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On the Anti-canonical Geometry of $\mathbb{Q}$-Fano 3-folds

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(Weak) $\mathbb{Q}$-Fano varieties

This talk is based on the joint work with Meng Chen.
Everything is over $\mathbb{C}$.

Definition

Let X be a normal projective variety.
X is called a *weak* $\mathbb{Q}$*-Fano variety* if

(1) X has at worst $\mathbb{Q}$-factorial terminal singularities;

(2) $-K_X$ is nef and big.

A weak $\mathbb{Q}$-Fano variety X is said to be $\mathbb{Q}$*-Fano* if $\rho(X) = 1$.

According to Minimal Model Program, $\mathbb{Q}$-Fano varieties form a fundamental class in birational geometry.

Birationality problem

Given a (weak) $\mathbb{Q}$-Fano n-fold X, the *m-th anti-canonical map* φ_{-m} is the rational map defined by the linear system $|-mK_X|$. By definition, φ_{-m} is a birational map when m is sufficiently large.

Birationality Problem

Find a number m_n, independent of X, which stably guarantees the birationality of φ_{-m_n}.

- $m_2 = 3$.

Theorem (Kawamata, Kollár–Miyaoka–Mori–Takagi)

(Weak) $\mathbb{Q}$-Fano 3-folds form a bounded family.
$\Longrightarrow$ *m_3 exists.*

Main Problem

Find the optimal constant c such that φ_{-m} is birational onto its image for all $m \geq c$ and for all (weak) $\mathbb{Q}$-Fano 3-folds.

- The behavior of φ_{-m} is not necessarily birational invariant! For example, $|-K_{\mathbb{P}^2}|$ gives a birational map while $|-K_{S_2}|$ does not for S_2 a del Pezzo surface of degree 2.

Effective results

(1) (X smooth) $c \leq 5$ [Ando] and $c \leq 4$ [Fukuda].
(2) (General case) $c \leq 3r_X + 10$ [M. Chen].

- $r_X \leq 840$.

Nonpencil problem

Second Problem

For a (weak) $\mathbb{Q}$-Fano 3-folds X, find an integer $\delta = \delta(X)$ such that $\dim \overline{\varphi_{-\delta}(X)} > 1$.

- This problem is essential for the birationality problem.
- $|-mK_X|$ is said to *be composed with a pencil of surfaces* if $\dim \overline{\varphi_{-m}(X)} = 1$.

Theorem (Kollár)

Let Y be a 3-fold of general type on which $|nK_Y|$ is composed of a pencil for some integer $n > 0$. Then there exists $m \leq 11n + 5$ such that $|mK_Y|$ is not composed of a pencil.

- A direct application of the semi-positivity of $f_*\omega^l_{Y/B}$.
- Not available for $\mathbb{Q}$-Fanos!

Main results 1

Theorem P1 (M. Chen–J.)

Let X be a $\mathbb{Q}$-Fano 3-fold.
Then there exists an integer $n_1 \leq 10$ such that $\dim \overline{\varphi_{-n_1}(X)} > 1$.
In particular, $\delta(X) \leq 10$.

Theorem B1 (M. Chen–J.)

Let X be a $\mathbb{Q}$-Fano 3-fold.
Then φ_{-m} is birational onto its image for all $m \geq 39$.
In particular, $c \leq 39$

- $\rho = 1$ is essential in Theorem P1.
- $\exists X$ s.t $\delta(X) \geq 9$ and $c \geq 33$.

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Examples

$\delta(X) \geq 9$:

Example (Iano-Fletcher)

$X_{24,30} \subset \mathbb{P}(1, 8, 9, 10, 12, 15)$.
Then $\dim \overline{\varphi_{-9}(X)} > 1$ while $\dim \overline{\varphi_{-8}(X)} = 1$ since $P_{-8} = 2$.

$c \geq 33$:

Example (Iano-Fletcher)

$X_{33} \subset \mathbb{P}(1, 5, 6, 22, 33)$.
Then φ_{-m} is birational onto its image for $m \geq 33$, but φ_{-32} fails to be birational.

Main results 2

For arbitrary weak $\mathbb{Q}$-Fano 3-folds, our method of Theorem P1 is not valid hence our results are weaker.

Theorem P2 (M. Chen–J.)

Let X be a weak $\mathbb{Q}$-Fano 3-fold.
Then $\dim \overline{\varphi_{-n_2}(X)} > 1$ for all $n_2 \geq 71$.

Theorem B2 (M. Chen–J.)

Let X be a weak $\mathbb{Q}$-Fano 3-fold.
Then φ_{-m} is birational onto its image for all $m \geq 97$.

