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Stability conditions and Fourier-Mukai transformations on $K3$ surfaces with Picard rank 1

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1 Backgrounds

Let M and W are projective manifolds over $\mathbb{C}$. If there is an equivalence between the bounded derived categories $D^b(M)$ and $D^b(W)$ of coherent sheaves on M and W, then W is said to be a Fourier-Mukai partner (shortly FM partner) of M. The following conjecture was proposed by Kawamata:

Conjecture 1.1 ([11]). *Let M be a projective manifold. The set* $\mathrm{FM}(M)$ *of isomorphism classes of FM partners of M is a finite set.*

We have to remark that the conjecture holds if $\dim M \leq 2$ or M is abelian variety by [10, mainly in Chapter 5], [2], [11], [14] and [6]. Motivated the conjecture, we wish to discuss relations between $D^b(M)$ and $\mathrm{FM}(M)$. Here the word "relations" does not have a specifying meaning. For instance, the following question suggests an example of "relations":

Question. Can we prove the conjecture in terms of the autoequivalence group $\mathrm{Aut}(D^b(M))$ of $D^b(M)$?

For the question, we have the following very partial answer:

Proposition 1.2. *Let M be a projective manifold. Suppose that* $\mathrm{Aut}(D^b(M))$ *is generated* *by:*

$$\mathrm{Aut}(D^b(M)) = \langle f_*, L, [1] | L \in \mathrm{Pic}(M), f \in \mathrm{Aut}(M) \rangle,$$

where $[1]$ *is the shift functor on* $D^b(M)$. *Then* M *has the only trivial FM partner* M *itself.*

Proof. Let $\Phi : D^b(M) \to D^b(W)$ be an equivalence between projective manifolds and let L_W be a very ample line bundle on W. By the assumption, the functor $F := \Phi^{-1} \circ (\otimes L_W) \circ \Phi$ is generated by

$$T(M) := \langle f_*, L, [1] | L \in \mathrm{Pic}(M), f \in \mathrm{Aut}(M) \rangle,$$

$$\begin{array}{ccc} D(M) & \xrightarrow{\Phi} & D(W) \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle L_W \otimes (-)} \\ D(M) & \xrightarrow{\Phi} & D(W). \end{array}$$

Since kL_W has a global section for any positive integer $k \in \mathbb{Z}_{>0}$, we can make a morphism $E \to E \otimes kL_W$ for any $E \in D(W)$. In particular since kL_W is base point free, we can choose the section so that the morphism $E \to E \otimes kL_W$ is not 0. Thus, for any $k \in \mathbb{Z}_{>0}$ and $E \in D(W)$, we have $\mathrm{Hom}_{D(Y)}(E, E \otimes kL_W) \neq 0$. Thus, for any closed point $x \in M$, we have

$$\mathrm{Hom}_{D(M)}(\mathcal{O}_x, F^k(\mathcal{O}_x)) \cong \mathrm{Hom}_{D(W)}(\Phi(\mathcal{O}_x), \Phi(\mathcal{O}_x) \otimes kL_W) \neq 0.$$

Since $F \in T(M)$ we have

$$F(-) = L_M \otimes f_*(-)[n],$$

where $f \in \mathrm{Aut}(M)$, $L_M \in \mathrm{Pic}(M)$ and $n \in \mathbb{Z}$. We wish to prove that $n = 0$ and $f = id_M$.

Suppose to the contrary that $n \neq 0$. Since $n \neq 0$, for sufficiently large $\ell \in \mathbb{Z}_{>0}$, we have $\mathrm{Hom}_{D(M)}(\mathcal{O}_x, F^\ell(\mathcal{O}_x)) = 0$ where F^ℓ is the ℓ times composition of F. This is contradiction. Hence n should be 0.

We assume that $f \neq id_M$. Then there is a closed point $x \in M$ such that $f(x) \neq x$. Since $F(\mathcal{O}_x) = \mathcal{O}_{f(x)}$, we have

$$\mathrm{Hom}_{D(M)}(\mathcal{O}_x, F(\mathcal{O}_x)) = 0.$$

This is contradiction.

