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SPECIAL BIRATIONAL TRANSFORMATIONS OF TYPE $(2, 1)$

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(JOINT WORK WITH JUN-MUK HWANG)

1. Introduction

This is a report on my joint work ([FH]) with Jun-Muk Hwang from KIAS. We are interested in the classification of *special birational transformations*: namely a birational transformation $\Phi : \mathbb{P}^n \dashrightarrow Z \subset \mathbb{P}^N$ where $Z \subset \mathbb{P}^N$ is a nonsingular projective variety of Picard number 1 such that the base locus $S \subset \mathbb{P}^n$ of Φ is irreducible and nonsingular. The map Φ is called of *type* (a, b) if the rational map Φ is given by a linear system belonging to $\mathcal{O}_{\mathbb{P}^n}(a)$ and the inverse rational map Φ^{-1} is given by a linear system belonging to $\mathcal{O}_Z(b)$.

When $Z = \mathbb{P}^n$, such a transformation is called a *special Cremona transformation*. In [ES], Ein and Shepherd-Barron have classified special Cremona transformations of type $(2, 2)$, by proving that the base locus S is one of the four Severi varieties classified by Zak ([Z]). Recently, special Cremona transformations on $\mathbb{P}^n$ of type $(2, 3)$ or of type $(2, 5)$ or when n is odd have been classified by Russo ([R]).

Let $\Phi : \mathbb{P}^n \dashrightarrow Z$ be a special Cremona transformation of type $(2, 2)$. If we restrict Φ to a general hyperplane section, we obtain a special birational transformation of type $(2, 2)$ $\Phi' : \mathbb{P}^{n-1} \dashrightarrow \mathbb{Q}^{n-1}$ with base locus being a hyperplane section of a Severi variety. Conversely, every special birational transformation of type $(2, 2)$ from $\mathbb{P}^n$ to $\mathbb{Q}^n$ is of this form, as shown by Staglianò ([St]).

Motivated by the paper of [AS2] (and by the lack of examples), we have the following

Speculation 1.1. Let $\Phi : \mathbb{P}^n \dashrightarrow Z$ be a special birational transformation of type $(2, b)$. If Z is different to $\mathbb{P}^n$ and $\mathbb{Q}^n$, then $b \leq 2$. Furthermore if $b = 2$, then n is even and $Z \simeq \mathrm{Gr}(2, n/2 + 2)$.

Here are some supporting results for this speculation: 1) If Z is a linear section of some rational homogeneous manifolds, then Speculation 1.1 holds by [AS2] (Corollary 3.21). 2) Speculation 1.1 with $b = 2$ holds if Z is a *LQEL* manifold by [AS2] (Theorem 3.6). 3) Speculation 1.1 with $b = 2$ holds if Z is a hypersurface by [Li].

In our paper ([FH]), we have obtained a complete classification for special birational transformations of type $(2,1)$, hence confirming Speculation 1.1 for the case when $b=1$. This classification can be described in terms of the classification of the base locus $S \subset \mathbb{P}^n$, which is contained in a hyperplane $\mathbb{P}^{n-1} \subset \mathbb{P}^n$ for the type $(2,1)$. Our main result is the following.

Theorem 1.2 ([FH])**.** *The base locus $S^d \subset \mathbb{P}^{n-1}$ of a special birational transformations of type $(2,1)$ is projectively equivalent to one of the following:*

(a) $\mathbb{Q}^d \subset \mathbb{P}^{d+1}$ *for* $d \geq 1$*;*
(b) $\mathbb{P}^1 \times \mathbb{P}^{d-1} \subset \mathbb{P}^{2d-1}$ *for* $d \geq 3$*;*
(c) *the 6-dimensional Grassmannian* $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$*;*
(d) *the 10-dimensional Spinor variety* $\mathbb{S}_5 \subset \mathbb{P}^{15}$*;*
(e) *a nonsingular codimension* ≤ 2 *linear section of* $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$*;*
(f) *a nonsingular codimension* ≤ 3 *linear section of* $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$.

The corresponding Z is given by

(a') $\mathbb{Q}^{d+2}$ *for* $d \geq 1$*;*
(b') $\mathrm{Gr}(2, d+2)$ *for* $d \geq 3$*;*
(c') *the 10-dimensional Spinor variety* $\mathbb{S}_5$*;*
(d') *the 16-dimensional E_6-variety* $\mathbb{OP}^2$*;*
(e') *a nonsingular codimension* ≤ 2 *linear section of* $\mathrm{Gr}(2,5)$*;*
(f') *a $\mathbb{P}^4$-general linear section of* $\mathbb{S}_5 \subset \mathbb{P}^{15}$ *of codimension* ≤ 3.

