorname{Spec} \mathbb{F}_p$ of *N-marked supersingular K3 surfaces*, that is, a moduli space for pairs $(X, \imath)$ consisting of a supersingular K3 surface X together with an isometric embedding $\imath : N \to \mathrm{NS}(X)$. Then, the assignment $X \mapsto H^2_{\mathrm{cris}}(X/W)$ gives rise to the *period map*

$$\pi_N : \mathcal{S}_N \to \mathcal{M}_N$$

for N-marked supersingular K3 surfaces. If $p \geq 5$, then π_N is étale, but only locally of finite type and not separated by [Og83]. To get a period map that is an isomorphism, one has to consider the moduli space $\mathcal{P}_N \to \mathcal{M}_N$

of N-marked supersingular K3 crystals together with ample (or effective) cones as introduced and described in [Og83]. One of the most important applications is the following Torelli theorem.

Theorem (Ogus [Og83])**.** *Two supersingular K3 surfaces in characteristic $p \geq 5$ are isomorphic if and only if their associated supersingular K3 crystals are isomorphic.*

We refer to [Og79] and [Og83] for details and precise statements, as well as to [Li14] for a recent overview.

The formal Brauer group

In this section, we rephrase supersingularity for K3 surfaces in terms of its formal Brauer group. Let us recall that Artin and Mazur [AM77] associated to a proper variety X over some field k the functors from local Artinian k-algbras with residue field k to Abelian groups

$$\begin{array}{cccc} \Phi^i_{X/k} & : & (\mathrm{Art}/k) & \to & (\text{Abelian groups}) \\ & & S & \mapsto & \ker\left(H^i_{\text{ét}}(X \times_k S, \mathbb{G}_m) \to H^i_{\text{ét}}(X, \mathbb{G}_m) \right) \end{array}$$

For example, since $H^1(-, \mathbb{G}_m)$ is isomorphic to the Picard group, it is not difficult to see that $\Phi^1_{X/k}$ is the completion of the Picard scheme $\mathrm{Pic}_{X/k}$ along its zero-section, which is why $\Phi^1_{X/k}$ is called the *formal Picard group*, denoted $\widehat{\mathrm{Pic}}_{X/k}$. Thus, $\Phi^1_{X/k}$ encodes the infinitesimal structure of $\mathrm{Pic}_{X/k}$ around the zero-section, such as (non-)reducedness, dimension, the Lie algebra,... For a K3 surface, the formal Picard group is zero, and thus, carries no interesting information about the surface.

In general, Artin and Mazur established a deformation-obstruction theory à la Schlessinger for the functors $\Phi^i_{X/k}$, where the tangent space is $H^i(\mathcal{O}_X)$ and the obstruction space is $H^{i+1}(\mathcal{O}_X)$. Thus, if X is a K3 surface, then $\Phi^2_{X/k}$ is pro-representable by a one-dimensional formal scheme that is formally smooth and an Abelian group object in the category formal schemes, that is, by an Abelian formal group law of dimension one. Since $H^2(-, \mathbb{G}_m)$ is the (cohomological) Brauer group, we have the following definition.

Definition. The formal group law $\Phi^2_{X/k}$ is called the *formal Brauer group* of X, denoted $\widehat{\mathrm{Br}}_{X/k}$.

Quite generally, if G is a one-dimensional formal group law over a field k, then $G = \mathrm{Spf}\, k[[t]]$ and the multiplication map $\mu : G \times G \to G$ gives rise to a map of completed k-algebras $\mu^{\#} : k[[t]] \to k[[x]]\hat{\otimes}k[[y]] = k[[x, y]]$. Then, $f(x, y) := \mu^{\#}(t)$ is formal power series that completely encodes the group law. The easiest examples are the following.

