Title: Special birational transformations of type (2,1)

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SPECIAL BIRATIONAL TRANSFORMATIONS OF
TYPE (2, 1)

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(JOINT WORK WITH JUN-MUK HWANG)

1. Introduction

This is a report on my joint work ([FH]) with Jun-Muk Hwang from KIAS. We are interested in the classification of special birational transformations: namely a birational transformation \( \Phi : \mathbb{P}^n \dashrightarrow Z \subset \mathbb{P}^N \) where \( Z \subset \mathbb{P}^N \) is a nonsingular projective variety of Picard number 1 such that the base locus \( S \subset \mathbb{P}^n \) of \( \Phi \) is irreducible and nonsingular. The map \( \Phi \) is called of type \((a, b)\) if the rational map \( \Phi \) is given by a linear system belonging to \( \mathcal{O}_{\mathbb{P}^n}(a) \) and the inverse rational map \( \Phi^{-1} \) is given by a linear system belonging to \( \mathcal{O}_Z(b) \).

When \( Z = \mathbb{P}^n \), such a transformation is called a special Cremona transformation. In [ES], Ein and Shepherd-Barron have classified special Cremona transformations of type \((2, 2)\), by proving that the base locus \( S \) is one of the four Severi varieties classified by Zak ([Z]). Recently, special Cremona transformations on \( \mathbb{P}^n \) of type \((2, 3)\) or of type \((2, 5)\) or when \( n \) is odd have been classified by Russo ([R]).

Let \( \Phi : \mathbb{P}^n \dashrightarrow Z \) be a special Cremona transformation of type \((2, 2)\). If we restrict \( \Phi \) to a general hyperplane section, we obtain a special birational transformation of type \((2, 2)\) \( \Phi' : \mathbb{P}^{n-1} \dashrightarrow \mathbb{Q}^{n-1} \) with base locus being a hyperplane section of a Severi variety. Conversely, every special birational transformation of type \((2, 2)\) from \( \mathbb{P}^n \) to \( \mathbb{Q}^n \) is of this form, as shown by Staglianò ([St]).

Motivated by the paper of [AS2] (and by the lack of examples), we have the following

**Speculation 1.1.** Let \( \Phi : \mathbb{P}^n \dashrightarrow Z \) be a special birational transformation of type \((2, b)\). If \( Z \) is different to \( \mathbb{P}^n \) and \( \mathbb{Q}^n \), then \( b \leq 2 \). Furthermore if \( b = 2 \), then \( n \) is even and \( Z \simeq \text{Gr}(2, n/2 + 2) \).

Here are some supporting results for this speculation: 1) If \( Z \) is a linear section of some rational homogeneous manifolds, then Speculation 1.1 holds by [AS2] (Corollary 3.21). 2) Speculation 1.1 with \( b = 2 \) holds if \( Z \) is a LQEL manifold by [AS2] (Theorem 3.6). 3) Speculation 1.1 with \( b = 2 \) holds if \( Z \) is a hypersurface by [Li].

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In our paper ([FH]), we have obtained a complete classification for special birational transformations of type \((2,1)\), hence confirming Speculation 1.1 for the case when \(b = 1\). This classification can be described in terms of the classification of the base locus \(S \subset \mathbb{P}^n\), which is contained in a hyperplane \(\mathbb{P}^{n-1} \subset \mathbb{P}^n\) for the type \((2,1)\). Our main result is the following.

**Theorem 1.2 ([FH]).** The base locus \(S^d \subset \mathbb{P}^{n-1}\) of a special birational transformations of type \((2,1)\) is projectively equivalent to one of the following:

1. \(\mathbb{Q}^d \subset \mathbb{P}^{d+1}\) for \(d \geq 1\);
2. \(\mathbb{P}^1 \times \mathbb{P}^{d-1} \subset \mathbb{P}^{2d-1}\) for \(d \geq 3\);
3. the 6-dimensional Grassmannian \(\text{Gr}(2,5) \subset \mathbb{P}^9\);
4. the 10-dimensional Spinor variety \(S_5 \subset \mathbb{P}^{15}\);
5. a nonsingular codimension \(\leq 2\) linear section of \(\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5\);
6. a nonsingular codimension \(\leq 3\) linear section of \(\text{Gr}(2,5) \subset \mathbb{P}^9\).