The birational map Φ is given by $H^0(\mathbb{P}^n, \mathcal{I}_S(2))$ except in case (b) with $d \geq 6$, where it can be given by some subspaces of $H^0(\mathbb{P}^n, \mathcal{I}_S(2))$.

By [IP] (Remark 3.3.2), all non-singular linear sections of $\mathrm{Gr}(2,5)$ of a fixed codimension ≤ 3 are projectively equivalent. All non-singular hyperplane sections of $\mathbb{S}_5$ are projectively equivalent. We can show that smooth codimension 2 linear sections of $\mathbb{S}_5$ are isomorphic but smooth codimension 3 linear sections of $\mathbb{S}_5$ have moduli. Thus in the case (f'), these linear sections of codimension 3 are not general, but only general among those containing a $\mathbb{P}^4$.

Consider the case (b) with $d \geq 6$. The corresponding birational transformation is $\phi^\circ : \mathbb{P}(V) \simeq \mathbb{P}^{2d} \dashrightarrow Z := \mathrm{Gr}(2, d+2) \subset \mathbb{P}(V \oplus W)$ with $Y = \mathrm{Gr}(2,d) \subset \mathbb{P}W = \mathbb{P}(\wedge^2 \mathbb{C}^d)$. Note that $Z \cap \mathbb{P}W = Y$ and $\mathrm{Sec}(Z) \cap \mathbb{P}W = \mathrm{Sec}(Y)$. Take any linear subspace $L \subset W$ such that $\mathbb{P}L \cap \mathrm{Sec}(Y) = \emptyset$, then $\mathbb{P}L \cap \mathrm{Sec}(Z) = \emptyset$. Let $p_L : \mathbb{P}(V \oplus W) \dashrightarrow \mathbb{P}(V \oplus W/L)$ be the projection from $\mathbb{P}L$. Then p_L sends isomorphically Z to $Z_L \subset \mathbb{P}(V \oplus W/L)$ and Y to $Y_L \subset \mathbb{P}(W/L)$. The map $\phi^\circ_L := p_L \circ \phi^\circ : \mathbb{P}(V) \dashrightarrow Z_L$ is a special birational transformation of type $(2,1)$ with $Y(\sigma_L) = Y_L \simeq Y \subset \mathbb{P}(W/L)$.

2. Outline of the proof

2.1. **Basic properties of special birational transformation of type** $(2,1)$**.** For a projective subvariety $S \subset \mathbb{P}^n$, its secant variety $\mathrm{Sec}(S)$ is the closure of the loci of lines which intersect S at two or more points. The secant defective is given by $\delta = 2\dim(S) + 1 - \dim \mathrm{Sec}(S)$. For $u \in \mathrm{Sec}(S) \setminus S$, the entry locus Σ_u is the closure of the locus on S of secant lines through u. We call $S \subset \mathbb{P}^n$ is a *QEL*-manifold if it is smooth and Σ_u is a smooth hyperquadric in its linear span $C_u := \langle \Sigma_u \rangle$.

From now on, let $\Phi : S^d \subset \mathbb{P}^n \dashrightarrow Z$ be a special birational transformation of type $(2,1)$. As easily seen, for a line $\ell \subset \mathbb{P}^n$ not contained in S, the image $\Phi(\ell)$ is a point (resp. a line, a conic) if $\ell \cdot C$ is two points (resp. one point, empty).

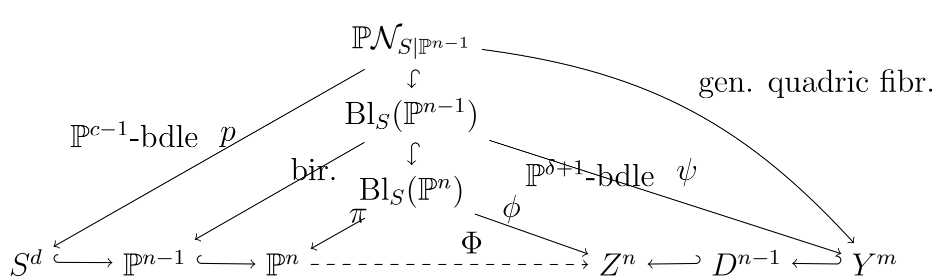
Lemma 2.1 ([ES])**.** *$S \subset \mathrm{Sec}(S) = \mathbb{P}^{n-1}$ is a QEL-manifold. As a consequence, the secant defective of S is $\delta = 2d + 2 - n$.*