Strategy

- Birationality Problem:
  We give a birationality criterion.
  We reduce the problem to curve by birationality principle.
  Then we can relate the problem to some invariant.
  To cut down the dimension, we need to consider nonpencil problem.
- Nonpencil Problem:
  We reduce the problem to the behavior of Hilbert polynomial of $-K_X$.
  For general case, we estimate the Hilbert polynomial by Reid's Riemann–Roch formula.
  For $\mathbb{Q}$-Fano case, we generalize a theorem of Alexeev and use a method developed by J. A. Chen and M. Chen to treat the behavior of Hilbert polynomial.

Birationality

Reduction to a smooth model

Let X be a weak $\mathbb{Q}$-Fano 3-fold. $\pi : Y \to X$ is a resolution.

Theorem

For any $m > 0$, φ_{-m} is birational if and only if $\Phi_{|K_Y + \lceil (m+1)\pi^*(-K_X)\rceil|}$ is birational.

Set $\Lambda_m = |K_Y + \lceil (m+1)\pi^*(-K_X)\rceil|$.

Settings

- Take m_0 s.t. $h^0(-m_0K_X) \geq 2$.
- Take $m_1 \geq m_0$ with $h^0(-m_1K_X) \geq 2$, $|-m_1K_X|$ and $|-m_0K_X|$ are not composed with the same pencil.

$$\begin{tikzcd} \Gamma \arrow[d] & Y \arrow[l, "f"'] \arrow[r, "f'"] \arrow[d, "\pi"] \arrow[dl, "|M_{-m_0}|"'] \arrow[dr, "|M_{-m_1}|"] & \Gamma' \arrow[d] \\ Z & X \arrow[l, dashed] \arrow[r, dashed] & Z' \end{tikzcd}$$

  $|-m_1K_X|$ and $|-m_0K_X|$ are *composed with the same pencil* if $\Gamma = \Gamma' = \mathbb{P}^1$ and $f = f'$.
  Here $M_{-m_i} := \mathrm{Mov}|\lfloor \pi^*(-m_iK_X)\rfloor|$. (base point free)
- S: a *generic irreducible element* of $|M_{-m_0}|$. (smooth)

$$\text{i.e} \begin{cases} \text{a general element of } |M_{-m_0}|, & \text{if } \dim \Gamma > 1; \\ \text{a general fiber of } f, & \text{if } \dim \Gamma = 1. \end{cases}$$

- Note that $-m_0\pi^*(K_X) \geq M_{-m_0} \geq S$.
- $\mu_0 := \inf\{t \in \mathbb{Q}^+ \mid t\pi^*(-K_X) - S \sim_{\mathbb{Q}} \geq 0\} \leq m_0$.
- C: a generic irreducible element of $|G|$.
  $|G| := |M_{-m_1}|_S|$ is a base point free linear system on S.
- Note that $-m_1\pi^*(K_X)|_S \geq M_{-m_1}|_S \geq C$.
- $\zeta := (\pi^*(-K_X) \cdot C)_Y = (\pi^*(-K_X)|_S \cdot C)_S$.
- $\varepsilon(m) := (m + 1 - \mu_0 - m_1)\zeta$.

Key theorem

Assumption

(1) The linear system Λ_m distinguishes different generic irreducible elements of $|M_{-m_0}|$ (namely, $\Phi_{\Lambda_m}(S') \neq \Phi_{\Lambda_m}(S'')$ for two different generic irreducible elements S', S'' of $|M_{-m_0}|$).

(2) The linear system $\Lambda_m|_S$ distinguishes different generic irreducible elements of $|G| = |M_{-m_1}|_S|$ on S.

Key Theorem

If Assumption is satisfied and $\varepsilon(m) > 2$, then φ_{-m} is birational onto its image.

Sketch of proof

Birationality: $\varphi_{-m} \Leftarrow \Phi_{\Lambda_m} \Leftarrow \Phi_{\Lambda_m}|_S \Leftarrow \Phi_{\Lambda_m}|_C \Leftarrow \Phi_{|K_C+\lceil \mathcal{D}_m \rceil|}$

- $-\mu_0\pi^*(K_X) = S + F$ (for simplicity).
- $-m_1\pi^*(K_X)|_S = C + H$.

$$
\begin{tikzcd}
|K_Y + \lceil (m+1)\pi^*(-K_X) \rceil| \arrow[r, phantom, "\succeq"] & |K_Y + \lceil (m+1)\pi^*(-K_X) - F \rceil| \arrow[d, "vanishing"] \\
|K_S + \lceil \mathcal{L}_m - H \rceil| \arrow[d, "vanishing"] \arrow[r, phantom, "\preceq"] & |K_S + \lceil (m+1)\pi^*(-K_X) - F - S \rceil|_S| \\
|K_C + \lceil \mathcal{L}_m - H - C \rceil|_C| \arrow[r, phantom, "\succeq"] & |K_C + \lceil \mathcal{D}_m \rceil|
\end{tikzcd}
$$

- $\mathcal{L}_m = ((m+1)\pi^*(-K_X) - F - S)|_S$.
- $\mathcal{D}_m = (\mathcal{L}_m - H - C)|_C$ with $\deg \mathcal{D}_m = \varepsilon(m) > 2$.