- If $f(x, y) = x + y$, then $G = \widehat{\mathbb{G}}_a$, the *formal additive group law.*
- If $f(x, y) = x+y+xy$, then $G = \widehat{\mathbb{G}}_m$, the *formal multiplicative group law.*

It turns out that $f(x,y)$ is congruent to $x+y$ modulo higher order terms, that associativity of G translates into $f(f(x,y),z) = f(x,f(y,z))$, and that being Abelian translates into $f(x,y) = f(y,x)$. Now, if k is a field, then all one-dimensional group laws are Abelian, and if, moreover, k is of characteristic zero, then they are all isomorphic to $\widehat{\mathbb{G}}_a$. If k is algebraically closed of positive characteristic p, then the multiplication-by-p-map $[p] : G \to G$ gives rise to a discrete invariant of G: namely, on the level of k-algebras, $[p]$ corresponds to a map $[p]^\# : k[[x]] \to k[[t]]$, which is completely determined by its value at $g(t) := [p]^\#(x) \in k[[t]]$. It turns out that either $g(t) = 0$ or that it is of the form $g(t) = t^{p^h}$ plus higher order terms for some integer $h \geq 1$. By definition, h is called the *height* of the formal group law, and one sets $h = \infty$ in first case. For example, we have

$$h(\widehat{\mathbb{G}}_a) \;=\; \infty \quad\text{ and }\quad h(\widehat{\mathbb{G}}_m) \;=\; 1\,.$$

In fact, there exists a formal group law over k of height h for every positive integer h. Moreover, two one-dimensional formal group laws over an algebraically closed field of positive characteristic are isomorphic if and only if they have the same height.

Another way of classifying such formal group laws (which has the advantage of generalizing to Abelian formal group laws of larger dimension) is the following: associated to a one-dimensional formal group law G over an algebraically closed field k of positive characteristic, there exists its *Cartier–Dieudonné module* DG, which is a module over the Witt ring $W = W(k)$ together with two operators, called Frobenius and Verschiebung. For example, the height h of G is equal to the number of generators of DG as W-module. The point is that for a smooth and proper variety X, the Cartier–Dieudonné module $D\widehat{\mathrm{Br}}_{X/k}$ controls the slope < 1 sub-F-isocrystal of $H^2_{\mathrm{cris}}(X/W) \otimes K$. For a thorough introduction to formal group laws, their classification, and Cartier–Dieudonné theory, we refer the interested reader to [Ha12]. For a K3 surface, these general facts translate into the following.

Proposition. *A K3 surface X over an algebraically closed field k of positive characteristic is Artin-supersingular if and only if $\widehat{\mathrm{Br}}_{X/k}$ is of infinite height.*

In fact, the condition $h(\widehat{\mathrm{Br}}_{X/k}) = \infty$ was Artin's original definition of supersingularity for K3 surfaces in [Ar74]. For details and references, we refer to [Ar74] and [Li14].

Moving torsors and purely inseparable isogenies

Now, we use the formal Brauer group to construct for a supersingular K3 surface over k a very special deformation over $\mathrm{Spec}\, k[[t]]$ that has the property that special and the generic fiber of this family are related by purely inseparable isogenies. More precisely, let $X \to \mathbb{P}^1$ be a K3 surface together with an elliptic fibration that has a section, called the zero-section. Let $A \to \mathbb{P}^1$ be the identity component of the associated Néron model, which arises from $X \to \mathbb{P}^1$ by removing the components of the fibers that do not meet the zero-section, as well as removing the remaining singularities of the fibers. Note that $A \to \mathbb{P}^1$ is a relative group scheme. If S is the (formal) spectrum of some complete, local, and Noetherian k-algebra with residue field k, then we call a Cartesian diagram of A-torsors

$$\begin{array}{ccc} A & \to & \mathcal{A} \\ \downarrow & & \downarrow \\ \mathbb{P}^1 & \to & \mathbb{P}^1 \times S \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \to & S \end{array}$$

a *moving torsor family*. For technical reasons (algebraizability), we will also assume that there exists some relative invertible sheaf of some fixed degree N on $\mathcal{A} \to \mathbb{P}^1 \times S$.