From this, we deduce that for $u \in \mathbb{P}^{n-1}$ general, $C_u \simeq \mathbb{P}^{\delta+1}$ is contracted to one point by Φ. The following implies that for any $u \in \mathbb{P}^{n-1} \setminus S$, we always have $C_u \simeq \mathbb{P}^{\delta+1}$. This property is called *strong QEL* in [AS2].

Lemma 2.2. *The subvariety $Y := \Phi(\mathbb{P}^{n-1}) \subset Z$ is smooth.*

The poof uses in a crucial way the smoothness of Z. Take any point $y \in \mathbb{P}^n \setminus \mathrm{Sec}(S)$, we consider the $\mathbb{C}^*$-action on $\mathbb{P}^n$ with orbits being lines through y. It fixes the point y and the hyperplabe $\mathbb{P}^{n-1} = \mathrm{Sec}(S)$. This action induces a $\mathbb{C}^*$-action on Z, with Y being an irreducible component of the fixed locus, hence Y is smooth since Z is smooth.

By [ES] (Theorem 1.1), $\mathrm{Bl}_S(\mathbb{P}^n)$ is also the blow-up of Z along Y, hence the map $\psi : \mathrm{Bl}_S(\mathbb{P}^{n-1}) \to Y$ is a $\mathbb{P}^{\delta+1}$-bundle. Let $c := n - 1 - d$ be the codimension of S in $\mathbb{P}^{n-1}$. The notations and basic properties of our situation are summarised in the following diagram.



An easy argument using previous diagram shows that

Proposition 2.3. *The Euler numbers of S and Y are related by*

$$\chi(S) = \frac{(\delta+2)\chi(Y) - n}{n-d-2}.$$

2.2. **Classification of Y.** A key observation is that the fibers of $p : \mathbb{P}\mathcal{N}_{S|\mathbb{P}^{n-1}} \to S$ are mapped to linear subspaces in Y and such $\mathbb{P}^{c-1}$'s cover Y. On the other hand, we have $\dim Y = n-1-(\delta+1) = 2(c-1)$ by Lemma 2.1. By [S] and note that Y has Picard number 1, we obtain

Proposition 2.4. *The projective subvariety $Y \subset \mathbb{P}^{a-1}$ is isomorphic to one of the following:*

- (Y1) $\mathbb{P}^{2(c-1)} \cong Y = \mathbb{P}^{a-1}$;
- (Y2) *a nonsingular quadric hypersurface* $\mathbb{Q}^{2(c-1)} \cong Y \subset \mathbb{P}^{a-1} \cong \mathbb{P}^{2c-1}$; *or*
- (Y3) *a biregular projection of the Plĉker embedding* $\mathrm{Gr}(2, \mathbb{C}^{c+1}) \subset \mathbb{P}(\wedge^2\mathbb{C}^{c+1})$.

We have the following topological consequence.

Corollary 2.5. *All odd Betti numbers of S vanish. In particular, the Euler number of S satisfies $\chi(S) \geq d+1$.*

2.3. **Classification when $Y = \mathbb{P}^{2(c-1)}$.**

Proposition 2.6. *Assume that $Y = \mathbb{P}^{2(c-1)}$, then $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to one of the following:*

- (i) $\mathbb{Q}^d \subset \mathbb{P}^{d+1}$;
- (ii) $\mathbb{P}^1 \times \mathbb{P}^2 \subset \mathbb{P}^5$;
- (iii) $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$;
- (iv) *a general hyperplane section of (ii);*
- (v) *a general codimension ≤ 2 linear section of (iii);*
- (vi) *a general codimension-3 linear section of (iii);*
- (vii) *a general codimension-2 linear section of (ii).*

Here is the idea for a proof different to that presented in [FH]. If $c = 1$, then $S \subset \mathbb{P}^{n-1}$ is a hyperquadric and we are done. Assume now $c \geq 2$. Let $g(S)$ be the sectional genus of S. In this case, we have $a-1 = 2(c-1)$, hence by Theorem 1 [AS1], we have either $\deg(S) = 2, c = 2, g(S) = 0$ or $\deg(S) = 5, c = 3, g(S) = 1$. Now the Proposition follows from the classification of low degree varieties or from the results on QEL-manifolds obtained by Russo in [R].