Applications

Theorem (Birationality criterion)

Let X be a weak $\mathbb{Q}$-Fano 3-fold.
Let ν_0 be an integer such that $h^0(-\nu_0 K_X) > 0$.
Keep the same notation as above.
Then φ_{-m} is birational if one of the following holds:

(i) $m \geq \max\{m_0 + m_1 + a(m_0), \lfloor 3\mu_0 \rfloor + 3m_1\}$;

(ii) $m \geq \max\{m_0 + m_1 + a(m_0), \lfloor \frac{5}{3}\mu_0 + \frac{5}{3}m_1 \rfloor, \lfloor \mu_0 \rfloor + m_1 + 2r_{\max}\}$;

(iii) $m \geq \max\{m_0 + m_1 + a(m_0), \lfloor \mu_0 \rfloor + m_1 + 2\nu_0 r_{\max}\}$,

where $r_{\max}$ is the maximum of local indices of singularities and

$$a(m_0) = \begin{cases} 6, & m_0 \geq 2; \\ 1, & m_0 = 1. \end{cases}$$

Example (12 examples)

$X_{6d} \subset \mathbb{P}(1, a, b, 2d, 3d)$ with $1 \le a \le b$ and $d = a + b$.
$r_{\max} = d$, $\nu_0 = 1$, $m_0 = \mu_0 = a$ and $m_1 = b$.
Then φ_{-3d} is birational but $\varphi_{-(3d-1)}$ is not.
On the other hand,
$3d = \lfloor 3\mu_0 \rfloor + 3m_1 = \lfloor \mu_0 \rfloor + m_1 + 2r_{\max} = \lfloor \mu_0 \rfloor + m_1 + 2\nu_0 r_{\max}$,
φ_{-m} is birational for $m \ge 3d$ by criterion.

Theorem (Fukuda, M. Chen–J.)

Let X be a weak $\mathbb{Q}$-Fano 3-fold with Gorenstein singularities. Then φ_{-m} is birational onto its image for all $m \ge 4$.

When is $|-mK_X|$ not composed with a pencil of surfaces? I

Let X be a weak $\mathbb{Q}$-Fano 3-fold.

Proposition

If $P_{-m} > r_X(-K_X)^3 m + 1$ for some integer m, then $|-mK_X|$ is not composed with a pencil.

- Reid's formula:

  $$P_{-n}(X) = \frac{1}{12}n(n+1)(2n+1)(-K_X^3)+(2n+1)-l(-n) \approx \frac{1}{6}(-K_X^3)n^3$$

  where $l(-n) = l(n+1) = \sum_i \sum_{j=1}^{n} \frac{\overline{jb_i}(r_i - \overline{jb_i})}{2r_i}$.
- Need to estimate a lower bound of $l(-n)$
- Basically, $m > \sqrt{6r_X}$ is enough.
- $r_X \leq 840$ by $\sum(r_i - \frac{1}{r_i}) \leq 24$.

When is $|-mK_X|$ not composed with a pencil of surfaces? II

Key Theorems

Let X be a $\mathbb{Q}$-Fano 3-fold. Set $P_{-m} := h^0(-mK_X)$.

Theorem (Alexeev)

A general element of $|-K_X|$ is irreducible and reduced. Futhermore, if $P_{-1} \geq 3$, then $|-K_X|$ does not have fixed part and is not composed with a pencil of surfaces.

- We only need to treat the case $P_{-1} \leq 2$.

Key Theorem 1

Fix a positive integer m such that $P_{-m} > 0$. Assume that, for each pair $(b, r) \in B_X$ (corresponding to a cyclic quotient singularity of type $\frac{1}{r}(1, -1, b)$), one of the following conditions is satisfied:

(1) $m \equiv 0, \pm 1 \mod r$;

(2) $m \equiv -2 \mod r$ and $b = \lfloor \frac{r}{2} \rfloor$;

(3) $m \equiv 2 \mod r$ and $\overline{3b} \geq r$;

(4) $m \equiv 3 \mod r$ and $\overline{4b} \geq r$;

(5) $m \equiv 4 \mod r$ and $\overline{b}(r - \overline{b}) \geq \overline{4b}(r - \overline{4b})$ and $\overline{b}(r - \overline{b}) + \overline{2b}(r - \overline{2b}) \geq \overline{3b}(r - \overline{3b}) + \overline{4b}(r - \overline{4b})$.

Then a g.e. of $|-mK_X|$ is irreducible and reduced.

