

EXAMPLES OF MOVABLE DIVISORS ON A GENERALIZED KUMMER VARIETY AND AN APPLICATION

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1. Moduli of stable sheaves on abelian surfaces

In this article, the author would like to explain a few result on the space of global sections of line bundles on the moduli spaces of stable sheaves on abelian surfaces. In particular, we shall explain the global sections of line bundles form a locally free sheaf on the moduli of polarized abelian surfaces, under some conditions. This is a joint work with Bolognese, Marian and Opear [2]. For this purpose, we shall explain basic results on the moduli of stable sheaves on abelian surfaces. In particular, we shall explain the Bogomolov decomposition of the moduli spaces and their second cohomology groups.

We start with a topological invariant of the moduli spaces.

1.1. **Mukai lattice.** Let X be an abelian surface. A Mukai lattice of X consists of $H^{2*}(X,\mathbb{Z}) := \bigoplus_{i=0}^{2} H^{2i}(X,\mathbb{Z})$ and an integral bilinear form $\langle\ ,\ \rangle$ on $H^{2*}(X,\mathbb{Z})$:

$$\langle x_0+x_1+x_2\varrho_X, y_0+y_1+y_2\varrho_X\rangle := (x_1,y_1)-x_0y_2-x_2y_0 \in \mathbb{Z},$$

where $x_1,y_1 \in H^2(X,\mathbb{Z})$, $x_0,x_2,y_0,y_2 \in \mathbb{Z}$ and $\varrho_X \in H^4(X,\mathbb{Z})$ is the fundamental class of X. We also introduce the algebraic Mukai lattice as the pair of $H^*(X,\mathbb{Z})_{\mathrm{alg}} := \mathbb{Z}\oplus \mathrm{NS}(X)\oplus\mathbb{Z}$ and $\langle\ ,\ \rangle$ on $H^*(X,\mathbb{Z})_{\mathrm{alg}}$. It is the Hodge $(1,1)$ part of a natural weight 2 Hodge structure on $H^{2*}(X,\mathbb{Z})$. For $x=x_0+x_1+x_2\varrho_X$ with $x_0,x_2\in\mathbb{Z}$ and $x_1\in H^2(X,\mathbb{Z})$, we also write $x=(x_0,x_1,x_2)$. For $E\in\mathbf{D}(X)$, $v(E):=\mathrm{ch}(E)$ denotes the Mukai vector of E.

1.2. **Moduli spaces.**

Definition 1.1. Let H be an ample divisor on X. For $v=(r,\xi,a)\in H^*(X,\mathbb{Z})_{\mathrm{alg}}$ with $r>0$, $M_H(v)$ denotes the moduli space of semi-stable sheaves E of $v(E)=v$ with respect to H.

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$v = (r, \xi, a) \in H^*(X, \mathbb{Z})_{\rm alg}$ is primitive if $H^*(X, \mathbb{Z})_{\rm alg}/\mathbb{Z}v$ is torsion free, that is, $\gcd(r, \xi, a) = 1$. From now on, we assume that v is primitive. For a general H in the ample cone $\operatorname{Amp}(X)$ of X, $M_H(v)$ consists of stable sheaves. Then by a work of Mukai [10], $M_H(v)$ is a smooth projective variety of dimension $\langle v, v \rangle + 2$ with a holomorphic symplectic form. In particular, the canonical line bundle is trivial.

1.3. **Bogomolov decomposition.** For smooth projective manifolds with trivial canonical line bundles, we have a Bogomolov decomposition. In order to explain the Bogomolov decomposition of $M_H(v)$ ([11]), we first explain the Albanese map of $M_H(v)$. Let $\widehat{X} := \operatorname{Pic}^0(X)$ be the dual abelian variety of X and $\mathbf{P}$ the Poincaré line bundle on $\widehat{X} \times X$. For an element $E_0 \in \overline{M}_H(v)$, let $\alpha : \overline{M}_H(v) \to X$ be the morphism such that

$$\alpha(E) := \det p_{\widehat{X}!}(p_X^*(E - E_0) \otimes (\mathbf{P} - \mathcal{O}_{\widehat{X} \times X})) \in \operatorname{Pic}^0(\widehat{X}) = X,$$

and $\det : \overline{M}_H(v) \to \widehat{X}$ the morphism sending E to $\det E \otimes \det E_0^\vee \in \widehat{X}$, where $p_X : \widehat{X} \times X \to X$ and $p_{\widehat{X}} : \widehat{X} \times X \to \widehat{X}$ are projections. We set $\mathfrak{a}_v := \alpha \times \det$.

