<table>
<thead>
<tr>
<th>Title</th>
<th>Special birational transformations of type (2,1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Fu, Baohua</td>
</tr>
<tr>
<td>Citation</td>
<td>代数幾何学シンポジウム記録 2014: 12-17</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2014</td>
</tr>
<tr>
<td>URL</td>
<td><a href="http://hdl.handle.net/2433/215026">http://hdl.handle.net/2433/215026</a></td>
</tr>
<tr>
<td>Type</td>
<td>Departmental Bulletin Paper</td>
</tr>
<tr>
<td>Textversion</td>
<td>publisher</td>
</tr>
</tbody>
</table>

Kyoto University
SPECIAL BIRATIONAL TRANSFORMATIONS OF TYPE \((2, 1)\)

BAOHUA FU

(JOINT WORK WITH JUN-MUK HWANG)

1. Introduction

This is a report on my joint work ([FH]) with Jun-Muk Hwang from KIAS. We are interested in the classification of special birational transformations: namely a birational transformation \(\Phi : \mathbb{P}^n \rightarrow Z \subset \mathbb{P}^N\) where \(Z \subset \mathbb{P}^N\) is a nonsingular projective variety of Picard number 1 such that the base locus \(S \subset \mathbb{P}^n\) of \(\Phi\) is irreducible and nonsingular. The map \(\Phi\) is called of type \((a, b)\) if the rational map \(\Phi\) is given by a linear system belonging to \(\mathcal{O}_{\mathbb{P}^n}(a)\) and the inverse rational map \(\Phi^{-1}\) is given by a linear system belonging to \(\mathcal{O}_Z(b)\).

When \(Z = \mathbb{P}^n\), such a transformation is called a special Cremona transformation. In [ES], Ein and Shepherd-Barron have classified special Cremona transformations of type \((2, 2)\), by proving that the base locus \(S\) is one of the four Severi varieties classified by Zak ([Z]). Recently, special Cremona transformations on \(\mathbb{P}^n\) of type \((2, 3)\) or of type \((2, 5)\) or when \(n\) is odd have been classified by Russo ([R]).

Let \(\Phi : \mathbb{P}^n \rightarrow Z\) be a special Cremona transformation of type \((2, 2)\), and \(Z\) is different to \(\mathbb{P}^n\) and \(\mathbb{Q}^n\). If \(\Phi\) is of this form, as shown by Staglianò ([St]).

Motivated by the paper of [AS2] (and by the lack of examples), we have the following

Speculation 1.1. Let \(\Phi : \mathbb{P}^n \rightarrow Z\) be a special birational transformation of type \((2, b)\). If \(Z\) is different to \(\mathbb{P}^n\) and \(\mathbb{Q}^n\), then \(b \leq 2\). Furthermore if \(b = 2\), then \(n\) is even and \(Z \cong \text{Gr}(2, n/2 + 2)\).

Here are some supporting results for this speculation: 1) If \(Z\) is a linear section of some rational homogeneous manifolds, then Speculation 1.1 holds by [AS2] (Corollary 3.21). 2) Speculation 1.1 with \(b = 2\) holds if \(Z\) is a LQEL manifold by [AS2] (Theorem 3.6). 3) Speculation 1.1 with \(b = 2\) holds if \(Z\) is a hypersurface by [Li].
In our paper ([FH]), we have obtained a complete classification for special birational transformations of type $(2, 1)$, hence confirming Speculation 1.1 for the case when $b = 1$. This classification can be described in terms of the classification of the base locus $S \subset P^n$, which is contained in a hyperplane $P^{n-1} \subset P^n$ for the type $(2, 1)$. Our main result is the following.

**Theorem 1.2** ([FH]). The base locus $S^d \subset P^{n-1}$ of a special birational transformations of type $(2, 1)$ is projectively equivalent to one of the following:

1. $Q^d \subset P^{d+1}$ for $d \geq 1$;
2. $P^1 \times P^{d-1} \subset P^{2d-1}$ for $d \geq 3$;
3. the 6-dimensional Grassmannian $Gr(2, 5) \subset P^9$;
4. the 10-dimensional Spinor variety $S_5 \subset P^{15}$;
5. a nonsingular codimension $\leq 2$ linear section of $P^1 \times P^2 \subset P^5$;
6. a nonsingular codimension $\leq 3$ linear section of $Gr(2, 5) \subset P^9$.