2.4. **Classification when $Y = \mathbb{Q}^{2(c-1)}$.** The following result is an easy consequence of Proposition 2.3.

Proposition 2.7. *For a special quadratic manifold $S^d \subset \mathbb{P}^{n-1}$ with codimension c and $n \geq 3$, assume that $Y = \mathbb{Q}^{2(c-1)}$. Then either $\delta \geq \frac{d}{2}$ or $\delta = \frac{d}{2} - 1$.*

Proposition 2.8. *For a special quadratic manifold $S \subset \mathbb{P}^{n-1}$ of codimension c, assume that $Y = \mathbb{Q}^{2(c-1)}$. Then $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to one of the following*

(i) *the Segre 4-fold $\mathbb{P}^1 \times \mathbb{P}^3 \subset \mathbb{P}^7$;*
(ii) *the 10-dimensional Spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$.*

By [R], d-dimensional QEL-manifolds with $\delta \geq d/2$ have been completely classified. Hence by Proposition 2.7, the remaining case is $\delta = \frac{d}{2} - 1$. If $\delta \leq 2$, then we get $d \leq 6$ and we do a case-by-case check. Now assume $\delta \geq 3$. By the divisibility theorem of [R], $d - \delta = \delta + 2$ is divisible by $2^{[\frac{\delta-1}{2}]}$ which implies that we have only two possibilities: $(\delta, d, n) = (4, 10, 18)$ or $(\delta, d, n) = (6, 14, 24)$. For the first case, we can show the VMRT of S is $\mathbb{P}^1 \times \mathbb{P}^4 \subset \mathbb{P}^9$, hence S is isomorphic to $\mathrm{Gr}(2, 7)$ by [Mo], which cannot be embedded linearly normally into $\mathbb{P}^{17}$.

The case $(\delta, d, n) = (6, 14, 24)$ is difficult to be ruled out. The contradiction comes from a delicate structure on intersections of entry loci of S. At one hand, we show that every two general entry loci $\Sigma_i, i = 1, 2$ of S through a general point $x \in S$ will intersect along $\mathbb{P}^1$, hence the lines on Σ_i through x give two entry loci on the VMRT $\mathcal{C}_x$ of S at x, intersecting at a single point (corresponding to the line $\Sigma_1 \cap \Sigma_2$). On the other hand, we can show that $\mathcal{C}_x$ is a codimension 2 linear section of the 10-dimensional spinor variety $\mathbb{S}_5 \subset \mathbb{P}^{15}$. Any two entry loci of such a linear section intersect along a $\mathbb{P}^k$ with $k \geq 1$ if non-empty, which is a contradiction.

2.5. **Classification when $Y = \mathrm{Gr}(2, c+1)$.** To handle the case (Y3) of Proposition 2.4, we need the following result.

Proposition 2.9 (Theorem 2.17 [AS2])**.** *Let $S' \subset \mathbb{P}^{2k}$ be a quadratic manifold. Assume that the rational map $\psi^o : S' \dashrightarrow Y(S')$ is birational and $Y(S')$ is biregular to $\mathrm{Gr}(2, k+2)$. Then $S' \subset \mathbb{P}^{2k}$ is one of the rational normal scrolls. In particular, S' has Picard number 2 and is covered by lines.*

Proposition 2.10. *Let $S \subset \mathbb{P}^{n-1}$ be a special quadratic manifold of codimension $c \geq 4$ with $Y(S) = \mathrm{Gr}(2, c+1)$. Then $S \subset \mathbb{P}^{n-1}$ is projectively equivalent to $\mathbb{P}^1 \times \mathbb{P}^c \subset \mathbb{P}^{2c+1}$.*

The idea of the proof is the following: we take a general codimension-$(\delta + 1)$ linear section $S' \subset \mathbb{P}^{n-\delta-2}$ of $S \subset \mathbb{P}^{n-1}$. This gives a birational map $\psi^\circ : S' \subset \mathbb{P}^{n-\delta-2} \dashrightarrow Y(S') = \mathrm{Gr}(2, c+1)$, hence S' is a rational normal scroll by 2.9. Except a few lower dimensional cases, this implies that S has Picard number 2 with $\delta > 0$. But then S becomes a conic-connected manifold with Picard number 2, which have been completely classified by [IR], and then we are done.

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