On the anti-canonical geometry of $\mathbb{Q}$-Fano 3-folds

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Citation
代数幾何学シンポジウム記録 代数幾何学シンポジウム記録

Issue Date
2014

URL
http://hdl.handle.net/2433/215020

Type
Departmental Bulletin Paper
On the Anti-canonical Geometry of $\mathbb{Q}$-Fano 3-folds

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Kinosaki Symposium on Algebraic Geometry

This talk is based on the joint work with Meng Chen. Everything is over $\mathbb{C}$.

Definition

Let $X$ be a normal projective variety.

$X$ is called a weak $\mathbb{Q}$-Fano variety if

1. $X$ has at worst $\mathbb{Q}$-factorial terminal singularities;
2. $-K_X$ is nef and big.

A weak $\mathbb{Q}$-Fano variety $X$ is said to be $\mathbb{Q}$-Fano if $\rho(X) = 1$.

According to Minimal Model Program, $\mathbb{Q}$-Fano varieties form a fundamental class in birational geometry.
Given a (weak) $\mathbb{Q}$-Fano $n$-fold $X$, the $m$-th anti-canonical map $\varphi_{-m}$ is the rational map defined by the linear system $| - mK_X|$. By definition, $\varphi_{-m}$ is a birational map when $m$ is sufficiently large.

**Birationality Problem**

Find a number $m_n$, independent of $X$, which stably guarantees the birationality of $\varphi_{-m_n}$.

- $m_2 = 3$.

**Theorem (Kawamata, Kollár–Miyaoka–Mori–Takagi)**

(Weak) $\mathbb{Q}$-Fano 3-folds form a bounded family.

$\implies m_3$ exists.

**Main Problem**

Find the optimal constant $c$ such that $\varphi_{-m}$ is birational onto its image for all $m \geq c$ and for all (weak) $\mathbb{Q}$-Fano 3-folds.

- The behavior of $\varphi_{-m}$ is not necessarily birational invariant! For example, $| - K_{P^2}|$ gives a birational map while $| - K_{S_2}|$ does not for $S_2$ a del Pezzo surface of degree 2.

**Effective results**

1. (X smooth) $c \leq 5$ [Ando] and $c \leq 4$ [Fukuda].
2. (General case) $c \leq 3r_X + 10$ [M. Chen].

- $r_X \leq 840$. 

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Nonpencil problem

Second Problem
For a (weak) $\mathbb{Q}$-Fano 3-folds $X$, find an integer $\delta = \delta(X)$ such that $\dim \varphi_{-\delta}(X) > 1$.

- This problem is essential for the birationality problem.
- $| - mK_X |$ is said to be composed with a pencil of surfaces if $\dim \varphi_{-m}(X) = 1$.

Theorem (Kollár)
Let $Y$ be a 3-fold of general type on which $| nK_Y |$ is composed of a pencil for some integer $n > 0$. Then there exists $m \leq 11n + 5$ such that $| mK_Y |$ is not composed of a pencil.

- A direct application of the semi-positivity of $f_*\omega_Y^i/B$.
- Not available for $\mathbb{Q}$-Fanos!

Main results 1

Theorem P1 (M. Chen--J.)
Let $X$ be a $\mathbb{Q}$-Fano 3-fold. Then there exists an integer $n_1 \leq 10$ such that $\dim \varphi_{-n_1}(X) > 1$. In particular, $\delta(X) \leq 10$.

Theorem B1 (M. Chen--J.)
Let $X$ be a $\mathbb{Q}$-Fano 3-fold. Then $\varphi_{-m}$ is birational onto its image for all $m \geq 39$. In particular, $c \leq 39$.

- $\rho = 1$ is essential in Theorem P1.
- $\exists X$ s.t $\delta(X) \geq 9$ and $c \geq 33$. 
Examples

\[ \delta(X) \geq 9: \]

**Example (Iano-Fletcher)**

\[ X_{24,30} \subset \mathbb{P}(1, 8, 9, 10, 12, 15). \]
Then \( \dim \varphi_{-9}(X) > 1 \) while \( \dim \varphi_{-8}(X) = 1 \) since \( P_{-8} = 2 \).

**Example (Iano-Fletcher)**

\[ X_{33} \subset \mathbb{P}(1, 5, 6, 22, 33). \]
Then \( \varphi_{-m} \) is birational onto its image for \( m \geq 33 \), but \( \varphi_{-32} \) fails to be birational.

Main results 2

For arbitrary weak \( Q \)-Fano 3-folds, our method of Theorem P1 is not valid hence our results are weaker.

**Theorem P2 (M. Chen–J.)**

Let \( X \) be a weak \( Q \)-Fano 3-fold. Then \( \dim \varphi_{-n_2}(X) > 1 \) for all \( n_2 \geq 71 \).