The corresponding $Z$ is given by

1. $Q^{d+2}$ for $d \geq 1$;
2. $Gr(2, d + 2)$ for $d \geq 3$;
3. the 10-dimensional Spinor variety $S_5$;
4. the 16-dimensional $E_6$-variety $OP^2$;
5. a nonsingular codimension $\leq 2$ linear section of $Gr(2, 5)$;
6. a $P^4$-general linear section of $S_5 \subset P^{15}$ of codimension $\leq 3$.

The birational map $\Phi$ is given by $H^0(P^n, I_S(2))$ except in case (b) with $d \geq 6$, where it can be given by some subspaces of $H^0(P^n, I_S(2))$.

By [IP] (Remark 3.3.2), all non-singular linear sections of $Gr(2, 5)$ of a fixed codimension $\leq 3$ are projectively equivalent. All non-singular hyperplane sections of $S_5$ are projectively equivalent. We can show that smooth codimension 2 linear sections of $S_5$ are isomorphic but smooth codimension 3 linear sections of $S_5$ have moduli. Thus in the case (f'), these linear sections of codimension 3 are not general, but only general among those containing a $P^4$.

Consider the case (b) with $d \geq 6$. The corresponding birational transformation is $\phi^o : P(V) \simeq P^{2d} \dashrightarrow Z := Gr(2, d + 2) \subset P(V \oplus W)$ with $Y = Gr(2, d) \subset PW = P(\wedge^2 C^d)$. Note that $Z \cap PW = Y$ and $Sec(Z) \cap PW = Sec(Y)$. Take any linear subspace $L \subset W$ such that $PL \cap Sec(Y) = 0$, then $PL \cap Sec(Z) = 0$. Let $p_L : P(V \oplus W) \dashrightarrow P(V \oplus W/L)$ be the projection from $PL$. Then $p_L$ sends isomorphically $Z$ to $Z_L \subset P(V \oplus W/L)$ and $Y$ to $Y_L \subset P(W/L)$. The map $\phi^o_L := p_L \circ \phi^o : P(V) \dashrightarrow Z_L$ is a special birational transformation of type $(2, 1)$ with $Y(\sigma_L) = Y_L \simeq Y \subset P(W/L)$. 
2. Outline of the proof

2.1. Basic properties of special birational transformation of type (2, 1). For a projective subvariety $S \subset \mathbb{P}^n$, its secant variety $\text{Sec}(S)$ is the closure of the loci of lines which intersect $S$ at two or more points. The secant defective is given by $\delta = 2 \dim(S) + 1 - \dim \text{Sec}(S)$.

For $u \in \text{Sec}(S) \setminus S$, the entry locus $\Sigma_u$ is the closure of the locus on $S$ of secant lines through $u$. We call $S \subset \mathbb{P}^n$ is a QEL-manifold if it is smooth and $\Sigma_u$ is a smooth hyperquadric in its linear span $C_u := \langle \Sigma_u \rangle$.

From now on, let $S_d \subset \mathbb{P}^n \dasharrow Z$ be a special birational transformation of type (2, 1). As easily seen, for a line $\ell \subset \mathbb{P}^n$ not contained in $S$, the image $\Phi(\ell)$ is a point (resp. a line, a conic) if $\ell \cdot C$ is two points (resp. one point, empty).

Lemma 2.1 ([ES]). $S \subset \text{Sec}(S) = \mathbb{P}^{n-1}$ is a QEL-manifold. As a consequence, the secant defective of $S$ is $\delta = 2d + 2 - n$.

From this, we deduce that for $u \in \mathbb{P}^{n-1}$ general, $C_u \simeq \mathbb{P}^{d+1}$ is contracted to one point by $\Phi$. The following implies that for any $u \in \mathbb{P}^{n-1} \setminus S$, we always have $C_u \simeq \mathbb{P}^{d+1}$. This property is called strong QEL in [AS2].

