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Kyoto University
Stability conditions and Fourier-Mukai transformations on $K3$ surfaces with Picard rank 1

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1 Backgrounds

Let $M$ and $W$ are projective manifolds over $\mathbb{C}$. If there is an equivalence between the bounded derived categories $D^b(M)$ and $D^b(W)$ of coherent sheaves on $M$ and $W$, then $W$ is said to be a Fourier-Mukai partner (shortly FM partner) of $M$. The following conjecture was proposed by Kawamata:

**Conjecture 1.1** ([11]). Let $M$ be a projective manifold. The set $\text{FM}(M)$ of isomorphism classes of FM partners of $M$ is a finite set.

We have to remark that the conjecture holds if $\dim M \leq 2$ or $M$ is abelian variety by [10, mainly in Chapter 5], [2], [11], [14] and [6]. Motivated the conjecture, we wish to discuss relations between $D^b(M)$ and $\text{FM}(M)$. Here the word “relations” does not have a specifying meaning. For instance, the following question suggests an example of “relations”:

**Question.** Can we prove the conjecture in terms of the autoequivalence group $\text{Aut}(D^b(M))$ of $D^b(M)$?

For the question, we have the following very partial answer:

**Proposition 1.2.** Let $M$ be a projective manifold. Suppose that $\text{Aut}(D^b(M))$ is generated
by:
\[ \text{Aut}(D^b(M)) = \langle f_*, L, [1] | L \in \text{Pic}(M), f \in \text{Aut}(M) \rangle, \]

where [1] is the shift functor on \( D^b(M) \). Then \( M \) has the only trivial FM partner \( M \) itself.

Proof. Let \( \Phi : D^b(M) \to D^b(W) \) be an equivalence between projective manifolds and let \( L_W \) be a very ample line bundle on \( W \). By the assumption, the functor \( F := \Phi^{-1} \circ (\otimes L_W) \circ \Phi \) is generated by
\[ T(M) := \langle f_*, L, [1] | L \in \text{Pic}(M), f \in \text{Aut}(M) \rangle, \]

Since \( kL_W \) has a global section for any positive integer \( k \in \mathbb{Z}_{>0} \), we can make a morphism \( E \to E \otimes kL_W \) for any \( E \in D(W) \). In particular since \( kL_W \) is base point free, we can choose the section so that the morphism \( E \to E \otimes kL_W \) is not 0. Thus, for any closed point \( x \in M \), we have
\[ \text{Hom}_{D(Y)}(E, E \otimes kL_W) \neq 0. \]

Hence \( n \) should be 0.

We assume that \( f \neq id_M \). Then there is a closed point \( x \in M \) such that \( f(x) \neq x \). Since \( F(O_x) = O_{f(x)} \), we have
\[ \text{Hom}_{D(M)}(O_x, F^k(O_x)) = 0. \]

This is contradiction.

Since \( F(-) = L_M \otimes (-)[n] \), where \( f \in \text{Aut}(M) \), \( L_M \in \text{Pic}(M) \) and \( n \in \mathbb{Z} \). We wish to prove that \( n = 0 \) and \( f = id_M \).

Since \( F(-) = L_M \otimes f_*(-)[n] \), where \( f \in \text{Aut}(M) \), \( L_M \in \text{Pic}(M) \) and \( n \in \mathbb{Z} \). We wish to prove that \( n = 0 \) and \( f = id_M \).

Suppose to the contrary that \( n \neq 0 \). Since \( n \neq 0 \), for sufficiently large \( \ell \in \mathbb{Z}_{>0} \), we have
\[ \text{Hom}_{D(M)}(O_x, F^\ell(O_x)) = 0 \]
where \( F^\ell \) is the \( \ell \) times composition of \( F \). This is contradiction. Hence \( n \) should be 0.

We assume that \( f \neq id_M \). Then there is a closed point \( x \in M \) such that \( f(x) \neq x \). Since \( F(O_x) = O_{f(x)} \), we have
\[ \text{Hom}_{D(M)}(O_x, F(O_x)) = 0. \]

This is contradiction.