Proposition 1.1. *Assume that H is a general polarization. Then*

$$\mathfrak{a}_v : M_H(v) \to X \times \widehat{X}$$

is the Albanese map of $M_H(v)$.

Definition 1.2. We set $K_H(v) := \mathfrak{a}_v^{-1}((0, 0))$.

Theorem 1.2. *Let v be a primitive Mukai vector with $\langle v^2 \rangle \geq 6$. Let H be a general ample divisor with respect to v. Then*

(1) $\mathfrak{a}_v$ *is a locally trivial fibration with a fiber $K_H(v)$.*
(2) $K_H(v)$ *is an irreducible symplectic manifold of dimension $\langle v^2 \rangle - 2$ whose deformation class is determined by $\langle v^2 \rangle$.*

If $v = (1, 0, -n)$, then $M_H(v)$ parametrizes torsion free sheaves of rank 1. Thus we have a decomposition $M_H(v) = \widehat{X} \times \operatorname{Hilb}_X^n$. Then $\mathfrak{a}_v$ is the map

$$\begin{array}{ccccc} \widehat{X} \times \operatorname{Hilb}_X^n & \to & \widehat{X} \times S^n X & \to & \widehat{X} \times X \\ (L, I_{\{x_1, \ldots, x_n\}}) & \mapsto & (L, \{x_1, ..., x_n\}) & \mapsto & (L, \sum_i x_i). \end{array}$$

In this case, $K_H(v)$ is the *generalized Kummer variety* constructed by Beauville [1] if $\langle v^2 \rangle \geq 6$.

Remark 1.1. If $\langle v^2 \rangle = 4$, then $K_H(v)$ is a Kummer surface of X. This is the reason why $K_H(v)$ is called a generalized Kummer variety.

We set $n := \langle v^2 \rangle/2$. Let $\nu : X \times \widehat{X} \to X \times \widehat{X}$ be the n times map and we shall consider the fiber product

$$
\begin{array}{ccc}
M_H(v) \times_{X \times \widehat{X}} X \times \widehat{X} & \longrightarrow & M_H(v) \\
\Big\downarrow & & \Big\downarrow \mathfrak{a}_v \\
X \times \widehat{X} & \xrightarrow{\nu} & X \times \widehat{X}.
\end{array}
\tag{1.1}
$$

Then we have an isomorphism:

$$
K_H(v) \times X \times \widehat{X} \to M_H(v) \times_{X \times \widehat{X}} X \times \widehat{X}, \tag{1.2}
$$

which is a Bogomolov decomposition of $M_H(v)$.

2. The second cohomology groups and the Picard groups of $M_H(v)$ and $K_H(v)$

There is a natural homomorphism

$$
\theta_v : v^\perp \to H^2(M_H(v), \mathbb{Z}). \tag{2.1}
$$

If there is a universal family $\mathbf{E}$ on $M_H(v) \times X$, then

$$
\theta_v(x) = c_1 \left(p_{M_H(v)*} (\mathrm{ch}(\mathbf{E}) p_X^*(x^\vee)) \right), \tag{2.2}
$$

where $p_{M_H(v)} : M_H(v) \times X \to M_H(v)$ and $p_X : M_H(v) \times X \to X$ are projections. By the construction, θ_v preserves the Hodge structure.

Theorem 2.1 ([11]). *Let $v = (r, \xi, a) \in H^*(X, \mathbb{Z})_{\mathrm{alg}}$ be a primitive Mukai vector such that $r > 0$. We assume that $\langle v^2 \rangle \geq 6$. Then for a general ample line bundle H, the following holds.*

(1) *θ_v is injective.*

(2)

$$
H^2(M_H(v), \mathbb{Z}) = \theta_v(v^\perp) \oplus \mathfrak{a}_v^* H^2(X \times \widehat{X}, \mathbb{Z}). \tag{2.3}
$$

