<table>
<thead>
<tr>
<th>Title</th>
<th>An inequality of Noether type for algebraic threefolds</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Chen, Jungkai A.; Chen, Meng</td>
</tr>
<tr>
<td>Citation</td>
<td>代数幾何学シンポジウム記録 (2014), 2014: 39-42</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2014</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/215022">http://hdl.handle.net/2433/215022</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
AN INEQUALITY OF NOETHER TYPE FOR ALGEBRAIC THREEFOLDS

JUNGKAI A. CHEN AND MENG CHEN

Abstract. We study the Noether type of inequality for projective threefolds of general type.

1. Introduction

Let $X$ be a minimal projective 3-fold of general type. It has been an open problem whether the inequality if true:

$$K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}. \quad (1.1) \quad \{\text{ineq}\}$$

Here is a brief history of this problem:

1. M. Kobayashi [10] constructed some examples satisfying $K_X^3 = \frac{4}{3}p_g - \frac{10}{3}$ in 1992;
2. M. Chen [5] proved Inequality (1.1) for canonically polarized 3-folds in 2004;
3. Catanese–Chen–Zhang [1] proved Inequality (1.1) for smooth minimal 3-folds of general type in 2006;
4. J. Chen and M. Chen [3] proved Inequality (1.1) for Gorenstein minimal 3-folds of general type in 2015.

The aim of this talk is to announce our main statement that Inequality (1.1) is true.

2. The discrepancy of a special resolution for linear systems

First of all, we recall the following result of the first author:

**Theorem 2.1.** ([2, Theorem 1.3]) Let $X$ be an algebraic 3-fold with at worst terminal singularities. For any terminal singularity $P \in X$, there exists a sequence of birational morphisms:

$$\tau_P : Y = X_m \rightarrow X_{m-1} \rightarrow \ldots \rightarrow X_1 \rightarrow X_0 = X,$$

such that $Y$ is smooth on $\tau_P^{-1}(P)$ and, for all $i$, the morphism $\pi_i : X_{i+1} \rightarrow X_i$ is a divisorial contraction to a singular point $P_i \in X_i$ of index $r_i \geq 1$ with discrepancy $1/r_i$.

**Definition 2.2.** Given a terminal singularity $P \in X$, the birational morphism $\tau_P : Y \rightarrow X \ni P$ constructed as above is called a feasible resolution of $P \in X$. 
Suppose that $|M|$ is a moving linear system (i.e. without fixed part) on the given projective terminal 3-fold $X$ with $\text{Bs}|M| \neq \emptyset$. Similar to usual elimination of indeterminacies, we can have a feasible elimination of indeterminacies as follows:

(i) Given a terminal singularity $P \in X$, we may define $d(P \in X) := \min\{m|X_m \to \cdots \to X_1 \to X_0\text{ is a feasible resolution.}\}$, and set

$$d(X) := \sum_{P \in \text{Sing}(X)} d(P \in X).$$

Note that $d(X) = 0$ if and only if $X$ is non-singular.

(ii) By induction on $d(X)$, this process must terminates in finite steps. We will end up with a partial resolutions $Y = X_k \to \cdots \to X_1 \to X$ so that $\text{Bs}|M_Y| \cap \text{Sing}(Y) = \emptyset$, where $M_Y$ is the proper transform of $M$ on $Y$.

(iii) If $|M_Y| \neq \emptyset$, then $\text{Bs}|M_Y|$ consists of smooth points. We then consider the usual elimination of indeterminacies over $\text{Bs}|M_Y|$, say $Z = X_n \to \cdots \to X_k = Y$, which is composed of a sequence of blow-ups along smooth points or curves by Hironaka’s big theorem.

(iv) Thus we end up with a possibly singular 3-fold $Z = X_n$, so that $|M_n|$ is base point free. We call

$$\{\text{Gres}\}$$

$$\mu : Z = X_n \to \cdots \to X_k = Y \to \cdots \to X$$

(2.1)

a feasible elimination of indeterminacies of $|M|$. Note that a general member $S \in |M_Z|$ is smooth by Bertini’s Theorem.

For any $i > 0$, let $E_i$ be the exceptional divisor of $X_i \to X_{i-1}$. Let $K_i$ be the canonical divisor of $X_i$. For $i > j$ we write $K_{X_i/X_j} = K_i - \pi^*_{i,j}K_j$, where $\pi_{i,j} : X_i \to X_j$ is the induced map. We also denote $K_{Z/X} := K_Z - \mu^*(K_X)$ and $K_{Y/X}$ similarly.