Since $F(-) = L_M \otimes (-)$, we have for any positive integer k,

$$\Phi(\mathcal{O}_x) \otimes kL_W = \Phi(\mathcal{O}_x \otimes kL_M) = \Phi(\mathcal{O}_x).$$

Thus each Hilbert polynomial of $H^i(\Phi(\mathcal{O}_x))$ with respect to L_W is constant. Since L_W is very ample, it follows that $\dim \mathrm{Supp}(H^i(\Phi(\mathcal{O}_x))) = 0$. Thus $\dim \mathrm{Supp}(\Phi(\mathcal{O}_x)) = 0$. By [10, Lemma 4.5], we have

$$\Phi(\mathcal{O}_x) = \mathcal{O}_{y_x}[n_x],$$

for some $y_x \in Y$ and $n_x \in \mathbb{Z}$. Since any equivalence has a Fourier-Mukai kernel by [14, Theorem 2.18], we see that n_x is locally constant. Hence, n_x is constant. So we put $n_x = n$. Since skyscraper sheaves are stable under the equivalence $\Phi : D^b(M) \to D^b(W)$, W is isomorphic to M (for instance, see [10, Corollary 5.23]). □

Proposition 1.2 gives an evidence that there should be a relation between $\mathrm{Aut}(D^b(M))$ and $\mathrm{FM}(M)$, but there is a problem: we do not have better ideas to discuss the relation since the evidence is clearly weak. Under this situation, a study of Hosono-Oguiso-Lian-Yau (for short, HLOY) gives us a cue for our problem.

Theorem 1.3 ([8]). *Let X be a projective $K3$ surface with $\mathrm{NS}(X) = \mathbb{Z}L$. Put $L^2 = 2d$. The following equation holds:$\#\mathrm{FM}(X) = [\mathrm{AL}_d : \mathrm{Fr}_d]$. Here AL_d and Fr_d are respectively Atkin-Lehner group and Fricke group of level d (defined in latter).*

Roughly speaking, AL_d and Fr_d are discrete subgroup of $PSL_2(\mathbb{R})$. In the next section we introduce the relation between $\mathrm{FM}(X)$ and AL_d.

2 HLOY's observation

The main aim of this section is the introduction of HLOY's observation. Before the introduction, we recall the definition of the Atkin-Lehner group and the Fricke group.

2.1 Atkin-Lehner groups

As usual we put

$$\Gamma_0(d) = \{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in PSL_2(\mathbb{Z}) | \gamma \in d\mathbb{Z}\}.$$

For integers $s, d \in \mathbb{Z}$ we define the symbol $s||d$ by

$$s||d \overset{\text{def}}{\iff} s|d \text{ and } \gcd(s, \frac{d}{s}) = 1. \tag{2.1}$$

Suppose $s||d$. We put

$$W_s = \{\frac{1}{\sqrt{s}}\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} \in PSL_2(\mathbb{R}) | \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(d/s) \text{ and } \delta \in s\mathbb{Z}\}.$$

W_s is also given as

$$W_s = \{\begin{pmatrix} \alpha\sqrt{s} & \frac{\beta}{\sqrt{s}} \\ \gamma\frac{d}{s}\sqrt{s} & \delta\sqrt{s} \end{pmatrix} \in PSL_2(\mathbb{R}) | \alpha, \beta, \gamma \text{ and } \delta \in \mathbb{Z}\}.$$

In particular we see $W_1 = \Gamma_0(d)$.

For cosets W_s one can check the following:

Lemma 2.1 ([5]). *Each W_s is a single coset under the multiplication of W_1 and is in the normalizer of $\Gamma_0(d)$ in $PSL_2(\mathbb{R})$. In addition the coset classes W_s and $W_{s'}$ satisfies the following rule of products:*

$$W_s^2 = W_1, W_s W_{s'} = W_{s'} W_s = W_{s*s'},$$

where $s * s' = \frac{ss'}{\gcd(s,s')^2}$

Definition 2.2. We define subsets of $PSL_2(\mathbb{R})$ by

$$\mathrm{AL}_d := \bigsqcup_{s||d} W_s \text{ and } \mathrm{Fr}_d := W_1 \sqcup W_d.$$

By Lemma 2.1, we see that AL_d and Fr_d are subgroups of $PSL_2(\mathbb{R})$. We call AL_d and Fr_d respectively the *Atkin-Lehner group* and the *Fricke group* of level d.