**Theorem B2 (M. Chen–J.)**

Let \( X \) be a weak \( Q \)-Fano 3-fold. Then \( \varphi_{-m} \) is birational onto its image for all \( m \geq 97 \).
Birationality Problem:
We give a birationality criterion.
We reduce the problem to curve by birationality principle.
Then we can relate the problem to some invariant.
To cut down the dimension, we need to consider non-pencil problem.

Non-pencil Problem:
We reduce the problem to the behavior of Hilbert polynomial
of $-K_X$.
For general case, we estimate the Hilbert polynomial by Reid’s Riemann–Roch formula.
For $\mathbb{Q}$-Fano case, we generalize a theorem of Alexeev and use a method developed by J. A. Chen and M. Chen to treat the behavior of Hilbert polynomial.
**Reduction to a smooth model**

Let $X$ be a weak $\mathbb{Q}$-Fano 3-fold. $\pi : Y \to X$ is a resolution.

**Theorem**

For any $m > 0$, $\varphi_{-m}$ is birational if and only if $\Phi_{|K_Y + [(m+1)\pi^*(-K_X)]|}$ is birational.

Set $\Lambda_m = |K_Y + [(m + 1)\pi^*(-K_X)]|.$

**Settings**

- Take $m_0$ s.t. $h^0(-m_0K_X) \geq 2$.
- Take $m_1 \geq m_0$ with $h^0(-m_1K_X) \geq 2$.
  $|-m_1K_X|$ and $|-m_0K_X|$ are not composed with the same pencil.

  \[
  \begin{array}{ccc}
  \Gamma & \xrightarrow{f} & Y \\
  |M_{-m_0}| & \downarrow & |M_{-m_1}| \\
  Z & \xleftarrow{\sim}\Gamma' & X \xrightarrow{\sim} Z'
  \end{array}
  \]

  $|-m_1K_X|$ and $|-m_0K_X|$ are composed with the same pencil

  if $\Gamma = \Gamma' = \mathbb{P}^1$ and $f = f'$.

  Here $M_{-m} := \text{Mov}(|\pi^*(-m_iK_X)|)$. (base point free)
- $S$: a generic irreducible element of $|M_{-m_0}|$. (smooth)
  i.e. \begin{itemize}
  \item a general element of $|M_{-m_0}|$, if $\dim \Gamma > 1$;
  \item a general fiber of $f$, if $\dim \Gamma = 1$.
  \end{itemize}
Note that \(-m_0\pi^*(K_X) \geq M_{-m_0} \geq S\).

\[ \mu_0 := \inf\{ t \in \mathbb{Q}^+ \mid t\pi^*(-K_X) - S \sim_\mathbb{Q} 0 \} \leq m_0. \]

\(C\): a generic irreducible element of \(|G|\).

\(|G| := |M_{-m_1}|_S|\) is a base point free linear system on \(S\).

Note that \(-m_1\pi^*(K_X)|_S \geq M_{-m_1}|_S \geq C\).

\(\zeta := (\pi^*(-K_X) \cdot C)|_S = (\pi^*(-K_X)|_S \cdot C)_S\).

\(\varepsilon(m) := (m + 1 - \mu_0 - m_1)\zeta\).

---

**Key Theorem**

**Assumption**

1. The linear system \(\Lambda_m\) distinguishes different generic irreducible elements of \(|M_{-m_0}|\) (namely, \(\Phi_{\Lambda_m}(S') \neq \Phi_{\Lambda_m}(S'')\) for two different generic irreducible elements \(S', S''\) of \(|M_{-m_0}|\)).

2. The linear system \(\Lambda_m|_S\) distinguishes different generic irreducible elements of \(|G| = |M_{-m_1}|_S|\) on \(S\).

**Key Theorem**

If Assumption is satisfied and \(\varepsilon(m) > 2\), then \(\varphi_{-m}\) is birational onto its image.
Sketch of proof

Birationality: $\varphi_m \leftarrow \Phi_{A_m} \leftarrow \Phi_{A_m}|S \leftarrow \Phi_{A_m}|C \leftarrow \Phi_{|KC + |D_m|}$

- $-\mu_0 \pi^*(K_X) = S + F$ (for simplicity).
- $-m_1 \pi^*(K_X)|_S = C + H.$

$|K_X + ((m + 1) \pi^*(-K_X))| \geq |K_Y + ((m + 1) \pi^*(-K_X) - F)|$

\[ |K_S + [L_m - H]| \geq |K_S + [(m + 1) \pi^*(-K_X) - F - S]|_S \]

\[ |K_C + [L_m - H - C]|_C | \geq |K_C + |D_m| | \]

- $L_m = ((m + 1) \pi^*(-K_X) - F - S)|_S.$
- $D_m = (L_m - H - C)|_C$ with $\deg D_m = \varepsilon(m) > 2.$

Applications

Theorem (Birationality criterion)

Let $X$ be a weak $\mathbb{Q}$-Fano 3-fold. Let $\nu_0$ be an integer such that $h^0(-\nu_0 K_X) > 0.$ Keep the same notation as above.

