代数幾何学シンポジウム記録
2014年度 pp.32-38

Classification of Calabi-Yau threefolds of type K

Kenji Hashimoto

December 31, 2014

Abstract

Any Calabi–Yau threefold X with infinite fundamental group admits an étale Galois covering either by an abelian threefold or by the product of a K3 surface and an elliptic curve. We call X of type A in the former case and of type K in the latter case. We provide the full classification of Calabi–Yau threefolds of type K, based on Oguiso and Sakurai's work [8]. Together with a refinement of their result on Calabi–Yau threefolds of type A, we finally complete the classification of Calabi–Yau threefolds with infinite fundamental group. This is a joint work with A. Kanazawa [5].

1 Introduction

In this note, a Calabi–Yau threefold is a smooth complex projective threefold X with trivial canonical bundle and $H^1(X, \mathcal{O}_X) = 0$. Let X be a Calabi–Yau threefold with infinite fundamental group. Then the Bogomolov decomposition theorem [2] implies that X admits an étale Galois covering either by an abelian threefold or by the product of a K3 surface and an elliptic curve. We call X of type A in the former case and of type K in the latter case. Among such coverings, there exists a unique smallest one up to isomorphism as a covering [3]. We call the smallest covering the minimal splitting covering of X. Our main result is the following:

Theorem 1.1 ([5, Theorem 3.1])**.** There exist exactly eight Calabi–Yau threefolds of type K up to deformation equivalence. The equivalence class is uniquely determined by the Galois group G of the minimal splitting covering. Moreover, the Galois group is isomorphic to one of the following combinations of cyclic and dihedral groups

$$C_2,\ C_2 \times C_2,\ C_2 \times C_2 \times C_2,\ D_6,\ D_8,\ D_{10},\ D_{12} \text{ or } C_2 \times D_8.$$

We also provide the full classification of Calabi–Yau threefolds of type A, again based on Oguiso and Sakurai's work [8]. It turns out that there exist exactly six deformation classes of Calabi–Yau threefolds of type A. Together with our main result, we finally complete the full classification of Calabi–Yau threefolds with infinite fundamental group:

Theorem 1.2 ([5, Theorem 6.4]). There exist exactly fourteen deformation classes of Calabi–Yau threefolds with infinite fundamental group. More precisely, six of them are of type A, and eight of them are of type K.

2 Calabi–Yau actions

We begin with a brief review of Oguiso and Sakurai's work [8]. Let X be a Calabi–Yau threefold of type K and let $\pi\colon S \times E \to X$ be the minimal splitting covering with Galois group G. Thanks to a result of Beauville [3], we have a canonical isomorphism $\mathrm{Aut}(S \times E) \cong \mathrm{Aut}(S) \times \mathrm{Aut}(E)$. The action of G on $S \times E$ is characterized by the following:

Proposition 2.1 (Oguiso–Sakurai [8, Lemma 2.28]). Define $H := \mathrm{Ker}(G \to \mathrm{GL}(H^{2,0}(S)))$ and take any $\iota \in G \setminus H$. Then the following hold.

1. $\mathrm{ord}(\iota) = 2$ and $G = H \rtimes \langle \iota \rangle$, where the semi-direct product structure is given by $\iota h \iota = h^{-1}$ for any $h \in H$.
2. $g|_S$ is an Enriques involution, i.e. an involution without fixed points, for any $g \in G \setminus H$.
3. $\iota|_E = -1_E$ and $H|_E = \langle t_a \rangle \times \langle t_b \rangle \cong C_n \times C_m$ under an appropriate origin of E, where t_a and t_b are translations of order n and m respectively such that $n|m$. Moreover we have $(n,m) \in \{(1,k)\,(1 \le k \le 6),\ (2,2),\ (2,4),\ (3,3)\}$.

Conversely, if the conditions in the proposition above are satisfied, the quotient $(S \times E)/G$ becomes a Calabi–Yau threefold of type K.

Theorem 2.2 (Oguiso–Sakurai [8, Theorem 2.23]). Let X be a Calabi–Yau threefold of type K. Let $S \times E \to X$ be the minimal splitting covering and G its Galois group. Then the following hold.

