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Automorphisms of O'Grady's Sixdimensional Manifold Acting Trivially on Cohomology

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1. Introduction

Let X be a smooth projective variety and σ an automorphism of X. Then σ acts via pullback of cocycles on the second cohomology group. In particular, we obtain a representation

$$\nu\colon \mathrm{Aut}(X) \to O(H^2(X,\mathbb{Z})),$$

called the *cohomological representation,* where $O(H^2(X,\mathbb{Z}))$ is a suitable subgroup of $GL(H^2(X,\mathbb{Z}))$, e.g. Hodge isomorphisms etc. In the case of K3 surfaces the famous Torelli theorem states that this representation is injective and this fact has been vastly used to study the geometry, i.e. the automorphisms of K3 surfaces by Nikulin, Mukai, Kondo and many others.

2. IHS manifolds

Definition 2.1. A compact kähler manifold X is called *irreducible holomorphic symplectic* (IHS for short), if $\pi_1(X) = \{1\}$ and $H^0(\Omega^2_X)$ is generated by a nowhere degenerate holomorphic two-form.

Example 2.2.
- If $\dim X = 2$, then $X = S$ a K3 surface.
- Let S be a K3, then $X := S^{[n]} = Hilb^n(S)$ is a $2n$-dimensional IHS introduced by Beauville.
- More generally in many cases moduli spaces of sheaves $M(v)$ on K3s have been shown (by Mukai and others) to be IHS deformation equivalent to $Hilb^n(S)$ (for some n).
- Let A be an abelian surface (or more generally a complex 2-torus) then it Hilbert scheme $A^{[n]}$ is not simply connected but admits an Albanese map

$$K^n(A) \to A^{[n]} \xrightarrow{Alb} A,$$

with isomorphic $2n-2$-dimensional fibres, which are IHS manifolds called *generalised Kummer manifolds*and have been introduce by Beauville. Note that for $n = 2$, $K^2(A)$ is nothing but the Kummer surface associated with A.

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– More generally, by work of Yoshioka, any moduli space of sheaves $M(v)$ on an abelian surface admits an Albanese map

$$K(v) \to M(v) \xrightarrow{Alb} A \times A^{\vee},$$

where $A^{\vee}$ denotes the dual torus, and again in many cases the fibres $K(v)$ are IHS manifolds deformation equivalent to $K^n(A)$.

Remark 2.3. The above statements about moduli spaces of sheaves $M(v)$ hold only if the vector v of numerical invariants (the so called *Mukai vector*) is primitive as an element in a certain lattice. Sheaf theoretically this means that no semistable sheaf $\mathcal{F}$ with these invariants can be written as a direct sum $\mathcal{F} = \bigoplus_i \mathcal{F}_i$ where all the $\mathcal{F}_i$ have the same invariants.

On the contrary, we have the following result for a special case where v is not primitive.

Theorem 2.4 O'Grady, Perego, Rapagnetta. *Let v be primitive such that $\dim M(v) = 4$, then $M(2v)$ and its albanese fibre $K(2v)$ are 2-factorial singular symplectic varieties of dimension ten respectively six, which admit symplectic resolutions obtained by blowing up the singular locus with its reduced scheme structure:*

$$\begin{array}{ccccc}
\widetilde{K}(2v) & \longrightarrow & \widetilde{M}(2v) & \overset{Alb}{\searrow} & \\
\downarrow & & \downarrow & & \\
K(2v) & \longrightarrow & M(2v) & \xrightarrow[Alb]{} & A \times A^{\vee},
\end{array}$$

and $\widetilde{K}(2v)$ is a six dimensional IHS not deformation equivalent to any of the other known examples.

Remark 2.5. Any manifold obtained in such a way is denotes by *manifold of Og_6-type.* There is a similar construction involving moduli spaces of sheaves for non-primitive Mukai vectors on K3 surfaces. In this case there is no Albanese map and we obtain new tendimensional manifolds, which are referred to as *manifolds of Og_{10}-type.*

3. The cohomological representaion $\nu\colon \mathrm{Aut}(X) \to O(H^2(X,\mathbb{Z}))$

Let us start by a fundamental result for the understanding of this representation:

Theorem 3.1 Hassett–Tschinkel. *The kernel of ν is a deformation invariant of the manifold X.*

Example 3.2.
- If $X = S$ a K3 or $X = S^{[n]}$, then $\ker \nu$ is trivial. (Beauville, [Beau83])
- If $X = K^n(A)$, then $\ker \nu \cong \langle A[n], \pm \mathrm{id}_A \rangle$, where $A[n]$ denotes the n-torsion points of the surface A. The automorphisms are induced by the translation- and $(-\mathrm{id})$-automorphisms of the abelian surface. (Boissière–Nieper-Wisskirchen–Sarti, [BNS11])
- If X is of Og_{10}-type, then $\ker \nu$ is trivial by [MW14].