Then $\varphi_m$ is birational if one of the following holds:

(i) $m \geq \max\{m_0 + m_1 + a(m_0), 3\mu_0 + 3m_1\};$
(ii) $m \geq \max\{m_0 + m_1 + a(m_0), \frac{5}{3}\mu_0 + \frac{5}{3}m_1, |\mu_0| + m_1 + 2r_{\text{max}}\};$
(iii) $m \geq \max\{m_0 + m_1 + a(m_0), |\mu_0| + m_1 + 2\nu_0 r_{\text{max}}\},$

where $r_{\text{max}}$ is the maximum of local indices of singularities and $a(m_0) = \begin{cases} 6, & m_0 \geq 2; \\ 1, & m_0 = 1. \end{cases}$
Example (12 examples)

$X_{6d} \subset \mathbb{P}(1, a, b, 2d, 3d)$ with $1 \leq a \leq b$ and $d = a + b$.

$r_{\text{max}} = d$, $\nu_0 = 1$, $m_0 = \mu_0 = a$ and $m_1 = b$.

Then $\varphi_{-3d}$ is birational but $\varphi_{-(3d-1)}$ is not.

On the other hand,

$3d = [3\mu_0] + 3m_1 = [\mu_0] + m_1 + 2r_{\text{max}} = [\mu_0] + m_1 + 2\nu_0 r_{\text{max}}$,

$\varphi_{-m}$ is birational for $m \geq 3d$ by criterion.

Theorem (Fukuda, M. Chen–J.)

Let $X$ be a weak $\mathbb{Q}$-Fano 3-fold with Gorenstein singularities.

Then $\varphi_{-m}$ is birational onto its image for all $m \geq 4$.

When is $| - mK_X |$ not composed with a pencil of surfaces? I
Let $X$ be a weak $\mathbb{Q}$-Fano 3-fold.

**Proposition**

If $P_m > r_X(-K_X)^3 m + 1$ for some integer $m$, then $|mK_X|$ is not composed with a pencil.

- Reid’s formula:
  
  $$P_{-n}(X) = \frac{1}{12} n(n+1)(2n+1)(-K_X^3)+(2n+1)-l(-n) \approx \frac{1}{6}(-K_X^3)n^3$$

  where $l(-n) = l(n+1) = \sum_i \sum_{j=1}^n \frac{b_i(r_i - b_j)}{2r_i}$.

- Need to estimate a lower bound of $l(-n)$

- Basically, $m > \sqrt{6r_X}$ is enough.

- $r_X \leq 840$ by $\sum (r_i - \frac{1}{r_i}) \leq 24$.

When is $| - mK_X |$ not composed with a pencil of surfaces? II
Let $X$ be a $\mathbb{Q}$-Fano 3-fold. Set $P_m := h^0(-mK_X)$.

**Theorem (Alexeev)**

A general element of $|-K_X|$ is irreducible and reduced. Furthermore, if $P_{-1} \geq 3$, then $|-K_X|$ does not have fixed part and is not composed with a pencil of surfaces.

- We only need to treat the case $P_{-1} \leq 2$.

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**Key Theorem 1**

Fix a positive integer $m$ such that $P_{-m} > 0$. Assume that, for each pair $(b, r) \in B_X$ (corresponding to a cyclic quotient singularity of type $\frac{1}{r}(1, -1, b)$), one of the following conditions is satisfied:

1. $m \equiv 0, \pm 1 \mod r$;
2. $m \equiv -2 \mod r$ and $b = \lfloor \frac{2}{r} \rfloor$;
3. $m \equiv 2 \mod r$ and $3b \geq r$;
4. $m \equiv 3 \mod r$ and $4b \geq r$;
5. $m \equiv 4 \mod r$ and $\overline{b}(r - \overline{b}) \geq 4\overline{b}(r - 4\overline{b})$ and $\overline{b}(r - \overline{b}) + 2\overline{b}(r - 2\overline{b}) \geq 3\overline{b}(r - 3\overline{b}) + 4\overline{b}(r - 4\overline{b})$.

Then a general element of $|-mK_X|$ is irreducible and reduced.
Key Theorem 2

Assume that a g.e. of $| - mK_X|$ is irreducible and reduced.

$n_0 := \min \{ n \in \mathbb{Z}^+ \mid P_{-nm} \geq 2 \}$.