1. G is isomorphic to one of the following:

   $C_2,\ C_2 \times C_2,\ C_2 \times C_2 \times C_2,\ D_6,\ D_8,\ D_{10},\ D_{12},\ C_2 \times D_8$, or $(C_3 \times C_3) \rtimes C_2$.

2. In each case the Picard number $\rho(X)$ of X is uniquely determined by G and is calculated as $\rho(X) = 11, 7, 5, 5, 4, 3, 3, 3, 3$ respectively.
3. The cases $G \cong C_2,\ C_2 \times C_2,\ C_2 \times C_2 \times C_2$ really occur.

It has not been settled yet whether or not there exist Calabi–Yau threefolds of type K with Galois group $G \cong D_{2n}$ $(3 \leq n \leq 6)$, $C_2 \times D_8$ or $(C_3 \times C_3) \rtimes C_2$. We settle this classification problem of Calabi–Yau threefolds of type K. In particular, it turns out that the case $G \cong (C_3 \times C_3) \rtimes C_2$ does not occur and that the other cases in the theorem above really occur.

Example 2.3 (Enriques Calabi–Yau threefold)**.** Let S be a K3 surface with an Enriques involution ι and E an elliptic curve with the negation -1_E. The free quotient

$$X := (S \times E)/\langle(\iota, -1_E)\rangle$$

is the simplest Calabi–Yau threefold of type K, known as the Enriques Calabi–Yau threefold.

By Proposition 2.1, the classification problem of Calabi–Yau threefolds of type K is reduced to that of *Calabi–Yau actions* defined by the following:

Definition 2.4. Let G be a finite group. We say that an action of G on a K3 surface S is a Calabi–Yau action if the following hold.

1. $G = H \rtimes \langle\iota\rangle$ for some $H \cong C_n \times C_m$ with $(n, m) \in \{(1, k)\ (1 \leq k \leq 6),\ (2, 2),\ (2, 4),\ (3, 3)\}$, and ι with $\mathrm{ord}(\iota) = 2$. The semi-direct product structure is given by $\iota h \iota = h^{-1}$ for any $h \in H$.
2. H acts on S symplectically, that is, the induced action of H on $H^{2,0}(S)$ is trivial, and any $g \in G \setminus H$ acts on S as an Enriques involution.

3 Construction and uniqueness for the case $G = D_{10}$

Now we discuss about construction and uniqueness of Calabi–Yau G-actions for the case $G = D_{10}$ as an example. In this case, we have $H = C_5$. Recall that a generic K3 surface with the simplest Calabi–Yau action, namely an Enriques involution, is realized as a Horikawa model. Similarly, a generic K3 surface equipped with a Calabi–Yau G-action is realized as an "H-equivariant" Horikawa model, as follows.

Let h be a generator of H. Recall that we have $G = H \rtimes \langle\iota\rangle$ with $\mathrm{ord}(\iota) = 2$. Set $Z = \mathbb{P}^1 \times \mathbb{P}^1$. Let $F = F(x, y, z, w) \in H^0(\mathcal{O}_Z(4, 4))$ be a homogeneous polynomial of bidegree $(4, 4)$. Let ζ_5 be a primitive fifth root of unity. Assume that F is G-invariant, where the action of G is defined by

$$h\colon (x, y, z, w) \mapsto (\zeta_5 x, \zeta_5^{-1} y, \zeta_5^2 z, \zeta_5^{-2} w), \quad \iota\colon (x, y, z, w) \mapsto (y, x, w, z). \tag{3.1}$$

Then F is a linear combination of the following polynomials:

$$x^4 z w^3 + y^4 z^3 w,\ x^3 y z^4 + x y^3 w^4,\ x^2 y^2 z^2 w^2. \tag{3.2}$$

If F is generic, the curve $B \subset Z$ defined by $F = 0$ is smooth and the induced action of any $g \in G \setminus H$ on B has no fixed points. Let S denote the double covering of Z branching along B. Then S is a K3 surface. Note that

$$(xdy - ydx) \wedge (zdw - wdz)/\sqrt{F} \tag{3.3}$$

gives a nowhere vanishing holomorphic 2-form on S. We can find a lift of the action of G on Z to that on S which is a Calabi–Yau action. The quotient $(S \times E)/G$ as in Section 2 is a Calabi–Yau threefold of type K. Let $\mathcal{L}$ denote the pullback of $\mathcal{O}_Z(1,1)$ by the natural map $S \to Z$. The K3 surface S with the polarization $\mathcal{L}$ is a polarized K3 surface of degree 4 (cf. the Key Lemma below).