Since \( F(-) = L_M \otimes (-) \), we have for any positive integer \( k \),
\[ \Phi(O_x) \otimes kL_W = \Phi(O_x \otimes kL_M) = \Phi(O_x). \]

Thus each Hilbert polynomial of \( H^i(\Phi(O_x)) \) with respect to \( L_W \) is constant. Since \( L_W \) is very ample, it follows that dim Supp(\( H^i(\Phi(O_x)) \)) = 0. Thus dim Supp(\( \Phi(O_x) \)) = 0. By [10,
Lemma 4.5, we have
\[ \Phi(O_x) = O_{y_x}[n_x], \]
for some \( y_x \in Y \) and \( n_x \in \mathbb{Z} \). Since any equivalence has a Fourier-Mukai kernel by [14, Theorem 2.18], we see that \( n_x \) is locally constant. Hence, \( n_x \) is constant. So we put \( n_x = n \). Since skyscraper sheaves are stable under the equivalence \( \Phi : D^b(M) \to D^b(W) \), \( W \) is isomorphic to \( M \) (for instance, see [10, Corollary 5.23]).

Proposition 1.2 gives an evidence that there should be a relation between \( \text{Aut}(D^b(M)) \) and \( \text{FM}(M) \), but there is a problem: we do not have better ideas to discuss the relation since the evidence is clearly weak. Under this situation, a study of Hosono-Oguiso-Lian-Yau (for short, HLOY) gives us a cue for our problem.

**Theorem 1.3** ([8]). Let \( X \) be a projective K3 surface with \( \text{NS}(X) = \mathbb{Z}L \). Put \( L^2 = 2d \). The following equation holds:
\[ \#\text{FM}(X) = |\text{AL}_d : \text{Fr}_d|. \]
Here \( \text{AL}_d \) and \( \text{Fr}_d \) are respectively Atkin-Lehner group and Fricke group of level \( d \) (defined in latter).

Roughly speaking, \( \text{AL}_d \) and \( \text{Fr}_d \) are discrete subgroup of \( \text{PSL}_2(\mathbb{R}) \). In the next section we introduce the relation between \( \text{FM}(X) \) and \( \text{AL}_d \).

## 2 HLOY’s observation

The main aim of this section is the introduction of HLOY’s observation. Before the introduction, we recall the definition of the Atkin-Lehner group and the Fricke group.

### 2.1 Atkin-Lehner groups

As usual we put
\[ \Gamma_0(d) = \{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \text{PSL}_2(\mathbb{Z}) | \gamma \in d\mathbb{Z} \}. \]

For integers \( s, d \in \mathbb{Z} \) we define the symbol \( s || d \) by
\[ s || d \iff s | d \text{ and } \gcd(s, \frac{d}{s}) = 1. \quad (2.1) \]

Suppose \( s || d \). We put
\[ W_s = \{ \frac{1}{\sqrt{s}} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} s & 0 \\ 0 & 1 \end{pmatrix} \in \text{PSL}_2(\mathbb{R}) \mid \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Gamma_0(d/s) \text{ and } \delta \in s\mathbb{Z} \}. \]
$W_s$ is also given as

$$W_s = \left\{ \begin{pmatrix} \alpha \sqrt{s} & \beta \\ \gamma \sqrt{s} & \delta \sqrt{s} \end{pmatrix} \in PSL_2(\mathbb{R}) | \alpha, \beta, \gamma \text{ and } \delta \in \mathbb{Z} \right\}.$$ 

In particular we see $W_1 = \Gamma_0(d)$.

For cosets $W_s$ one can check the following:

**Lemma 2.1 ([5]).** Each $W_s$ is a single coset under the multiplication of $W_1$ and is in the normalizer of $\Gamma_0(d)$ in $PSL_2(\mathbb{R})$. In addition the coset classes $W_s$ and $W_s'$ satisfies the following rule of products:

$$W_s^2 = W_1, W_s W_s' = W_s W_s W_s = W_{ss'},$$

where $s \ast s' = \frac{ss'}{\gcd(s, s')}^2$.