Now, moving torsor families are related to the formal Brauer group: namely, if $f : \mathcal{A} \to \mathbb{P}^1 \times S$ is a moving family of A-torsors, it corresponds to a class in $\ker(H^1_{\text{ét}}(\mathbb{P}^1 \times S, A) \to H^1_{\text{ét}}(\mathbb{P}^1, A))$. The condition on the existence of a relative degree-N invertible sheaf on f translates into being an N-torsion element in this kernel. From the Grothendieck–Leray spectral sequence

$$H^i_{\text{ét}}(\mathbb{P}^1 \times S, R^j f_* \mathbb{G}_m) \Rightarrow H^{i+j}_{\text{ét}}(\mathcal{A}, \mathbb{G}_m)$$

and the fact that $R^1 f_* \mathbb{G}_m$ is the relative Picard scheme, we see that there is a contribution of $H^1_{\text{ét}}(\mathbb{P}^1 \times S, A)$ to $H^2_{\text{ét}}(\mathcal{A}, \mathbb{G}_m)$. Putting these observations together, and chasing through several commutative diagrams and spectral sequences, we obtain the following proposition, part of which was already obtained in [AS73].

Proposition. *Let $X \to \mathbb{P}^1$ be a K3 surface with an elliptic fibration that has a section. Let R be a local, complete, and Noetherian k-algebra with residue field k. Then, there exists a bijection of sets between*

- *Moving-torsor families $f : \mathcal{A} \to \mathbb{P}^1 \times \operatorname{Spf} R$ such that there exists a relative invertible sheaf of degree N, and*
- *Elements of $\widehat{\operatorname{Br}}_{X/k}(R)[N]$, where $[N]$ denotes N-torsion.*

We shall be mainly interested in the case where $R = k[[t]]$, that is, non-infinitesimal families over DVR's. If G is a one-dimensional formal group law over k, and N is an integer, then it turns out that $G(k[[t]])[N]$ is non-zero if and only if k is of positive characteristic p, and $p|N$, and G is of infinite height.

Corollary. *Let $X \to \mathbb{P}^1$ be a K3 surface with an elliptic fibration that has a section. Then, non-trivial moving torsor families over $k[[t]]$ exist if and only if X is Artin-supersingular.*

In some form, this corollary must have been known already to Artin: in [Ar74], he found that the supersingular locus is 9-dimensional, conjectured that all supersingular K3's are elliptic, but then, in characteristic 2, he computed the space of supersingular K3's that are elliptic with a section to be of dimension 8. He explained this defect by what we called above moving torsor families, and remarked that "the unusual phenomenon of continuous families of homogeneous spaces occurs only for supersingular surfaces".

Now, given a non-trivial moving torsor family $\mathcal{A} \to \mathbb{P}^1 \times S$, it can be compactified, its singularities can be resolved, it is algebraizable, and we eventually obtain a family $\mathcal{X} \to S$ of supersingular K3 surfaces with special fiber X. Let us now assume that $S = \operatorname{Spec} R$ with $R = k[[t]]$, and that the family comes from a p-torsion element of $\widehat{\operatorname{Br}}_{X/k}(R)$. Then, the Picard group of the generic fiber injects into the Picard group of the special fiber and the cokernel is cyclic of order p, generated by the class of the zero-section of $X \to \mathbb{P}^1$ (the class of the zero-section does not extend to the generic fiber for otherwise it would trivialize the family of torsors, which would contradict the non-triviality of the family). Before proceeding, we have to introduce a new notion.

Definition. A *purely inseparable isogeny* between two K3 surfaces X, Y in characteristic p is a dominant, rational, and generically finite map $X \dashrightarrow Y$ such that the induced extension of function fields $k(Y) \subseteq k(X)$ is purely inseparable.