In particular,

$$
\mathrm{Pic}(M_H(v)/X \times \widehat{X}) \cong v^\perp \cap H^*(X, \mathbb{Z})_{\mathrm{alg}}.
$$

(3) *For $F \in \mathbf{D}(X)$ with $v(F) = x^\vee$, we have a line bundle Θ_F on $M_H(v)$ such that $c_1(\Theta_F) = -\theta_v(x)$. If there is a universal family, then*

$$
\Theta_F = \det p_{M_H(v)*} \left(\mathbf{E} \otimes p_X^*(F) \right)^\vee .
$$

Definition 2.1. For simplicity, we also denote the homomorphism

$$
v^\perp \to H^2(M_H(v), \mathbb{Z}) \to H^2(K_H(v), \mathbb{Z})
$$

by θ_v.

For an irreducible symplectic manifold M, Beauville constructed an integral bilinear form

$$q_M : H^2(M,\mathbb{Z}) \times H^2(M,\mathbb{Z}) \to \mathbb{Z}.$$

q_M is called *Beauville-Bogomolov-Fujiki form.* For $K_H(v)$, it is described by using Mukai lattice.

Theorem 2.2. *We have an isometry of Hodge structure*

$$\theta_v : (v^\perp, \langle\ ,\ \rangle) \to (H^2(K_H(v),\mathbb{Z}), q_{K_H(v)}). \tag{2.4}$$

Since $\mathrm{Pic}(K_H(v)) \cong \mathrm{NS}(K_H(v))$, $\Theta_{F|K_H(v)}$ depends only on $v(F)$.

Definition 2.2. We set $\Theta_w := \Theta_{F|K_H(v)}$, where $w = v(F)$.

We are interested in the space of global sections $H^0(M_H(v), \mathcal{O}(\theta_v(x)))$ and $H^0(K_H(v), \mathcal{O}(\theta_v(x)))$. One of the reason to study the space of global sections is the relation with strange duality, which is an observation of relations of global sections line bundles of two different moduli spaces. Before considering the space of global sections, we shall consider the holomorphic Euler characteristic of line bundles.

Theorem 2.3 (Marian-Oprea [5])**.** *Assume that Mukai vectors v, w satisfies $\langle v^\vee, w\rangle = 0$.*

(1) *Assume that w is primitive and H is general. Then*

$$\chi(M_H(w), \mathcal{O}(-\theta_w(v^\vee))) = \frac{1}{2}\frac{\langle v^2\rangle^2}{\langle v^2\rangle + \langle w^2\rangle}\binom{\langle v^2\rangle/2 + \langle w^2\rangle/2}{\langle v^2\rangle/2}. \tag{2.5}$$

(2) *Assume that v is primitive and H is general. Then*

$$\chi(K_H(v), \mathcal{O}(-\theta_v(w^\vee))) = \frac{1}{2}\frac{\langle v^2\rangle^2}{\langle v^2\rangle + \langle w^2\rangle}\binom{\langle v^2\rangle/2 + \langle w^2\rangle/2}{\langle v^2\rangle/2}. \tag{2.6}$$

Remark 2.1.
(1) Since $v^\vee \in w^\perp$ and $w^\vee \in v^\perp$, $\theta_w(v^\vee)$ and $\theta_v(w^\vee)$ are well-defined.
(2) For an irreducible symplectic manifold M, the holomorphic Euler characteristic of a line bundle L is a polynomial of $q_M(c_1(L)^2)$ depending only on the deformation classes.

If the higher cohomology groups vanish, then Theorem 2.3 shows the dimension of global sections. By the Kawamata-Viehweg vanishing theorem, if $-\theta_v(w^\vee)$ is nef and big, then the higher cohomology groups vanish. More generally, if there is a different minimal model such that $-\theta_v(w^\vee)$ is nef and big, then the higher cohomology groups vanish. So we are interested in the movable cones of the moduli spaces.