Given a $\mathbb{Q}$-Cartier divisor $D$ on $X$, let $D_i$ be the proper transform of $D$ in $X_i$. Similarly, we define $D_{X_i/X_j} := \pi^*_{i,j}D_j - D_i$ write $D_{Z/X} := \mu^*(D) - D_Z$. $D_{Y/X}$ is defined similarly.

{key}

**Theorem 2.3.** Let $|M|$ be a moving linear system on a projective terminal 3-fold $X$ and $D \in |M|$ be a general member. Let $\mu : Z = X_n \to X$ be the feasible elimination of indeterminacies as in (2.1). Then $D_{Y/X} \geq K_{Y/X}$ and $2D_{Z/Y} \geq K_{Z/Y}$.

**Lemma 2.4.** Keep the notation as above. Suppose that $\alpha_i + \beta_i = a(D_{Z/X}, E_i) + a(K_{Z/X}, E_i) \leq 2$, then $i \in J$.  

3. The case of canonical family of curves

Let $X$ be a projective minimal 3-fold of general type. We may assume that $p_g \geq 3$ and always consider the non-trivial canonical map $\varphi_1$. Set $d := \dim \varphi_1(X)$.

The following inequalities are already known:

I. if $p_g(X) \geq 3$, then $K_X^3 \geq 1$ and if $p_g(X) \geq 4$, then $K_X^3 \geq 2$ (cf. [7, Theorem 1.5]).

II. If $d = 2$ and $X$ is canonically fibred by curves of genus $g(C) \geq 3$, then $K_X^3 \geq 2p_g(X) - 4$ by [6, Theorem 4.1(ii)].

From now on, we consider $d = 2$ and $X$ is canonically fibred by curves $C$ of genus $g(C) = 2$.

Theorem 3.1. Let $X$ be a projective minimal smooth 3-fold of general type. Suppose that $d = 2$ and $X$ is canonically fibred by curves of genus $g(C) = 2$. Then

$$K_X^3 \geq \frac{1}{3}(4p_g(X) - 10).$$

The inequality is sharp.

4. The case of canonical family of surfaces

Assume $d = 1$. We have an induced fibration $f : X' \to \Gamma$. Take a general fiber $F$ of $f$. By assumption, we know that $H^0(X', f_*\omega_{X'})$ naturally generate an invertible sheaf $L \subset f_*\omega_{X'}$. We may always assume $p_g(X) \geq 5$. Thus, by [4, Theorem 1], we have

- either $0 \leq b := g(\Gamma) \leq 1$ (4.1) \{b2\}
- or $b > 1$ and $p_g(F) = 1.$ (4.2) \{b1\}

Denote by $\sigma : F \to F_0$ the birational contraction onto the minimal model. Recall that we have $\pi^*(K_X) \sim M + E' \equiv \theta F + E'$ where

$$\theta := \deg \mathcal{L} \geq p_g(X) - 1,$$

$|M| := \text{Mov}|K_X|$ and $E'$ is an effective $\mathbb{Q}$-divisor. When $K_{F_0}^2 \geq 2$, by Chen–Zhang [8, Lemma 3.7, Lemma 4.7], we have

$$K_X^3 \geq (1 - \frac{1}{p_g(X)})^2 \cdot K_{F_0}^2 \cdot (p_g(X) - 1) > \frac{4}{3}p_g(X) - \frac{10}{3}.$$

Since $p_g(X) > 0$, we have $p_g(F) > 0$. Thus, when $K_{F_0}^2 = 1$, the Noether inequality for surfaces implies $1 \leq p_g(F) \leq 2$.

Theorem 4.1. Let $X$ be a minimal projective 3-fold of general type. Assume $p_g(X) \geq 5$ and $d = 1$. If $F$ is a $(1,1)$ surface, then $K_X^3 \geq \frac{27}{20}p_g(X) - \frac{9}{5}$.

Remark 4.2. Since we are only concerned with the inequality $K_X^3 \geq \frac{4}{3}p_g(X) - \frac{10}{3}$, Theorem 4.1 may be improved to some extent.
Theorem 4.3. (=Claim) Let $X$ be a minimal projective 3-fold of general type. Assume $p_g(X) \geq 5$ and $d = 1$. If $F$ is a $(1, 2)$ surface, then $K_X^3 \geq \frac{3}{5}p_g(X) - \frac{10}{3}$.

References