**Definition 2.2.** We define subsets of $PSL_2(\mathbb{R})$ by

$$AL_d := \bigsqcup_{s \mid d} W_s \text{ and } Fr_d := W_1 \sqcup W_d.$$ 

By Lemma 2.1, we see that $AL_d$ and $Fr_d$ are subgroups of $PSL_2(\mathbb{R})$. We call $AL_d$ and $Fr_d$ respectively the Atkin-Lehner group and the Fricke group of level $d$.

**Remark 2.3.** $AL_d$ is the abelian normalizer group of $\Gamma_0(d)$ in $PSL_2(\mathbb{R})$. Since $W_s W_d = W_{\frac{s}{d}}$, the coset decomposition of $AL_d/ Fr_d$ is given by

$$AL_d / Fr_d = \bigsqcup_{s \mid d} (W_s \sqcup W_{\frac{s}{d}}).$$

### 2.2 HLOY’s observation

Let $X$ be a projective $K3$ surface. Recall that the total cohomology ring

$$H^*(X, \mathbb{Z}) = H^0(X, \mathbb{Z}) \oplus H^2(X, \mathbb{Z}) \oplus H^4(X, \mathbb{Z})$$

has a pure Hodge structure with weight 2 (For instance, see [10, Chapter 10]). Moreover $H^*(X, \mathbb{Z})$ has the Mukai pairing (or Euler pairing) given by

$$\langle r \oplus c \oplus s, r' \oplus c' \oplus s' \rangle = cc' - rs' - sr'.$$

The numerical Grothendieck group of $X$ is given by

$$\mathcal{N}(X) = H^0(X, \mathbb{Z}) \oplus NS(X) \oplus H^4(X, \mathbb{Z}) \subset H^*(X, \mathbb{Z}).$$
By the Hodge index theorem, the index of the Mukai pairing on $\mathcal{N}(X)$ is $(2, \rho(X))$. For objects $E \in D(X)$ we put $v(E) = \text{ch}(E)\sqrt{\text{td}_X}$ and call it the Mukai vector of $E$. One can check that $v(E) = r \oplus c \oplus s \in \mathcal{N}(X)$ and see that $r = \text{rank } E$, $c = c_1(E)$ and $s = \chi(X, E) - \text{rank } E$ by the Riemann-Roch theorem.

Let $\Phi : D(Y) \to D(X)$ be an equivalence between projective $K3$ surfaces. As is well-known, $\Phi$ induces a Hodge isometry $\Phi^H : H^*(Y, \mathbb{Z}) \to H^*(X, \mathbb{Z})$ in a standard way (For instance, see [10]). Since $\Phi^H$ is a Hodge isometry, we get an isometry $\Phi^N : \mathcal{N}(Y) \to \mathcal{N}(X)$ by restricting $\Phi^H$. Namely we have $\Phi^N = \Phi^H|_{\mathcal{N}(X)}$ Thus we obtain the representation of $\text{Aut}(D^b(X))$ on $O^+(\mathcal{N}(X) \otimes \mathbb{R})$ where $O^+(\mathcal{N}(X) \otimes \mathbb{R})$ is a subgroup of $O(\mathcal{N}(X) \otimes \mathbb{R})$ which preserves the orientation of positive 2 planes. Moreover $O^+(\mathcal{N}(X) \otimes \mathbb{R})/ \pm 1$ is isomorphic to $PSL_2(\mathbb{R})$ if $\text{rank } \text{NS}(X) = 1$. Thus under the assumption Picard rank $1$, we obtain the following representation:

$$\rho : \text{Aut}(D^b(X)) \to O^+(\mathcal{N}(X) \otimes \mathbb{R}) \to PSL_2(\mathbb{R}).$$

We can describe the image of $\rho : \text{Aut}(D^b(X)) \to PSL_2(\mathbb{R})$.

**Theorem 2.4 ([3], [9]).** Let $X$ be a $K3$ surface with $\text{NS}(X) = \mathbb{Z}L$. Put $L^2 = 2d$. The image of $\rho$ is the Fricke group of level $d$.

HLOY expected the following:

**Conjecture 2.5 ([8]).** Notations and assumptions are being as above. For any $\alpha \in \text{AL}_d$, there exists an equivalence $\Phi : D^b(Y) \to D^b(X)$ such that $\rho(\Phi) = \alpha$.