Coming back to our family $\mathcal{A} \to \mathbb{P}^1 \times S$, there exists a relative invertible sheaf of degree p (since we constructed it from a p-torsion element in the formal Brauer group), which gives rise to a multisection of degree p of that family. This multisection can be chosen to be purely inseparable over the base of the family, and thus, $\mathcal{A} \to \mathbb{P}^1 \times S$ can be trivialized after a purely inseparable base change of degree p. From this, it follows that the generic fiber and the special fiber of such a moving torsor family are related by purely inseparable isogenies. Putting all these observations together, we obtain the following result, which is the technical heart of [Li13].

Theorem. *Let $X \to \mathbb{P}^1$ be a supersingular K3 surface with an elliptic fibration that has a section. Then, there exists a smooth family of supersingular K3 surfaces*

$$\begin{array}{ccc} X & \to & \mathcal{X} \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \to & \operatorname{Spec} k[[t]] \end{array}$$

such that

- *there exists a short exact sequence*
$$0 \to \operatorname{Pic}(\mathcal{X}_{\overline{\eta}}) \to \operatorname{Pic}(X) \to \mathbb{Z}/p\mathbb{Z} \to 0,$$
whose cokernel is generated by the class of the zero-section of the fibration $X \to \mathbb{P}^1$. In particular,
$$\sigma_0(\mathcal{X}_{\overline{\eta}}) = \sigma_0(X) + 1,$$
and thus, the family has non-trivial moduli.

- *There exist purely inseparable isogenies*

$$\mathcal{X}_{\overline{\eta}}^{(1/p)} \dashrightarrow X \times_k \overline{\eta} \dashrightarrow \mathcal{X}_{\overline{\eta}}^{(p)}.$$

In particular, X is unirational if and only if $\mathcal{X}_{\overline{\eta}}$ is unirational.

Finally, in order for the previous theorem to be applicable, we need the existence of elliptic fibrations that have a section on a given supersingular K3 surface. This question turns out to be equivalent to the question whether the Néron–Severi lattice contains a hyperbolic plane. Since this question depends only on the isometry class of the Néron–Severi lattice of the surface, it only depends on the Artin invariant σ_0 of the supersingular K3 surface.

Proposition. *A supersingular K3 surface with $\sigma_0 \leq 9$ possesses an elliptic fibration that has a section.*

The upshot of this section is that given a supersingular K3 surface with Artin invariant $\sigma_0 \leq 9$, then there exists a one-dimensional deformation with non-trivial moduli (in fact, the Artin invariant goes up by 1 in the generic fiber), such that the property of being unirational is preserved. We refer to [Li13] and [Li14] for details.

Supersingular K3 surfaces are unirational

The idea to prove the unirationality of supersingular K3 surfaces is to use the results of the previous section to fill up the whole moduli space of supersingular K3 surfaces with these moving torsor families, along which the property of being unirationality is preserved. Once this is achieved, the question of whether *all* supersingular K3 surfaces in characteristic p are unirational becomes equivalent to asking whether *one* supersingular K3 surface in characteristic p is unirational. But the latter is true by Shioda's result [Sh77] on the unirationality of supersingular Kummer surfaces.

First, we study the moving torsor families of the previous section on the level of supersingular K3 crystals. Since the Artin invariant in these families goes up by 1, it is not surprising that this relates the moduli spaces $\mathcal{M}_N$ and $\mathcal{M}_{N_+}$ of marked supersingular K3 crystals if N and N_+ denote supersingular K3 lattices, whose Artin invariants differ by 1.

Theorem. *Let N and N_+ be supersingular K3 lattices in odd characteristic such that $\sigma_0(N_+) = \sigma_0(N) + 1$. Then, a choice of hyperbolic plane $U \subset N$ gives rise to a morphism*

$$\mathcal{M}_{N_+} \to \mathcal{M}_N$$

which is a $\mathbb{P}^1$-bundle with a distinguished section.