Key Theorem 2

Assume that a g.e. of $|-mK_X|$ is irreducible and reduced.
$n_0 := \min\{n \in \mathbb{Z}^+ \mid P_{-nm} \geq 2\}$.
$l_0 := \min\{l = sn_0 + t \mid s \in \mathbb{Z}_{>0}, 0 \leq t \leq n_0 - 1, P_{-lm} > s + 1\}$.
Then $|-l_0 mK_X|$ does not have fixed part and is not composed with a pencil of surfaces.

Example

P_{-m}	P_{-2m}	P_{-3m}	P_{-4m}	P_{-5m}	P_{-6m}	P_{-7m}	P_{-8m}
1	1	2	2	2	3	3	4
2	3	4	5	6	7	8	10

Examples

Example (Iano-Fletcher)

- $X_{42} \subset \mathbb{P}(1^2, 6, 14, 21)$, $m = 1$.

P_{-1}	P_{-2}	P_{-3}	P_{-4}	P_{-5}	P_{-6}	P_{-7}	P_{-8}
2	3	4	5	6	8	10	12

Hence $\dim \overline{\varphi_{-6}(X_{42})} > 1$. $(n_0 = 1, l_0 = 6)$

- $X_{24,30} \subset \mathbb{P}(1, 8, 9, 10, 12, 15)$, $m = 1$.

P_{-1}	P_{-2}	P_{-3}	P_{-4}	P_{-5}	P_{-6}	P_{-7}	P_{-8}	P_{-9}
1	1	1	1	1	1	1	2	3

Hence $\dim \overline{\varphi_{-9}(X)} > 1$. $(n_0 = 8, l_0 = 9)$

Example (Iano-Fletcher, Altinok–Reid)

- $X_{12,14} \subset \mathbb{P}(2,3,4,5,6,7)$. $B_X = \{7 \times (1,2), 2 \times (1,3), (2,5)\}$.
- $X \subset \mathbb{P}(2,3,4,5,6,7,8,9)$. $B_X = \{7 \times (1,2), (1,3), (3,8)\}$.

$m = 2$.

P_{-2}	P_{-4}	P_{-6}	P_{-8}
1	2	4	6

Hence $\dim \overline{\varphi_{-6}(X)} > 1$. $(n_0 = 2, l_0 = 3)$

Strategy

- Divide all $\mathbb{Q}$-Fano 3-folds into several families, roughly speaking, by the value of P_{-1} and P_{-2}.
- For each family, take a proper m satisfying the condition of KEY1.
- Applying KEY2 to m, we are able to find the number l_0 and so $l_0 m$ is what we need.
- In order to find an upper bound of l_0, we may assume $l_0 \geq 9$, then by KEY2, we know the value of $P_{-m}, P_{-2m}, P_{-3m}, \ldots, P_{-8m}$.
- By Chen–Chen's method on the analysis of baskets, we can recover all possibilities for baskets of singularities from the numerical behavior of Hilbert polynomial.

- The baskets with bad behavior is very few and easy to treat.

Example

Assume $P_{-1} = 1$. Take $m = 1$. Assume $l_0 \geq 9$. Then

n_0	P_{-1}	P_{-2}	P_{-3}	P_{-4}	P_{-5}	P_{-6}	P_{-7}	P_{-8}
2	1	2	2	3	3	4	4	5
3	1	1	2	2	2	3	3	3
4	1	1	1	2	2	2	2	3
5	1	1	1	1	2	2	2	2
6	1	1	1	1	1	2	2	2
7	1	1	1	1	1	1	2	2
8	1	1	1	1	1	1	1	2

By Chen–Chen's method, only possible baskets are
$B = \{(1,2),(2,5),(1,3),(1,4),(1,s)\}$ with $s = 9, 10, 11$

Conclusion

Theorem

Let X be a $\mathbb{Q}$-Fano 3-fold.

- If $P_{-1} \geq 3$, then $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geq 1$. (optimal)
- If $P_{-1} = 2$, then $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geq 6$. (optimal)
- If $P_{-1} = 1$, then $\dim \overline{\varphi_{-m}(X)} > 1$ for all $m \geq 9$. (optimal)
- If $P_{-1} = 0$, then there exists $m_1 \leq 10$ s.t. $\dim \overline{\varphi_{-m_1}(X)} > 1$.

- We do not know if this result is optimal since very few examples with $P_{-1} = 0$ are known. There are 4 possible baskets for which we have to take $m_1 = 10$. If one can confirm either the existence or non-existence of these 4 baskets, the result becomes optimal.

Theorem (M. Chen–J.)

Let X be a $\mathbb{Q}$-Fano 3-fold.
Then φ_{-m} is birational onto its image for all $m \geq 39$.

Theorem (M. Chen–J.)

Let X be a weak $\mathbb{Q}$-Fano 3-fold.
Then φ_{-m} is birational onto its image for all $m \geq 97$.

Thank you!