The corresponding \(Z\) is given by

1. \(\mathbb{Q}^{d+2}\) for \(d \geq 1\);
2. \(\text{Gr}(2,d+2)\) for \(d \geq 3\);
3. the 10-dimensional Spinor variety \(S_5\);
4. the 16-dimensional \(E_6\)-variety \(\mathbb{O}\mathbb{P}^2\);
5. a nonsingular codimension \(\leq 2\) linear section of \(\text{Gr}(2,5)\);
6. a \(\mathbb{P}^4\)-general linear section of \(S_5 \subset \mathbb{P}^{15}\) of codimension \(\leq 3\).

The birational map \(\Phi\) is given by \(H^0(\mathbb{P}^n, \mathcal{I}_S(2))\) except in case (b) with \(d \geq 6\), where it can be given by some subspaces of \(H^0(\mathbb{P}^n, \mathcal{I}_S(2))\).

By [IP] (Remark 3.3.2), all non-singular linear sections of \(\text{Gr}(2,5)\) of a fixed codimension \(\leq 3\) are projectively equivalent. All non-singular hyperplane sections of \(S_5\) are projectively equivalent. We can show that smooth codimension 2 linear sections of \(S_5\) are isomorphic but smooth codimension 3 linear sections of \(S_5\) have moduli. Thus in the case (f'), these linear sections of codimension 3 are not general, but only general among those containing a \(\mathbb{P}^4\).

Consider the case (b) with \(d \geq 6\). The corresponding birational transformation is \(\phi^\circ: \mathbb{P}(V) \simeq \mathbb{P}^{2d} \dashrightarrow Z := \text{Gr}(2,d+2) \subset \mathbb{P}(V \oplus W)\) with \(Y = \text{Gr}(2,d) \subset \mathbb{P}W = \mathbb{P}(\wedge^2 \mathbb{C}^d)\). Note that \(Z \cap \mathbb{P}W = Y\) and \(\text{Sec}(Z) \cap \mathbb{P}W = \text{Sec}(Y)\). Take any linear subspace \(L \subset W\) such that \(\mathbb{P}L \cap \text{Sec}(Y) = \emptyset\), then \(\mathbb{P}L \cap \text{Sec}(Z) = \emptyset\). Let \(p_L: \mathbb{P}(V \oplus W) \dashrightarrow \mathbb{P}(V \oplus W/L)\) be the projection from \(\mathbb{P}L\). Then \(p_L\) sends isomorphically \(Z\) to \(Z_L \subset \mathbb{P}(V \oplus W/L)\) and \(Y\) to \(Y_L \subset \mathbb{P}(W/L)\). The map \(\phi_L^\circ := p_L \circ \phi^\circ: \mathbb{P}(V) \dashrightarrow Z_L\) is a special birational transformation of type \((2,1)\) with \(Y(\sigma_L) = Y_L \simeq Y \subset \mathbb{P}(W/L)\).
2. Outline of the proof

2.1. Basic properties of special birational transformation of type (2, 1). For a projective subvariety $S \subset \mathbb{P}^n$, its secant variety $\text{Sec}(S)$ is the closure of the loci of lines which intersect $S$ at two or more points. The secant defective is given by $\delta = 2 \dim(S) + 1 - \dim \text{Sec}(S)$. For $u \in \text{Sec}(S) \setminus S$, the entry locus $\Sigma_u$ is the closure of the locus on $S$ of secant lines through $u$. We call $S \subset \mathbb{P}^n$ is a QEL-manifold if it is smooth and $\Sigma_u$ is a smooth hyperquadric in its linear span $C_u := \langle \Sigma_u \rangle$.

From now on, let $\Phi : S^d \subset \mathbb{P}^n \rightarrow Z$ be a special birational transformation of type (2, 1). As easily seen, for a line $\ell \subset \mathbb{P}^n$ not contained in $S$, the image $\Phi(\ell)$ is a point (resp. a line, a conic) if $\ell \cdot C$ is two points (resp. one point, empty).

**Lemma 2.1 ([ES]).** $S \subset \text{Sec}(S) = \mathbb{P}^{n-1}$ is a QEL-manifold. As a consequence, the secant defective of $S$ is $\delta = 2d + 2 - n$.