Remark 2.2. $M_H(v)$ may be singular if $\gcd(r, d(H^2), a) \neq 1$. By choosing a small and a general $\beta \in \mathrm{NS}(X)_{\mathbb{Q}}$, we have a symplectic resolution $\phi : M_H^\beta(v) \to M_H(v)$, where $M_H^\beta(v)$ is the moduli of β-twisted semi-stable sheaves in the sense of Matsuki and Wentworth [7]. For $F \in \mathbf{D}(X)$ with $v(F^\vee) \in \mathbb{Z} \oplus \mathbb{Z}H \oplus \mathbb{Z} \cap v^\perp$, Θ_F is well-defined on $M_H(v)$ and $M_H^\beta(v)$. Since ϕ is a symplectic resolution,

$$\mathbf{R}\phi_*(\mathcal{O}_{M_H^\beta(v)}) = \mathcal{O}_{M_H(v)}.$$

Hence $H^i(M_H^\beta(v), \Theta_F) \cong H^i(M_H(v), \Theta_F)$. Similar claim also holds for $\Theta_{F|K_H(v)}$. In particular, similar claim to Theorem 2.3 also hold for singular cases. If $\Theta_{F|K_H^\beta(v)} \in \mathrm{Mov}(K_H^\beta(v))$, then

$$\dim H^0(K_H^\beta(v), \Theta_{F|K_H^\beta(v)}) = \chi(K_H^\beta(v), \Theta_{F|K_H^\beta(v)}).$$

2.1. **Movability.** Let $P^+(v^\perp)$ be the positive cone of $v^\perp$:

$$P^+(v^\perp) := \{x \in H^*(X, \mathbb{R})_{\mathrm{alg}} \mid \langle x^2 \rangle > 0,\ \langle x, h \rangle > 0\}, \tag{2.7}$$

where $\theta_v(h)$ is ample. Let $I_d(v)$, $d = 1, 2$ be the set of primitive isotropic Mukai vectors u such that $\langle u, v \rangle = \pm d$ and set $I(v) := I_1(v) \cup I_2(v)$.

Theorem 2.4 (Markman [6], Yoshioka [12]). *Assume that H is general. Let $\mathcal{C}$ be the connected component of $P^+(v^\perp) \setminus \cup_{u \in I(v)} u^\perp$ containing*

$$h = (0, rH, (\xi, H)) + \epsilon(-r, 0, a), \tag{2.8}$$

where $\epsilon > 0$ is sufficiently small. Then the interior of the movable cone $\mathrm{Mov}(M_H(v))$ is $\theta_v(\mathcal{C})$.

Remark 2.3. By the consruction of the moduli space $M_H(v)$, the line bundle $\theta_v(h)$ in (2.8) is ample.

For the proof of this result, the author used a modern and sophisticated method, that is, the geometry of objects in the derived category of coherent sheaves [3]. Roughly speaking, we have

$$\mathrm{Mov}(K_H(v)) = \bigcup_{K' \dashrightarrow K_H(v)} \mathrm{Nef}(K') \tag{2.9}$$

Moreover K' is the Bogomolov factor of a moduli space of stable complexes in the sense of Bridgeland. In particular, all K' are deformation equivalent and Theorem 2.3 (2) holds for corresponding line bundles.

Remark 2.4. For the case of twisted stability in Remark 2.2, we have

$$\mathrm{Mov}(K_H^\beta(v)) = \left\{ \theta_v(x) \;\middle|\; \begin{matrix} x \in P^+(v^\perp) \text{ and } \langle x, u \rangle \geq 0 \\ \text{for } u \in I(v) \text{ with } \langle u, h \rangle > 0 \end{matrix} \right\},$$

where $h := \epsilon((-1, \frac{-a}{d(H^2)}H, 0) + (0, \frac{(H,\beta)}{(H^2)}H - \beta, 0)) + (0, rH, d(H^2))$, $1 \gg \epsilon > 0$ defines an ample class of $K_H^\beta(v)$.

2.2. **A typical movable divisor.** Let (X, H) be a polarized abelian surface. Since $\rho(X) = 1$ for a general pair (X, H), we assume that a primitive Mukai vecor v is of the form $v = (r, dH, a)$. For the moduli space $M_H(v)$, we know that $\theta_v((0, rH, d(H^2))$ is nef and big and gives a contraction

$$M_H(v) \to N_H(v)$$

from the Gieseker compactification $M_H(v)$ of the moduli of μ-stable vector bundles to the Uhlenbeck compactification $N_H(v)$ of the moduli of μ-stable vector bundles. Thus $\theta_v((0, rH, d(H^2)) \in \operatorname{Mov}(M_H(v))$.