The main result discussed in this note is the following:

Theorem 3.3 Mongardi–W. *Let X be of Og_6-type. Then*

$$\ker \nu \cong G_0 := \langle A[2], A^{\vee}[2] \rangle \cong (\mathbb{Z}/2\mathbb{Z})^{\times 8}.$$

4. Sketch of the Proof

4.1 Choice of X

By **Theorem 3.1** we can choose an arbitrary X of Og_6-type. Thus consider a generic genus two curve and let (A, H) be its Jacobian, polarised by a symmetric theta divisor. We choose v such that $M(2v)$ is isomorphic to the product of $A^\vee$ with the relative compactified Jacobian $J^4(|2H|)$ of degree four over the linear system $|2H|$. It follows that the corresponding $X = \widetilde{K}(2v)$ is of Og_6-type and admits a Lagrangian fibration over $|2H| \cong \mathbb{P}^3$.

4.2 G_0 acts on X

It is a straightforward consequence of [Yos99] (cf. diagram (1.8)) that we have an action of G_0 on $J^4(|2H|)$ preserving the Albanese fibre. Moreover it follows from the description of the singular locus that this action lifts to the blow-up. By the description of the second cohomology of moduli spaces of sheaves on abelian surfaces, it is clear that G_0 acts trivially on the second cohomology.

4.3 Action on the linear system

Now, in order to show the inclusion $\ker\nu \subseteq G_0$, we let $\sigma \in \ker\nu$ be arbitrary. We easily deduce that σ descends to an action of $K(2v)$ and the fibration $\pi\colon K(2v) \to |2H|$ is equivariant. We thus obtain an action of σ on $|2H|$.

4.4 Action of $A[2]$

It is easy to see that the action of σ on $|2H|$ must preserve the stratification of the linear system into analytic type of singularity. The detailed analysis of Rapagnetta (cf. [Rap07, Prop. 2.1.3]) shows that the projective dual of the singular Kummer surface associated with A forms one such stratum in $|2H|$ and another one is given by the nodes of this surface. Hence the action is actually given by translations of points in $A[2]$. Thus up to the action of $A[2]$ we may assume that σ acts fibrewise.

4.5 A rigid divisor

We prove that there is a rigid divisor on $K(2v)$. Such divisor must be preserved by any automorphism acting trivially on the second cohomology. We define a divisor D in $K(2v)$ as follows: The Jacobian $J^4(|2H|)$ carries a natural relative theta divisor $\Theta_{|2H|}$ of the fibration π. We set $D \coloneqq \Theta_{|2H|} \cap K(2v)$. This divisor allows the following birational description

$$D \doteq \{\mathcal{O}_C(p + \iota(p) + q + \iota(q)) \mid C \in |2H|, p, q \in C\},$$

where ι denotes the involution on every C induced by $-\mathrm{id}_A$. Now we define a map

$$\begin{aligned} D &\dashrightarrow Sym^2(Kum_s), \\ \mathcal{O}(p + \iota(p) + q + \iota(q)) &\mapsto p + q. \end{aligned}$$

The fibre over $p + q$ consists of curves in $|2H|$ that pass through p and q, hence is a $\mathbb{P}^1$. It is now a well-known fact that a divisor (in a symplectic variety), which admits a $\mathbb{P}^1$-fibration, is rigid.

4.6 Putting things together

Summarising we have shown that we may assume that σ acts fibrewise and preserves the relative theta divisor of π. We consider now the induced action on one general fibre. (Note that $A^\vee[2]$ naturally acts on these fibres by translation by points of order two.) Thus let $C \in |2H|$ be a smooth genus five curve. Note that it is given as the unramified double cover of a smooth genus three curve $\bar{C}$ in the singular kummer surface. The Albanese fibre $K(C)$ of the Jacobian $J^4(C)$ is given as the image of $J^2(\bar{C})$ via the pullback map. The action of σ on $K(C)$ cannot be given by -1 since this would yield a non-symplectic automorphisms of X. Thus it has to be given by translation of a point of finite order. It follows now from general facts about Prym varieties that any translation in $K(C)$ preserving the restriction of the theta divisor is given by a two-torsion point in the quotient $J^4(C)/K(C) \cong A$.

5. Outlook

The next important thing to do is to study the fixed locus of the automorphisms in G_0. Closely related is the question wether the action of G_0 on the full cohomology $H^*(X, \mathbb{Z})$ is also trivial or not.

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