From this, we deduce that for $u \in \mathbb{P}^{n-1}$ general, $C_u \simeq \mathbb{P}^{d+1}$ is contracted to one point by $\Phi$. The following implies that for any $u \in \mathbb{P}^{n-1} \setminus S$, we always have $C_u \simeq \mathbb{P}^{d+1}$. This property is called strong QEL in [AS2].

**Lemma 2.2.** The subvariety $Y := \Phi(\mathbb{P}^{n-1}) \subset Z$ is smooth.

The proof uses in a crucial way the smoothness of $Z$. Take any point $y \in \mathbb{P}^n \setminus \text{Sec}(S)$, we consider the $\mathbb{C}^*$-action on $\mathbb{P}^n$ with orbits being lines through $y$. It fixes the point $y$ and the hyperplane $\mathbb{P}^{n-1} = \text{Sec}(S)$. This action induces a $\mathbb{C}^*$-action on $Z$, with $Y$ being an irreducible component of the fixed locus, hence $Y$ is smooth since $Z$ is smooth.

By [ES] (Theorem 1.1), $\text{Bl}_S(\mathbb{P}^n)$ is also the blow-up of $Z$ along $Y$, hence the map $\psi : \text{Bl}_S(\mathbb{P}^{n-1}) \rightarrow Y$ is a $\mathbb{P}^{d+1}$-bundle. Let $c := n - 1 - d$ be the codimension of $S$ in $\mathbb{P}^{n-1}$. The notations and basic properties of our situation are summarised in the following diagram.

An easy argument using previous diagram shows that...
**Proposition 2.3.** The Euler numbers of $S$ and $Y$ are related by

$$\chi(S) = \frac{(\delta + 2)\chi(Y) - n}{n - d - 2}.$$ 

2.2. **Classification of $Y$.** A key observation is that the fibers of $p : \mathbb{P} N_{S|\mathbb{P}^{n-1}} \to S$ are mapped to linear subspaces in $Y$ and such $\mathbb{P}^{c-1}$'s cover $Y$. On the other hand, we have $\dim Y = n - 1 - (\delta + 1) = 2(c - 1)$ by Lemma 2.1. By [S] and note that $Y$ has Picard number 1, we obtain

**Proposition 2.4.** The projective subvariety $Y \subset \mathbb{P}^{a-1}$ is isomorphic to one of the following:

1. $\mathbb{P}^{2(c-1)} \cong Y = \mathbb{P}^{a-1}$;
2. a nonsingular quadric hypersurface $\mathbb{Q}^{2(c-1)} \cong Y \subset \mathbb{P}^{a-1} \cong \mathbb{P}^{2c-1}$; or
3. $a$ biregular projection of the Plücker embedding $\text{Gr}(2, \mathbb{C}^{c+1}) \subset \mathbb{P}(\wedge^2 \mathbb{C}^{c+1})$.

We have the following topological consequence.

**Corollary 2.5.** All odd Betti numbers of $S$ vanish. In particular, the Euler number of $S$ satisfies $\chi(S) \geq d + 1$.

2.3. **Classification when $Y = \mathbb{P}^{2(c-1)}$.**

**Proposition 2.6.** Assume that $Y = \mathbb{P}^{2(c-1)}$, then $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to one of the following:

1. $\mathbb{Q}^{d} \subset \mathbb{P}^{d+1}$;
2. $\mathbb{P}^{1} \times \mathbb{P}^{2} \subset \mathbb{P}^{5}$;
3. $\text{Gr}(2, 5) \subset \mathbb{P}^{9}$;
4. a general hyperplane section of (ii);
5. a general codimension $\leq 2$ linear section of (iii);
6. a general codimension-3 linear section of (iii);
7. a general codimension-2 linear section of (ii).

Here is the idea for a proof different to that presented in [FH]. If $c = 1$, then $S \subset \mathbb{P}^{n-1}$ is a hyperquadric and we are done. Assume now $c \geq 2$. Let $g(S)$ be the sectional genus of $S$. In this case, we have $a - 1 = 2(c - 1)$, hence by Theorem 1 [AS1], we have either $\deg(S) = 2, c = 2, g(S) = 0$ or $\deg(S) = 5, c = 3, g(S) = 1$. Now the Proposition follows from the classification of low degree varieties or from the results on QEL-manifolds obtained by Russo in [R].