We have to remark that the definition of $\rho$ is not clear since an equivalence is not necessary an autoequivalence. To define $\rho(\Phi)$ precisely, we use the assumption that Picard rank is $1$.

Note that the Picard rank of $Y$ is the same as that of $X$ if there is an equivalence $\Phi : D^b(Y) \to D^b(X)$. Moreover, if $\text{NS}(X) = \mathbb{Z}L_X$, then both numerical Grothendieck groups $\mathcal{N}(Y)$ and $\mathcal{N}(X)$ are canonically isomorphic to the lattice $(\mathbb{Z}^{\oplus 3}, \Sigma)$ where $\Sigma$ is

$$\Sigma = \begin{pmatrix}
0 & 0 & -1 \\
0 & 2d & 0 \\
-1 & 0 & 0
\end{pmatrix}.$$

Now we put these canonical isomorphism as follows:

$$c_X : \mathcal{N}(X) \to (\mathbb{Z}^{\oplus 3}, \Sigma) \text{ and } c_Y : \mathcal{N}(Y) \to (\mathbb{Z}^{\oplus 3}, \Sigma).$$

By using these canonical isomorphisms, we obtain an extended representation of $\Phi : D^b(Y) \to$
$D^b(X)$. Namely we have the following definition of $\rho$:

$$\rho(\Phi) := c_X \circ \Phi^N \circ c_Y^{-1}. \quad (2.2)$$

Main results of this article (and also of the talk) is the following:

**Theorem 2.6** ([13, Theorem 3.3]). Let $X$ be a projective K3 surface with $\text{NS}(X) = \mathbb{Z}L$ and $\Phi : D^b(Y) \to D^b(X)$ an equivalence. Put $L^2 = 2d$. Then Conjecture 2.5 holds. Moreover if the image of $\Phi : D^b(Y) \to D^b(X)$ by $\rho$ is in $\text{Fr}_d$, then $Y$ is isomorphic to $X$.

### 2.3 Relation between FM$(X)$ and AL$_d$/Fr$_d$

Let us recall Theorem 1.3. Basically the theorem gives us the equation of numbers. After Theorem 2.6, we have more concrete correspondence between FM$(X)$ and AL$_d$/Fr$_d$ though this correspondence is still far from our question introduced in §1.

By Theorem 2.4, $\text{Aut}(D^b(X))$ can be divided into 2 classes $\rho^{-1}(W_1) \sqcup \rho^{-1}(W_d)$. Moreover one can see that $\rho^{-1}(W_d)$ is represented by “spherical twist functors” on $D^b(X)$ (of general K3 surfaces). We do not give the definition of spherical twist functors but give a typical example $T_{O_X}$. (If you need the precise definition, you can see it, for instance, in [10, Chapter 8].

For the structure sheaf $O_X$ on $X$, we can define an autoequivalence $T_{O_X}$. The functor $T_{O_X}$ sends skyscraper sheaves $O_x$ of closed points $x \in X$ to the shift $I_x[1]$ of ideal sheaves $I_x$ of $x \in X$. So the functor $T_{O_X}$ can be regarded as an equivalence between $D^b(X)$ and the “dual” $D^b(X)^\vee$ of $D^b(X)$.

In addition, if we take $Y \in \text{FM}(X) \setminus \{X\}$, then there should be an equivalence $\Phi : D^b(Y) \to D^b(X)$ whose image by $\rho$ does not belong to $\text{Fr}_d$. Since $D^b(Y)$ has also the “dual” $D^b(Y)^\vee$, the image $\rho(\Phi)$ belongs to $W_s \sqcup W_d$ for $\geq s||d$. As a result, we obtain the following bijection between FM$(X)$ and AL$_d$/Fr$_d$:

$$\text{FM}(X) \to \text{AL}_d/\text{Fr}_d, Y \mapsto \rho(\Phi) \text{ where } \Phi : D^b(Y) \to D^b(X) \text{ an equivalence.}$$

### 3 Sketch of the proof of Theorem 2.6

We first recall the work of [7] which is an explicit construction of Fourier-Mukai partners of $X$ with $\text{NS}(X) = \mathbb{Z}L$. Put $L^2 = 2d$ as usual. We set the set $P_d$ by

$$P_d = \{r \in \mathbb{N}|r||d\} / \sim$$

where $r_1 \sim r_2$ if and only if $r_1 = r_2$ or $r_1 = \frac{d}{r_2}$. 