Although this is a statement about moduli spaces of supersingular K3 crystals only, and the proof is entirely independent of the previous section, the connection to the previous section is that a supersingular K3 surface is determined by its supersingular K3 crystal by Ogus' crystalline Torelli theorem. A choice of hyperbolic plane $U \subset N$ corresponds to equipping a supersingular K3 surface with an elliptic fibration that has a section, and then, the fibers of $\mathcal{M}_{N_+} \to \mathcal{M}_N$ correspond to the moving torsor families from the previous section, with the distinguished section being the special fiber of the moving torsor families.

In [Og79], Ogus showed that $\mathcal{M}_N \cong \mathrm{Spec}\mathbb{F}_{p^2}$ if $\sigma_0(N) = 1$. Geometrically, this means the following: there exists only one supersingular K3 surface with Artin invariant $\sigma_0 = 1$ and it can be defined over $\mathbb{F}_p$. However, its geometric Néron–Severi group cannot be defined over $\mathbb{F}_p$, but there exist models where this group can be defined over $\mathbb{F}_{p^2}$. This explains why the moduli space has two geometric components, despite of only one surface.

Corollary. *The moduli space $\mathcal{M}_N$ is an iterated $\mathbb{P}^1$-bundle over $\mathbb{F}_{p^2}$.*

We have seen above that the generic and the special fiber of a moving torsor family are related by purely inseparable isogenies. Using the just established $\mathbb{P}^1$-bundle structure, Ogus' Torelli theorem [Og83], and a little bit more work, it eventually follows that every supersingular K3 surface with Artin invariant σ_0 is related by a purely inseparable isogeny to one with Artin invariant $\sigma_0 - 1$. By induction, it follows that every supersingular K3 surface is related by a purely inseparable isogeny to one of Artin invariant $\sigma_0 = 1$. But then, there is only one such supersingular K3 surface with $\sigma_0 = 1$, which eventually allows us to relate all supersingular K3 surface to each other by purely inseparable isogenies.

Theorem. *Let X and Y be supersingular K3 surfaces in characteristic $p \geq 5$. Then, there exist purely inseparable isogenies $X \dashrightarrow Y$ and $Y \dashrightarrow X$.*

Since supersingular Kummer surfaces in characteristic $p \geq 3$ are unirational by Shioda's theorem [Sh77], we obtain the desired unirationality.

Theorem. *Supersingular K3 surfaces in characteristic $p \geq 5$ are unirational.*

We refer to [Li13] for proofs and detailed statements, as well as to [Li14] for an overview.

Finally, let us come back to and discuss isogenies of K3 surfaces: for a singular K3 surface X over $\mathbb{C}$ (that is, a K3 surface with Picard rank $\rho(X) = 20$ over $\mathbb{C}$), a classical theorem of Shioda and Inose [SI77] states that there exist rational and dominant maps, generically finite of degree 2

$$\mathrm{Km}(E \times E) \dashrightarrow X \dashrightarrow \mathrm{Km}(E \times E),$$

where E is an elliptic curve with complex multiplication, and where $\mathrm{Km}(E \times E)$ denotes the Kummer surface associated to the Abelian surface $E \times E$. (See also [Li13] for a Shioda–Inose classification result in odd characteristic.) Later, Morrison [Mo84], Mukai [Mu87], and Nikulin [Ni87], [Ni91] generalized these results to other types of complex K3 surfaces. One problem is the definition of isogeny for K3 surfaces, and we refer to [Mo87] for discussion. Now, the existence of purely inseparable isogenies between supersingular K3 surfaces above can also be rephrased by saying that for every supersingular K3 surface in characteristic $p \geq 5$, there exist rational, dominant, generically finite, and purely inseparable maps

$$\mathrm{Km}(E \times E) \dashrightarrow X \dashrightarrow \mathrm{Km}(E \times E)$$

where E is a supersingular elliptic curve. Thus, the result on purely inseparable isogenies between supersingular K3 surfaces nicely fits into the Shioda–Inose theorem.

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TU München, Zentrum Mathematik - M11, Boltzmannstr. 3, D-85748 Garching bei München, Germany

E-mail address: liedtke@ma.tum.de