Remark 2.3. AL_d is the abelian normalizer group of $\Gamma_0(d)$ in $PSL_2(\mathbb{R})$. Since $W_s W_d = W_{\frac{d}{s}}$, the coset decomposition of $\mathrm{AL}_d/\mathrm{Fr}_d$ is given by

$$\mathrm{AL}_d/\mathrm{Fr}_d = \bigsqcup_{s||d} (W_s \sqcup W_{\frac{d}{s}}).$$

2.2 HLOY's observation

Let X be a projective $K3$ surface. Recall that the total cohomology ring

$$H^*(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$$

has a pure Hodge structure with weight 2 (For instance, see [10, Chapter 10]). Moreover $H^*(X, \mathbb{Z})$ has the *Mukai pairing* (or *Euler pairing*) given by

$$\langle r \oplus c \oplus s, r' \oplus c' \oplus s' \rangle = cc' - rs' - sr'.$$

The numerical Grothendieck group of X is given by

$$\mathcal{N}(X) = H^0(X, \mathbb{Z}) \oplus \mathrm{NS}(X) \oplus H^4(X, \mathbb{Z}) \subset H^*(X, \mathbb{Z}).$$

By the Hodge index theorem, the index of the Mukai pairing on $\mathcal{N}(X)$ is $(2, \rho(X))$. For objects $E \in D(X)$ we put $v(E) = ch(E)\sqrt{td_X}$ and call it the *Mukai vector* of E. One can check that $v(E) = r \oplus c \oplus s \in \mathcal{N}(X)$ and see that $r = \operatorname{rank} E$, $c = c_1(E)$ and $s = \chi(X, E) - \operatorname{rank} E$ by the Riemann-Roch theorem.

Let $\Phi \colon D(Y) \to D(X)$ be an equivalence between projective $K3$ surfaces. As is well-known, Φ induces a Hodge isometry $\Phi^H \colon H^*(Y, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ in a standard way (For instance, see [10]). Since Φ^H is a Hodge isometry, we get an isometry $\Phi^N \colon \mathcal{N}(Y) \to \mathcal{N}(X)$ by restricting Φ^H. Namely we have $\Phi^N = \Phi^H|_{\mathcal{N}(X)}$ Thus we obtain the representation of $\operatorname{Aut}(D^b(X))$ on $O^+(\mathcal{N}(X) \otimes \mathbb{R})$ where $O^+(\mathcal{N}(X) \otimes R)$ is a subgroup of $O(\mathcal{N}(X) \otimes \mathbb{R})$ which preserves the orientation of positive 2 planes. Moreover $O^+(\mathcal{N}(X) \otimes \mathbb{R})/\pm 1$ is isomorphic to $PSL_2(\mathbb{R})$ if $\operatorname{rank} \operatorname{NS}(X) = 1$. Thus under the assumption Picard rank 1, we obtain the following representation:

$$\rho : \operatorname{Aut}(D^b(X)) \to O^+(\mathcal{N}(X) \otimes \mathbb{R}) \to PSL_2(\mathbb{R}).$$

We can describe the image of $\rho : \operatorname{Aut}(D^b(X)) \to PSL_2(\mathbb{R})$.

Theorem 2.4 ([3], [9])**.** *Let X be a $K3$ surface with $\operatorname{NS}(X) = \mathbb{Z}L$. Put $L^2 = 2d$. The image of ρ is the Fricke group of level d.*

HLOY expected the following:

Conjecture 2.5 ([8])**.** *Notations and assumptions are being as above. For any $\alpha \in \mathrm{AL}_d$, there exists an equivalence $\Phi : D^b(Y) \to D^b(X)$ such that "ρ"$(\Phi) = \alpha$.*

We have to remark that the definition of "ρ" is not clear since an equivalence is not necessary an autoequivalence. To define "ρ"(Φ) precisely, we use the assumption that Picard rank is 1.