6
**Theorem 3.1** ([7, Theorem 2.1]). Let $X$ be a projective $K3$ surface with $\text{NS}(X) = \mathbb{Z}L$. Put $L^2 = 2d$. There is a one to one correspondence between $P_d$ and the set $\text{FM}(X)$ of isomorphic classes of Fourier-Mukai partners of $X$:

$$P_d \ni r \mapsto M_L(r \oplus L \oplus \frac{d}{r}) \in \text{FM}(X).$$

Here $M_L(r \oplus L \oplus s)$ is the fine moduli space of $\mu_L$-stable sheaves with Mukai vector $r \oplus L \oplus s$. 

Suppose that one wishes to prove Theorem 2.6. Since there is a surjection $\rho : \text{Aut}(D^b(X)) \to \text{Fr}_d$ by Theorem 2.4, it is enough to find an equivalence $\Phi : D^b(Y) \to D^b(X)$ such that $\rho(\Phi) \in W_s$ for any $s \mid d$. Thus our claim is the following:

**Claim 3.2.** Notations are being as above. Let $Y$ be the fine moduli space of $\mu$-stable locally free sheaves on $X$ with Mukai vector $r \oplus L \oplus s$ and $E_Y$ a universal family. Define an equivalence $\Phi : D^b(Y) \to D^b(X)$ by

$$\Phi : D^b(Y) \to D^b(X) \Phi(\cdot) := \mathbb{R}\pi_Y^*(E_Y \otimes \pi_Y^*(-)).$$

Then $\rho(\Phi) \in W_s$.

To show the claim, we give an explicit description of the matrix $\rho(\Phi)$. Before giving the description in Proposition 3.3 below, for an arbitrary equivalence $\Phi : D^b(Y) \to D^b(X)$, we put $\text{NS}(X) = \mathbb{Z}L_X$ (resp. $\text{NS}(Y) = \mathbb{Z}L_Y$) and

$$v(\Phi(O_y)) = r_X \oplus n_X L_X \oplus s_X$$

(resp. $v(\Phi^{-1}(O_x)) = r_Y \oplus n_Y L_Y \oplus s_Y$).

**Lemma 3.3** ([12, Lemmas 3.1 and 3.2]). Let $\Phi : D(Y) \to D(X)$ be an equivalence between projective $K3$ surfaces of degree $X = 2d$ with Picard rank 1.

1. We have $r = r_X = r_Y$. Moreover if $r_X = 0$, then

$$\rho(\Phi) = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \text{ where } m \in \mathbb{Z}. $$

2. Suppose that $r \neq 0$. Then $\rho(\Phi)$ is given by

$$\rho(\Phi) = \begin{pmatrix} 1 & \frac{n_Y}{r} \\ 0 & \frac{1}{\sqrt{d|r|}} \end{pmatrix} \frac{1}{d|r|} \begin{pmatrix} 0 & -1 \\ d|r| & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{-n_X}{r} \\ 0 & 1 \end{pmatrix}.$$ 

Due to Lemma 3.3, we can prove Claim 3.2. Consequently we can also prove Theorem 2.6. The details are in [13].
4 The origin of Lemma 3.3

As you saw in the last section, a key ingredient for Theorem 2.6 is Lemma 3.3. In this section we explain that Lemma 3.3 is closely related to the space of stability conditions on $D^b(X)$ introduced by Bridgeland in [3].

4.1 Stability conditions on $K3$ surfaces

In this subsection, we give a brief review of stability conditions on a $K3$ surface $X$. We can regard a stability condition on $D^b(X)$ as a generalization of Gieseker stability (or $\mu$-stability) for coherent sheaves on $X$. One of the obstructions for the generalization is the fact that $D^b(X)$ is not an abelian category. Hence we can not define subobjects $F$ of $E \in D^b(X)$. To determine a subobject $F$ of $E \in D^b(X)$, we have to fix a full sub abelian category $A$ of $D^b(X)$ so-called the heart of a bounded $t$-structure. Thus the rough definition is the following:

Definition 4.1 ([4]). Let $A$ be the heart of a bounded $t$-structure on $D^b(X)$ and let $Z : \mathcal{N}(X) \to \mathbb{C}$ be a group homomorphism. If the pair $\sigma = (A, Z)$ has the "Harder-Narashimhan property", $\sigma$ is said to be a stability condition on $D^b(X)$. The set of stability conditions on $D^b(X)$ is denoted by $\text{Stab}(X)$.