Lemma 2.2. The subvariety $Y := \Phi(\mathbb{P}^{n-1}) \subset Z$ is smooth.

The proof uses in a crucial way the smoothness of $Z$. Take any point $y \in \mathbb{P}^n \setminus \text{Sec}(S)$, we consider the $\mathbb{C}^*$-action on $\mathbb{P}^n$ with orbits being lines through $y$. It fixes the point $y$ and the hyperplane $\mathbb{P}^{n-1} = \text{Sec}(S)$. This action induces a $\mathbb{C}^*$-action on $Z$, with $Y$ being an irreducible component of the fixed locus, hence $Y$ is smooth since $Z$ is smooth.

By [ES] (Theorem 1.1), $\text{Bl}_S(\mathbb{P}^n)$ is also the blow-up of $Z$ along $Y$, hence the map $\psi : \text{Bl}_S(\mathbb{P}^{n-1}) \rightarrow Y$ is a $\mathbb{P}^{d+1}$-bundle. Let $c := n - 1 - d$ be the codimension of $S$ in $\mathbb{P}^{n-1}$. The notations and basic properties of our situation are summarised in the following diagram.

An easy argument using previous diagram shows that...
Proposition 2.3. The Euler numbers of $S$ and $Y$ are related by
\[ \chi(S) = \frac{(\delta + 2)\chi(Y) - n}{n - d - 2}. \]

2.2. Classification of $Y$. A key observation is that the fibers of $p : \mathbb{P}N_S|_{\mathbb{P}^{n-1}} \to S$ are mapped to linear subspaces in $Y$ and such $\mathbb{P}^{c-1}$'s cover $Y$. On the other hand, we have $\dim Y = n - 1 - (\delta + 1) = 2(c - 1)$ by Lemma 2.1. By [S] and note that $Y$ has Picard number 1, we obtain

Proposition 2.4. The projective subvariety $Y \subset \mathbb{P}^{a-1}$ is isomorphic to one of the following:

(Y1) $\mathbb{P}^{2(c-1)} \cong Y = \mathbb{P}^{a-1}$;
(Y2) a nonsingular quadric hypersurface $\mathbb{Q}^{2(c-1)} \cong Y \subset \mathbb{P}^{a-1} \cong \mathbb{P}^{2c-1}$; or
(Y3) a biregular projection of the Plücker embedding $\text{Gr}(2, \mathbb{C}^{c+1}) \subset \mathbb{P}(\wedge^2 \mathbb{C}^{c+1})$.

We have the following topological consequence.

Corollary 2.5. All odd Betti numbers of $S$ vanish. In particular, the Euler number of $S$ satisfies $\chi(S) \geq d + 1$.

2.3. Classification when $Y = \mathbb{P}^{2(c-1)}$.

Proposition 2.6. Assume that $Y = \mathbb{P}^{2(c-1)}$, then $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to one of the following:

(i) $\mathbb{Q}^d \subset \mathbb{P}^{d+1}$;
(ii) $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$;
(iii) $\text{Gr}(2, 5) \subset \mathbb{P}^9$;
(iv) a general hyperplane section of (ii);
(v) a general codimension $\leq 2$ linear section of (iii);
(vi) a general codimension-3 linear section of (iii);
(vii) a general codimension-2 linear section of (ii).

Here is the idea for a proof different to that presented in [FH]. If $c = 1$, then $S \subset \mathbb{P}^{n-1}$ is a hyperquadric and we are done. Assume now $c \geq 2$. Let $g(S)$ be the sectional genus of $S$. In this case, we have $a - 1 = 2(c - 1)$, hence by Theorem 1 [AS1], we have either $\deg(S) = 2, c = 2, g(S) = 0$ or $\deg(S) = 5, c = 3, g(S) = 1$. Now the Proposition follows from the classification of low degree varieties or from the results on QEL-manifolds obtained by Russo in [R].

2.4. Classification when $Y = \mathbb{Q}^{2(c-1)}$. The following result is an easy consequence of Proposition 2.3.
Proposition 2.7. For a special quadratic manifold $S^d \subset \mathbb{P}^{n-1}$ with codimension $c$ and $n \geq 3$, assume that $Y = \mathbb{Q}^{2(c-1)}$. Then either $\delta \geq \frac{d}{2}$ or $\delta = \frac{d}{2} - 1$.