Note that the Picard rank of Y is the same as that of X if there is an equivalence $\Phi : D^b(Y) \to D^b(X)$. Moreover, if $\operatorname{NS}(X) = \mathbb{Z}L_X$, then both numerical Grothendieck groups $\mathcal{N}(Y)$ and $\mathcal{N}(X)$ are canonically isomorphic to the lattice $(\mathbb{Z}^{\oplus 3}, \Sigma)$ where Σ is

$$\Sigma = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 2d & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Now we put these canonical isomorphism as follows:

$$c_X \colon \mathcal{N}(X) \to (\mathbb{Z}^{\oplus 3}, \Sigma) \text{ and } c_Y \colon \mathcal{N}(Y) \to (\mathbb{Z}^{\oplus 3}, \Sigma).$$

By using these canonical isomorphisms, we obtain an extended representation of $\Phi : D^b(Y) \to$

$D^b(X)$. Namely we have the following definition of ρ:

$$\rho(\Phi) := c_X \circ \Phi^N \circ c_Y^{-1}. \tag{2.2}$$

Main results of this article (and also of the talk) is the following:

Theorem 2.6 ([13, Theorem 3.3]). *Let X be a projective $K3$ surface with $\mathrm{NS}(X) = \mathbb{Z}L$ and $\Phi : D^b(Y) \to D^b(X)$ an equivalence. Put $L^2 = 2d$. Then Conjecture 2.5 holds. Moreover if the image of $\Phi : D^b(Y) \to D^b(X)$ by ρ is in Fr_d, then Y is isomorphic to X.*

2.3 Relation between $\mathrm{FM}(X)$ and $\mathrm{AL}_d/\mathrm{Fr}_d$

Let us recall Theorem 1.3. Basically the theorem gives us the equation of numbers. After Theorem 2.6, we have more concrete correspondence between $\mathrm{FM}(X)$ and $\mathrm{AL}_d/\mathrm{Fr}_d$ though this correspondence is still far from our question introduced in §1.

By Theorem 2.4, $\mathrm{Aut}(D^b(X))$ can be divided into 2 classes $\rho^{-1}(W_1) \sqcup \rho^{-1}(W_d)$. Moreover one can see that $\rho^{-1}(W_d)$ is represented by "spherical twist functors" on $D^b(X)$ (of general K3 surfaces). We do not give the definition of spherical twist functors but give a typical example $T_{\mathcal{O}_X}$. (If you need the precise definition, you can see it, for instance, in [10, Chapter 8].) For the structure sheaf $\mathcal{O}_X$ on X, we can define an autoequivalence $T_{\mathcal{O}_X}$. The functor $T_{\mathcal{O}_X}$ sends skyscraper sheaves $\mathcal{O}_x$ of closed points $x \in X$ to the shift $\mathcal{I}_x[1]$ of ideal sheaves $\mathcal{I}_x$ of $x \in X$. So the functor $T_{\mathcal{O}_X}$ can be regarded as an equivalence between $D^b(X)$ and the "dual" $D^b(X)^\vee$ of $D^b(X)$.

In addition, if we take $Y \in \mathrm{FM}(X) \setminus \{X\}$, then there should be an equivalence $\Phi : D^b(Y) \to D^b(X)$ whose image by ρ does not belong to Fr_d. Since $D^b(Y)$ has also the "dual" $D^b(Y)^\vee$, the image $\rho(\Phi)$ belongs to $W_s \sqcup W_{\frac{d}{s}}$ for $^\exists s||d$. As a result, we obtain the following bijection between $\mathrm{FM}(X)$ and $\mathrm{AL}_d/\mathrm{Fr}_d$:

$$\mathrm{FM}(X) \to \mathrm{AL}_d/\mathrm{Fr}_d, Y \mapsto \rho(\Phi) \text{ where } \Phi\colon D^b(Y) \to D^b(X) \text{ an equvalence.}$$

3 Sketch of the proof of Theorem 2.6

We first recall the work of [7] which is an explicit construction of Fourier-Mukai partners of X with $\mathrm{NS}(X) = \mathbb{Z}L$. Put $L^2 = 2d$ as usual. We set the set P_d by

$$P_d = \{r \in \mathbb{N} | r||d\}/\sim$$

where $r_1 \sim r_2$ if and only if $r_1 = r_2$ or $r_1 = \frac{d}{r_2}$.