Remark 4.2. We do not explain stability conditions any more in this article. If you need more precise definition, we strongly recommend to read the original articles [3] and [4].

One of the most important properties is the non-emptiness of $\text{Stab}(X)$.

Theorem 4.3 ([4]). Let $X$ a $K3$ surface. Then $\text{Stab}(X)$ is not empty and each of nonempty connected components is a complex manifold. Moreover there is a connected component $\text{Stab}^\dagger(X)$ which is a covering space of a set $\mathcal{P}_0^+(X)$ defined as follows:

$$\mathcal{P}(X) = \{ v_r + \sqrt{-1}v_i \in \mathcal{N}(X) \otimes \mathbb{C}[v_r, v_i] \text{ spans a positive 2-plane} \}.$$ 

We take a connected component $\mathcal{P}^+(X)$ of $\mathcal{P}(X)$ containing $\exp(\mathcal{O}_X(1))$ of an ample line bundle on $X$ since $\mathcal{P}(X)$ has 2 connected components.

$$\mathcal{P}^+_0(X) := \mathcal{P}^+(X) \setminus \bigcup_{\delta : (-2)-vector in \mathcal{N}(X)} \langle \delta \rangle^\perp.$$ 

Here $\langle \delta \rangle^\perp$ is the orthogonal complement of $\delta$. 

8
Remark 4.4. Due to Bayer-Bridgeland [1], if the Picard rank of $X$ is 1 then $\text{Stab}^\dagger(X)$ is simply connected, that is $\text{Stab}^\dagger(X)$ is a universal cover of $\mathcal{P}_0^+(X)$.

4.2 $\text{Stab}(X)$ and Lemma 3.3

In this subsection we explain where a key idea for the proof of Lemma 3.3 comes from. Roughly speaking, it comes from the relation between the representation $\rho(\Phi)$ and $\text{Stab}(X)$.

Note that $\mathbb{C}$ can be regarded as a 2-dimensional $\mathbb{R}$-vector space in a canonical way. Thus we have a right action of $GL_2^+(\mathbb{R})$ on $N(X) \otimes \mathbb{C}$. Since $\mathcal{P}_0^+(X)$ is a subset of $N(X) \otimes \mathbb{C}$, we have a right action of $GL_2^+(\mathbb{R})$ on $\mathcal{P}_0^+(X)$.

Assume that the Picard rank of $X$ is 1. Then the quotient $\mathcal{P}_0^+(X)/GL_2^+(\mathbb{R})$ is isomorphic to an open and dense subset of the upper half plain $\mathbb{H}$. Thus, for an equivalence $\Phi : D^b(Y) \to D^b(X)$, we have the following commutative diagram:

$$
\begin{array}{ccc}
\text{Stab}^\dagger(Y) & \xrightarrow{\Phi^\dagger} & \text{Stab}^\dagger(X) \\
\downarrow & & \downarrow \\
\mathcal{P}_0^+(Y)/GL_2^+(\mathbb{R}) & \xrightarrow{\Phi^P} & \mathcal{P}_0^+(X)/GL_2^+(\mathbb{R}).
\end{array}
$$

Since the open embedding of $\mathcal{P}_0^+(X)/GL_2^+(\mathbb{R})$ to $\mathbb{H}$ is canonical, one can easily check the following:

Claim 4.5. Notations are being as above. The lower horizontal morphism $\Phi^P$ of the diagram is $\rho(\Phi)$ which we defined in (2.2).

This is the relation between $\text{Stab}(X)$ and the representation $\rho(\Phi)$. Due to Claim 4.5, the proof of Lemma 3.3 becomes easier.

Reference


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