Next we prove the uniqueness of a Calabi–Yau action of $G = D_{10}$. More precisely, we prove that a generic K3 surface with a Calabi–Yau G-action is realized as above. This implies that a Calabi–Yau threefold $(S \times E)/G$ of type K (as in Section 2) for $G = D_{10}$ is unique up to deformation. The key to proving the uniqueness is the following Key Lemma, which is also true for the other G including $(C_3 \times C_3) \rtimes C_2$.

Key Lemma. Let S be a K3 surface with a Calabi–Yau G-action. Then there exists a G-invariant element $v \in \mathrm{NS}(S)$ such that $v^2 = 4$.

The proof of the Key Lemma will be given in Section 4. Before proving the uniqueness, we consider projective models of K3 surfaces with a Calabi–Yau action.

Lemma 3.1 (Oguiso–Sakurai [8, Lemma 1.7])**.** Let S be a K3 surface with an action of a finite group G and let x be a G-invariant element in $\mathrm{NS}(S) \otimes \mathbb{R}$ with $x^2 > 0$. Then there exists $\gamma \in \mathrm{O}(H^2(S, \mathbb{Z}))$ such that $\gamma(H^{2,0}(S)) = H^{2,0}(S)$, $\gamma(x)$ is nef, and γ commutes with G.

Lemma 3.2. Let S be a K3 surface with a Calabi–Yau G-action. Assume that there exists a G-invariant element $v \in \mathrm{NS}(S)$ such that $v^2 = 4$. Then there exists a G-invariant line bundle $\mathcal{L}$ on S satisfying the following conditions.

1. $\mathcal{L}^2 = 4$ and $h^0(\mathcal{L}) = 4$.
2. The linear system $|\mathcal{L}|$ defined by $\mathcal{L}$ is base-point free and defines a map $\phi_{\mathcal{L}} \colon S \to \mathbb{P}^3$.
3. $\dim \phi_{\mathcal{L}}(S) = 2$.
4. The degree $\deg \phi_{\mathcal{L}}$ of the map $\phi_{\mathcal{L}} \colon S \to \phi_{\mathcal{L}}(S)$ is 2, and $\phi_{\mathcal{L}}(S)$ is isomorphic to either $\mathbb{P}^1 \times \mathbb{P}^1$ or a cone (i.e. a nodal quadric surface).

Proof. We may assume that v is nef by Lemma 3.1. Let $\mathcal{L}$ be a line bundle on S representing v. By Saint-Donat [9, Sections 4 and 8], we have $h^0(\mathcal{L}) = 4$ and either of the following occurs.

(a) $\mathcal{L}$ is base-point free, $\dim \phi_{\mathcal{L}}(S) = 2$, and $\deg \phi_{\mathcal{L}} = 1$ or 2. Any connected component of $\phi_{\mathcal{L}}^{-1}(p)$ for any p is either a point or an ADE-configuration.

(b) $\mathcal{L} \cong \mathcal{O}_S(3E + \Gamma)$ and $|\mathcal{L}| = \{D_1 + D_2 + D_3 + \Gamma \mid D_i \sim E\}$, where E and $\Gamma \cong \mathbb{P}^1$ are irreducible divisors such that $E^2 = 0$, $\Gamma^2 = -2$ and $E.\Gamma = 1$.