Theorem 3.1 ([7, Theorem 2.1]). *Let X be a projective $K3$ surface with $\mathrm{NS}(X) = \mathbb{Z}L$. Put $L^2 = 2d$. There is a one to one correspondence between P_d and the set $\mathrm{FM}(X)$ of isomorphic classes of Fourier-Mukai partners of X:*

$$P_d \ni r \mapsto M_L(r \oplus L \oplus \frac{d}{r}) \in \mathrm{FM}(X).$$

Here $M_L(r \oplus L \oplus s)$ is the fine moduli space of μ_L-stable sheaves with Mukai vector $r \oplus L \oplus s$.

Suppose that one wishes to prove Theorem 2.6. Since there is a surjection $\rho : \mathrm{Aut}(D^b(X)) \to \mathrm{Fr}_d$ by Theorem 2.4, it is enough to find an equivalence $\Phi : D^b(Y) \to D^b(X)$ such that $\rho(\Phi) \in W_s$ for any $s||d$. Thus our claim is the following:

Claim 3.2. *Notations are being as above. Let Y be the fine moduli space of μ-stable locally free sheaves on X with Mukai vector $r \oplus L \oplus s$ and $\mathcal{E}_Y$ a universal family. Define an equivalence $\Phi\colon D^b(Y) \to D^b(X)$ by*

$$\Phi : D^b(Y) \to D^b(X) \;\; \Phi(-) := \mathbb{R}\pi_X^*(\mathcal{E}_Y \otimes \pi_Y^*(-)).$$

Then $\rho(\Phi) \in W_s$.

To show the claim, we give an explicit description of the matrix $\rho(\Phi)$. Before giving the description in Proposition 3.3 below, for an arbitrary equivalence $\Phi : D^b(Y) \to D^b(X)$, we put $\mathrm{NS}(X) = \mathbb{Z}L_X$ (resp. $\mathrm{NS}(Y) = \mathbb{Z}L_Y$) and

$$v(\Phi(\mathcal{O}_y)) = r_X \oplus n_X L_X \oplus s_X (\text{resp. } v(\Phi^{-1}(\mathcal{O}_x)) = r_Y \oplus n_Y L_Y \oplus s_Y).$$

Lemma 3.3 ([12, Lemmas 3.1 and 3.2]). *Let $\Phi\colon D(Y) \to D(X)$ be an equivalence between projective $K3$ surfaces of $\deg X = 2d$ with Picard rank 1.*

1. *We have $r = r_X = r_Y$. Moreover if $r_X = 0$, then*

$$\rho(\Phi) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \text{ where } {}^{\exists}m \in \mathbb{Z}.$$

2. *Suppose that $r \neq 0$. Then $\rho(\Phi)$ is given by*

$$\rho(\Phi) = \begin{pmatrix} 1 & \frac{n_X}{r} \\ 0 & 1 \end{pmatrix} \frac{1}{\sqrt{d|r|}} \begin{pmatrix} 0 & -1 \\ d|r| & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{-n_Y}{r} \\ 0 & 1 \end{pmatrix}.$$

Due to Lemma 3.3, we can prove Claim 3.2. Consequently we can also prove Theorem 2.6. The details are in [13].

4 The origin of Lemma 3.3

As you saw in the last section, a key ingredient for Theorem 2.6 is Lemma 3.3. In this section we explain that Lemma 3.3 is closely related to the space of stability conditions on $D^b(X)$ introduced by Bridgeland in [3].

4.1 Stability conditions on $K3$ surfaces

In this subsection, we give a brief review of stability conditions on a $K3$ surface X. We can regard a stability condition on $D^b(X)$ as a generalization of Gieseker stability (or μ-stability) for coherent sheaves on X. One of the obstructions for the generalization is the fact that $D^b(X)$ is not an abelian category. Hence we can not define subobjects F of $E \in D^b(X)$. To determine a subobject F of $E \in D^b(X)$, we have to fix a full sub abelian category $\mathcal{A}$ of $D^b(X)$ so-called the heart of a bounded t-structure. Thus the rough definition is the following:

Definition 4.1 ([4])**.** Let $\mathcal{A}$ be the heart of a bounded t-structure on $D^b(X)$ and let $Z : \mathcal{N}(X) \to \mathbb{C}$ be a group homomorphism. If the pair $\sigma = (\mathcal{A}, Z)$ has the "Harder-Narashimhan property", σ is said to be a stability condition on $D^b(X)$. The set of stability conditions on $D^b(X)$ is denoted by $\mathrm{Stab}(X)$.