Proposition 2.8. For a special quadratic manifold $S \subset \mathbb{P}^{n-1}$ of codimension $c$, assume that $Y = \mathbb{Q}^{2(c-1)}$. Then $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to one of the following

(i) the Segre 4-fold $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$;
(ii) the 10-dimensional Spinor variety $S_5 \subset \mathbb{P}^{15}$.

By [R], $d$-dimensional QEL-manifolds with $\delta \geq d/2$ have been completely classified. Hence by Proposition 2.7, the remaining case is $\delta = \frac{d}{2} - 1$. If $\delta \leq 2$, then we get $d \leq 6$ and we do a case-by-case check. Now assume $\delta \geq 3$. By the divisibility theorem of [R], $d - \delta = \delta + 2$ is divisible by $2^{\frac{d-1}{2}}$ which implies that we have only two possibilities: $(\delta, d, n) = (4, 10, 18)$ or $(\delta, d, n) = (6, 14, 24)$. For the first case, we can show the VMRT of $S$ is $\mathbb{P}^1 \times \mathbb{P}^4 \subset \mathbb{P}^9$, hence $S$ is isomorphic to $\text{Gr}(2, 7)$ by [Mo], which cannot be embedded linearly normally into $\mathbb{P}^{17}$.

The case $(\delta, d, n) = (6, 14, 24)$ is difficult to be ruled out. The contradiction comes from a delicate structure on intersections of entry loci of $S$. At one hand, we show that every two general entry loci $\Sigma_i, i = 1, 2$ of $S$ through a general point $x \in S$ will intersect along $\mathbb{P}^1$, hence the lines on $\Sigma_i$ through $x$ give two entry loci on the VMRT $C_x$ of $S$ at $x$, intersecting at a single point (corresponding to the line $\Sigma_1 \cap \Sigma_2$). On the other hand, we can show that $C_x$ is a codimension 2 linear section of the 10-dimensional spinor variety $S_5 \subset \mathbb{P}^{15}$. Any two entry loci of such a linear section intersect along a $\mathbb{P}^k$ with $k \geq 1$ if non-empty, which is a contradiction.

2.5. Classification when $Y = \text{Gr}(2, c + 1)$. To handle the case $(Y3)$ of Proposition 2.4, we need the following result.

Proposition 2.9 (Theorem 2.17 [AS2]). Let $S' \subset \mathbb{P}^{2k}$ be a quadratic manifold. Assume that the rational map $\psi^o : S' \dashrightarrow Y(S')$ is birational and $Y(S')$ is biregular to $\text{Gr}(2, k + 2)$. Then $S' \subset \mathbb{P}^{2k}$ is of one of the rational normal scrolls. In particular, $S'$ has Picard number 2 and is covered by lines.

Proposition 2.10. Let $S \subset \mathbb{P}^{n-1}$ be a special quadratic manifold of codimension $c \geq 4$ with $Y(S) = \text{Gr}(2, c + 1)$. Then $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to $\mathbb{P}^1 \times \mathbb{P}^c \subset \mathbb{P}^{2c+1}$.

The idea of the proof is the following: we take a general codimension $(\delta + 1)$ linear section $S' \subset \mathbb{P}^{n-\delta-2}$ of $S \subset \mathbb{P}^{n-1}$. This gives a birational
map $\psi^s : S' \subset \mathbb{P}^{n-\delta-2} \to Y(S') = \text{Gr}(2, c+1)$, hence $S'$ is a rational normal scroll by 2.9. Except a few lower dimensional cases, this implies that $S$ has Picard number 2 with $\delta > 0$. But then $S$ becomes a conic-connected manifold with Picard number 2, which have been completely classified by [IR], and then we are done.

References


[Li] Qifeng Li: Quadro-quadric special birational transformations from projective spaces to smooth complete intersections, preprint


Baohua Fu
Institute of Mathematics, AMSS, Chinese Academy of Sciences,
55 ZhongGuangCun East Road, Beijing, 100190, China
bhfu@math.ac.cn