In Case (b), the base locus $\Gamma \cong \mathbb{P}^1$ of $|\mathcal{L}|$ is stable under the action of ι and thus ι has a fixed point in Γ, which is a contradiction. Hence Case (a) occurs. Since the fixed locus of any (projective) involution of $\mathbb{P}^3$ is at least 1-dimensional, there exists a fixed point p of the action of ι on $\phi_{\mathcal{L}}(S)$. If $\deg \phi_{\mathcal{L}} = 1$, then ι has a fixed point in S, which is a contradiction. Hence $\deg \phi_{\mathcal{L}} = 2$, and $\phi_{\mathcal{L}}(S)$ is an irreducible quadric surface in $\mathbb{P}^3$, which is either $\mathbb{P}^1 \times \mathbb{P}^1$ or a cone. □

Now we prove the uniqueness of a Calabi–Yau G-action. For a generic K3 surface S with a Calabi–Yau G-action, the image $\phi_{\mathcal{L}}(S)$ of S by the map $\phi_{\mathcal{L}}$ defined in Lemma 3.2 is isomorphic to $Z = \mathbb{P}^1 \times \mathbb{P}^1$ (see [5] for details). By direct computation, the map $\phi_{\mathcal{L}} \colon S \to Z$ is of the form given above. Note that if the action of G is defined by

$$h\colon (x, y, z, w) \mapsto (\zeta_5 x, \zeta_5^{-1} y, \zeta_5 z, \zeta_5^{-1} w), \quad \iota\colon (x, y, z, w) \mapsto (y, x, w, z), \tag{3.4}$$

then the branching curve B has "bad" singular points and the double covering S is not a K3 surface (even after resolution).

4 Proof of the Key Lemma

In this section we prove the Key Lemma for $G = D_{10}$. We use the lattice theory due to Nikulin [7]. The second integral cohomology $H^2(S, \mathbb{Z})$ of S with the cup product $\langle\ ,\ \rangle$ is considered as a lattice.

Recall that the action of H on S is symplectic. Hence the minimal resolution $\widetilde{S}$ of S/H is again a $K3$ surface. The action of ι on S induces that on $\widetilde{S}$. We have the following:

- The quotient S/H has exactly four singular points, each of which is of type A_4 (Nikulin [6]).
- ι acts on $\widetilde{S}$ as an Enriques involution.

For a lattice L, we define $L(n)$ to be the lattice obtained by multiplying the bilinear form of L by n. Since ι acts on $\widetilde{S}$ as an Enriques involution, the invariant part $H^2(S, \mathbb{Z})^\iota$ of $H^2(S, \mathbb{Z})$ by the action of ι is isomorphic to $U(2) \oplus E_8(-2)$ (see [1, Section VIII.19]), where U is the lattice defined by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We have the following:

- For the sublattice M of $H^2(\widetilde{S}, \mathbb{Z})$ which is generated by the exceptional curves of the map $\widetilde{S} \to S/H$, we have $M \cong A_4(-1)^{\oplus 4}$.
- $M^\iota \cong A_4(-2)^{\oplus 2}$.
- $(M^\iota)^\perp \cong U(2)$ or $U(10)$.

Here $(M^\iota)^\perp$ denotes the orthogonal part of M^ι in $H^2(S, \mathbb{Z})^\iota$. By Lemma 4.1 below, it follows that $(M^\iota)^\perp \cong U(10)$. Consider the pullback map

$$\pi^*\colon H^2(\widetilde{S}, \mathbb{Z}) \to H^2(S, \mathbb{Z})$$

induced by the natural rational map $\pi\colon S - \to \widetilde{S}$. For any $x, y \in M^\perp$ (in $H^2(\widetilde{S}, \mathbb{Q})$), we have $\langle \pi^* x, \pi^* y\rangle = |H|\langle x, y\rangle = 5 \cdot \langle x, y\rangle$. By Garbagnati [4, Proposition 2.4], we have

$$H^2(S, \mathbb{Z})^G = \pi^*\{x/5 \mid x \in (M^\iota)^\perp,\ \langle x/5, (M^\iota)^\perp\rangle \subset \mathbb{Z}\} \cong U(2). \qquad (4.1)$$

Since we have $H^2(S, \mathbb{Z})^G \subset H^2(S, \mathbb{Z})^\iota \subset \mathrm{NS}(S)$, the Key Lemma follows.

Lemma 4.1. There is no element $v \in (M^\iota)^\perp$ such that $v^2 = 4$.