Remark 4.2. We do not explain stability conditions any more in this article. If you need more precise definition, we strongly recommend to read the original articles [3] and [4].

One of the most important properties is the non-emptiness of $\mathrm{Stab}(X)$.

Theorem 4.3 ([4])**.** *Let X a $K3$ surface. Then* $\mathrm{Stab}(X)$ *is not empty and each of nonempty connected components is a complex manifold. Moreover there is a connected component* $\mathrm{Stab}^{\dagger}(X)$ *which is a covering space of a set $\mathcal{P}_0^+(X)$ defined as follows:*

$$\mathcal{P}(X) = \{v_r + \sqrt{-1}v_i \in \mathcal{N}(X) \otimes \mathbb{C} | \langle v_r, v_i \rangle \text{ spans a positive 2-plane}\}.$$

We take a connected component $\mathcal{P}^+(X)$ of $\mathcal{P}(X)$ containing $\exp(\mathcal{O}_X(1))$ *of an ample line bundle on X since $\mathcal{P}(X)$ has 2 connected components.*

$$\mathcal{P}_0^+(X) := \mathcal{P}^+(X) \setminus \bigcup_{\delta:(-2)\text{-vector in } \mathcal{N}(X)} \langle \delta \rangle^{\perp}.$$

Here $\langle \delta \rangle^{\perp}$ is the orthogonal complement of δ.

Remark 4.4. Due to Bayer-Bridgeland [1], if the Picard rank of X is 1 then $\mathrm{Stab}^{\dagger}(X)$ is simply connected, that is $\mathrm{Stab}^{\dagger}(X)$ is a universal cover of $\mathcal{P}_0^+(X)$.

4.2 $\mathrm{Stab}(X)$ and Lemma 3.3

In this subsection we explain where a key idea for the proof of Lemma 3.3 comes from. Roughly speaking, it comes from the relation between the representation $\rho(\Phi)$ and $\mathrm{Stab}(X)$.

Note that $\mathbb{C}$ can be regarded as a 2-dimensional $\mathbb{R}$-vector space in a canonical way. Thus we have a right action of $GL_2^+(\mathbb{R})$ on $\mathcal{N}(X) \otimes \mathbb{C}$. Since $\mathcal{P}_0^+(X)$ is a subset of $\mathcal{N}(X) \otimes \mathbb{C}$, we have a right action of $GL_2^+(\mathbb{R})$ on $\mathcal{P}_0^+(X)$

Assume that the Picard rank of X is 1. Then the quotient $\mathcal{P}_0^+(X)/GL_2^+(\mathbb{R})$ is isomorphic to an open and dense subset of the upper half plain $\mathbb{H}$. Thus, for an equivalence $\Phi : D^b(Y) \to D^b(X)$, we have the following commutative diagram:

$$\begin{array}{ccc} \mathrm{Stab}^{\dagger}(Y) & \xrightarrow{\Phi_*^S} & \mathrm{Stab}^{\dagger}(X) \\ \downarrow & & \downarrow \\ \mathcal{P}_0^+(Y)/GL_2^+(\mathbb{R}) & \xrightarrow{\Phi_*^P} & \mathcal{P}_0^+(X)/GL_2^+(\mathbb{R}). \end{array}$$

Since the open embedding of $\mathcal{P}_0^+(X)/GL_2^+(\mathbb{R})$ to $\mathbb{H}$ is canonical, one can easily check the following:

Claim 4.5. *Notations are being as above. The lower horizontal morphism Φ_*^P of the diagram is $\rho(\Phi)$ which we defined in* (2.2).

This is the relation between $\mathrm{Stab}(X)$ and the representation $\rho(\Phi)$. Due to Claim 4.5, the proof of Lemma 3.3 becomes easier.

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