2.4. **Classification when $Y = \mathbb{Q}^{2(c-1)}$.** The following result is an easy consequence of Proposition 2.3.
Proposition 2.7. For a special quadratic manifold $S^d \subset \mathbb{P}^{n-1}$ with codimension $c$ and $n \geq 3$, assume that $Y = \mathbb{Q}^{2(c-1)}$. Then either $\delta \geq \frac{d}{2}$ or $\delta = \frac{d}{2} - 1$.

Proposition 2.8. For a special quadratic manifold $S \subset \mathbb{P}^{n-1}$ of codimension $c$, assume that $Y = \mathbb{Q}^{2(c-1)}$. Then $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to one of the following

(i) the Segre 4-fold $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$;

(ii) the 10-dimensional Spinor variety $S_5 \subset \mathbb{P}^{15}$.

By [R], $d$-dimensional QEL-manifolds with $\delta \geq d/2$ have been completely classified. Hence by Proposition 2.7, the remaining case is $\delta = \frac{d}{2} - 1$. If $\delta \leq 2$, then we get $d \leq 6$ and we do a case-by-case check. Now assume $\delta \geq 3$. By the divisibility theorem of [R], $d - \delta = \delta + 2$ is divisible by $2^{\lceil \frac{\delta}{2} \rceil}$ which implies that we have only two possibilities: $(\delta, d, n) = (4, 10, 18)$ or $(\delta, d, n) = (6, 14, 24)$. For the first case, we can show the VMRT of $S$ is $\mathbb{P}^1 \times \mathbb{P}^4 \subset \mathbb{P}^9$, hence $S$ is isomorphic to $\text{Gr}(2, 7)$ by [Mo], which cannot be embedded linearly normally into $\mathbb{P}^{17}$.

The case $(\delta, d, n) = (6, 14, 24)$ is difficult to be ruled out. The contradiction comes from a delicate structure on intersections of entry loci of $S$. At one hand, we show that every two general entry loci $\Sigma_i, i = 1, 2$ of $S$ through a general point $x \in S$ will intersect along $\mathbb{P}^1$, hence the lines on $\Sigma_i$ through $x$ give two entry loci on the VMRT $C_x$ of $S$ at $x$, intersecting at a single point (corresponding to the line $\Sigma_1 \cap \Sigma_2$). On the other hand, we can show that $C_x$ is a codimension 2 linear section of the 10-dimensional spinor variety $S_5 \subset \mathbb{P}^{15}$. Any two entry loci of such a linear section intersect along a $\mathbb{P}^k$ with $k \geq 1$ if non-empty, which is a contradiction.

2.5. Classification when $Y = \text{Gr}(2, c + 1)$. To handle the case (Y3) of Proposition 2.4, we need the following result.

Proposition 2.9 (Theorem 2.17 [AS2]). Let $S' \subset \mathbb{P}^{2k}$ be a quadratic manifold. Assume that the rational map $\psi : S' \dashrightarrow Y(S')$ is birational and $Y(S')$ is biregular to $\text{Gr}(2, k + 2)$. Then $S' \subset \mathbb{P}^{2k}$ is one of the rational normal scrolls. In particular, $S'$ has Picard number 2 and is covered by lines.

Proposition 2.10. Let $S \subset \mathbb{P}^{n-1}$ be a special quadratic manifold of codimension $c \geq 4$ with $Y(S) = \text{Gr}(2, c + 1)$. Then $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to $\mathbb{P}^1 \times \mathbb{P}^c \subset \mathbb{P}^{2c+1}$.

The idea of the proof is the following: we take a general codimension-$(\delta + 1)$ linear section $S' \subset \mathbb{P}^{n-\delta-2}$ of $S \subset \mathbb{P}^{n-1}$. This gives a birational
map $\psi^c : S' \subset \mathbb{P}^{n-\delta-2} \dashrightarrow Y(S') = \text{Gr}(2, c+1)$, hence $S'$ is a rational normal scroll by 2.9. Except a few lower dimensional cases, this implies that $S$ has Picard number 2 with $\delta > 0$. But then $S$ becomes a conic-connected manifold with Picard number 2, which have been completely classified by [IR], and then we are done.

References


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