Proof. We assume that an element $v \in (M^\iota)^\perp$ satisfies $v^2 = 4$ and derive a contradiction. By Lemma 3.1, we may assume that v is nef (see [5] for details). Recall that the action of ι on S/H has no fixed points. By the same argument as in the proof of Proposition 3.2, the class v induces a morphism $\widetilde{f}\colon \widetilde{S} \to \mathbb{P}^3$ such that $\widetilde{f}(\widetilde{S})$ is a quadric surface and the degree of $\widetilde{f}$ is 2. Since we have $v \perp M$ by the assumption, the morphism $\widetilde{f}$ induces a morphism $f\colon S/H \to \mathbb{P}^3$. We may assume that $f(S/H) \cong \mathbb{P}^1 \times \mathbb{P}^1$ by taking a generic S (see [5] for details). The action of ι on S/H is of the form $\sigma\tau$, where σ is induced by a symplectic involution of $\widetilde{S}$ and τ is the covering transformation of f. Let $\overline{\tau} \in \mathrm{Aut}(S)$ be a lift of τ. Note that $\overline{\tau}$ normalizes H. Since f induces a generically one-to-one morphism $S/\langle H, \overline{\tau}\rangle \to \mathbb{P}^1 \times \mathbb{P}^1$, it follows that $S/\langle H, \overline{\tau}\rangle$ is smooth and that the action of τ fixes each singular point of S/H. Hence the actions of a generator of H and $\overline{\tau}$ are represented by the matrices $\begin{bmatrix} \zeta_5 & 0 \\ 0 & \zeta_5^{-1} \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ respectively, in local coordinates around a fixed point of the action of H on S. Therefore we have $\overline{\tau} h \overline{\tau} = h^{-1}$ for any $h \in H$ $(\star)$.

We checked that τ fixes each point in $\mathrm{Sing}(S/H)$. Since σ is a symplectic involution, the action of σ has exactly 8 fixed points $q_i \notin \mathrm{Sing}(S/H)$, $1 \le i \le 8$ (Nikulin [6]). Let $Q_i \subset S$ denote the inverse image of q_i. Then $|Q_i| = |H| = 5$. Take a point $p \in Q_i$. Since H acts on Q_i transitively, we can take a lift $\overline{\sigma} \in \mathrm{Aut}(S)$ of σ such that $\overline{\sigma} \cdot p = p$. The action of $\overline{\sigma}$ around p is locally identified with that of σ around q_i. Therefore $\mathrm{ord}(\overline{\sigma}) = 2$. Since $\overline{\sigma}\,\overline{\tau} \in H\iota$, the condition ($\star$) implies that $\overline{\sigma}$ commutes with H. Hence the action of $\overline{\sigma}$ on each Q_i is trivial or free and the fixed locus $S^{\overline{\sigma}}$ of the action of $\overline{\sigma}$ on S is the union of some Q_i. On the other hand, similarly to σ, we have $|S^{\overline{\sigma}}| = 8$. This is a contradiction. □

References

[1] W. P. Barth, K. Hulek, C. A. M. Peters and A. van de Ven, Compact complex surfaces, Second edition, Springer-Verlag, Berlin, 2004.

[2] A. Beauville, Variétés Kähleriennes dont la première classe de Chern est nulle, J. Diff. Geom. 18 (1983), no. 4, 755–782.

[3] A. Beauville, Some remarks on Kähler manifolds with $c_1 = 0$, in Classification of Algebraic and Analytic Manifolds, K. Ueno, ed., Progress Math. 39 (1983), 1–26.

[4] A. Garbagnati, Symplectic automorphisms on Kummer surfaces, Geom. Dedicata 145 (2010), 219–232.

[5] K. Hashimoto and A. Kanazawa, Calabi–Yau threefolds of type K (I) Classification, arXiv:1409.7601

[6] V. V. Nikulin, Finite groups of automorphisms of Kählerian surfaces of Type K3, Trudy Moskov. Mat. Obshch. 38 (1979), 75–137.

[7] V. V. Nikulin, Integer symmetric bilinear forms and some of their geometric applications, Izv. Akad. Nauk SSSR Ser. Mat. 43 (1979), no. 1, 111–177.

[8] K. Oguiso and J. Sakurai, Calabi–Yau threefolds of quotient type, Asian J. Math. 5 (2001), no.1, 43–77.

[9] B. Saint-Donat, Projective models of K3 surfaces, Math. Z, 189 (1985), 1083–1119.