$l_0 := \min \{ l = sn_0 + t \mid s \in \mathbb{Z}_{>0}, 0 \leq t \leq n_0 - 1, P_{-lm} > s + 1 \}$.

Then $| - l_0 mK_X|$ does not have fixed part and is not composed with a pencil of surfaces.

Example

<table>
<thead>
<tr>
<th>$P_{-m}$</th>
<th>$P_{-2m}$</th>
<th>$P_{-3m}$</th>
<th>$P_{-4m}$</th>
<th>$P_{-5m}$</th>
<th>$P_{-6m}$</th>
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</table>

Examples

Example (Iano-Fletcher)

- $X_{42} \subset \mathbb{P}(1^2, 6, 14, 21)$, $m = 1$.

<table>
<thead>
<tr>
<th>$P_{-1}$</th>
<th>$P_{-2}$</th>
<th>$P_{-3}$</th>
<th>$P_{-4}$</th>
<th>$P_{-5}$</th>
<th>$P_{-6}$</th>
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</table>

Hence $\dim \varphi_{-6}(X_{42}) > 1$. ($n_0 = 1, l_0 = 6$)

- $X_{24,30} \subset \mathbb{P}(1, 8, 9, 10, 12, 15)$, $m = 1$.

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<tr>
<th>$P_{-1}$</th>
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<th>$P_{-5}$</th>
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Hence $\dim \varphi_{-9}(X) > 1$. ($n_0 = 8, l_0 = 9$)
Example (Iano-Fletcher, Altinok–Reid)

- $X_{12,14} \subset \mathbb{P}(2, 3, 4, 5, 6, 7)$. $B_X = \{7 \times (1, 2), 2 \times (1, 3), (2, 5)\}$.
- $X \subset \mathbb{P}(2, 3, 4, 5, 6, 7, 8, 9)$. $B_X = \{7 \times (1, 2), (1, 3), (3, 8)\}$.

$m = 2$.

\[
\begin{array}{cccc}
P_{-2} & P_{-4} & P_{-6} & P_{-8} \\
1 & 2 & 4 & 6
\end{array}
\]

Hence $\dim \varphi_6(X) > 1$. ($n_0 = 2, l_0 = 3$)

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Strategy

- Divide all $\mathbb{Q}$-Fano 3-folds into several families, roughly speaking, by the value of $P_{-1}$ and $P_{-2}$.
- For each family, take a proper $m$ satisfying the condition of KEY1.
- Applying KEY2 to $m$, we are able to find the number $l_0$ and so $l_0m$ is what we need.
- In order to find an upper bound of $l_0$, we may assume $l_0 \geq 9$, then by KEY2, we know the value of $P_{-m}, P_{-2m}, P_{-3m}, \ldots, P_{-8m}$.
- By Chen–Chen’s method on the analysis of baskets, we can recover all possibilities for baskets of singularities from the numerical behavior of Hilbert polynomial.
• The baskets with bad behavior is very few and easy to treat.

**Example**

Assume \( P_{-1} = 1 \). Take \( m = 1 \). Assume \( l_0 \geq 9 \). Then

<table>
<thead>
<tr>
<th>( n_0 )</th>
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By Chen–Chen’s method, only possible baskets are

\( B = \{(1, 2), (2, 5), (1, 3), (1, 4), (1, s)\} \) with \( s = 9, 10, 11 \)

**Conclusion**

**Theorem**

Let \( X \) be a \( \mathbb{Q} \)-Fano 3-fold.

- If \( P_{-1} \geq 3 \), then \( \dim \varphi_{-m}(X) > 1 \) for all \( m \geq 1 \). (optimal)
- If \( P_{-1} = 2 \), then \( \dim \varphi_{-m}(X) > 1 \) for all \( m \geq 6 \). (optimal)
- If \( P_{-1} = 1 \), then \( \dim \varphi_{-m}(X) > 1 \) for all \( m \geq 9 \). (optimal)
- If \( P_{-1} = 0 \), then there exists \( m_1 \leq 10 \) s.t. \( \dim \varphi_{-m_1}(X) > 1 \).

We do not know if this result is optimal since very few examples with \( P_{-1} = 0 \) are known. There are 4 possible baskets for which we have to take \( m_1 = 10 \). If one can confirm either the existence or non-existence of these 4 baskets, the result becomes optimal.
Theorem (M. Chen–J.)
Let $X$ be a $\mathbb{Q}$-Fano 3-fold.
Then $\varphi_m$ is birational onto its image for all $m \geq 39$.

Theorem (M. Chen–J.)
Let $X$ be a weak $\mathbb{Q}$-Fano 3-fold.
Then $\varphi_m$ is birational onto its image for all $m \geq 97$.

Thank you!