3. Our results

3.1. **Movable divisors and the spaces of global secions.** Let (X, H) be a polarized abelian surface as in the previous subsection. Let $v = (r, dH, a)$ and $w = (r', d'H, a')$ be primitive Mukai vectors such that $r, r' \geq 0$, $d, d' > 0$ and $a, a' \leq -1$. Assume that $\langle v^2 \rangle, \langle w^2 \rangle \geq 6$. We also assume that $\langle v, w^\vee \rangle = -dd'(H^2) - ra' - r'a = 0$.

Proposition 3.1. *Assume that $r, r' \geq 0$, $d, d' > 0$ and $a, a' < 0$. Then $\theta_v(-w^\vee) \in \operatorname{Mov}(K_H(v))$ unless $d = 1$ and there is a divisor η such that $(H, \eta) = 1$ and $(\eta^2) = 0$. Moreover if $\theta_v(-w^\vee) \notin \operatorname{Mov}(K_H(v))$, then (1) $a = -1$, $d' + a' > 0$ or (2) $r = 1$, $d' - r' < 0$.*

Let $u := (p, \eta, q)$ be a primitive isotropic Mukai vector. Then $(\eta^2) = 2pq$. By classifying $u \in I(v)$ separating $(0, rH, d(H^2))$ and $(-1, \frac{-a}{d(H^2)}H, 0)$, we get Proposition 3.1 (see [2]). By the proof of this result, we also get the following corollary.

Corollary 3.2. *For a general stable sheaf $E \in M_H(r, dH, a)$, $\Phi^{\mathbf{P}[1]}_{X \to \widehat{X}}(E)$ is a stable sheaf. In particular, $\Phi^{\mathbf{P}[1]}_{X \to \widehat{X}}$ induces a birational map*

$$M_H(r, dH, a) \cdots \to M_{\widehat{H}}(-a, d\widehat{H}, -r)$$

for any X. Moreover $\Phi^{\mathbf{P}[1]}_{X \to \widehat{X}}$ induces an isomorphism

$$\operatorname{Mov}(K_H(r, dH, a)) \cong \operatorname{Mov}(K_{\widehat{H}}(-a, d\widehat{H}, -r))$$

unless (1) $v = (r, H, -1)$ or $v = (1, H, a)$ and (2) there is a divisor η such that $(\eta, H) = 1$, $(\eta^2) = 0$.

Let $\mathbf{E}$ be a universal family on $M_H(v) \times X$. For $F \in \mathbf{D}(X)$ with $F \in M_H(w)$, we set

$$\Theta_{F|K_H(v)} = \det \mathbf{R}p_{K_H(v)*}(\mathbf{E} \otimes p_X^*(F))^\vee \in \operatorname{Pic}(M_H(v)) \tag{3.1}$$

is independent of the choice of F. So we set

$$\Theta_w := \Theta_{F|K_H(v)}.$$

The following is a consequence of Proposition 3.1 and the Kawamata-Viehweg vanishing theorem.

Theorem 3.3 ([2, Theorem 4]). *We set $v = (r, dH, a)$ and $w = (r', d'H, a')$. Assume that $r, r' \geq 2$, $d, d' > 0$, $a, a' < 0$, and $(d, a) = (1, -1)$ implies (X, H) is not a product of polarized elliptic curves (C_1, h_1) and (C_2, h_2) with $\deg h_1 = 1$. Then*

$$h^0(K_H(v), \Theta_w) = \chi(K_H(v), \Theta_w)$$

and

$$h^0(M_H(v), \Theta_F) = \chi(M_H(v), \Theta_F).$$

In particular, the global sections of Θ_w and Θ_F form locally free sheaves on the moduli of polarized abelian surfaces.

The claim for $M_H(v)$ follows from the decomposition (1.1).

3.2. **Relation of two spaces of global sections.** In this subsection, we shall explain strange duality conjecture. Le Potier found a relation of the spaces of global sections on some moduli spaces of semi-stable sheaves on $\mathbb{P}^2$. O'Grady, Abe, Marian and Oprea found similar phenomena on other surfaces with trivial first betti numbers. If the first betti number is not zero, it seems to modify the formulation. In the following, we shall explain a formulation of strange duality for abelian surfaces.

We set

$$\mathcal{D} := \{(E, F) \in M_H(w) \times K_H(v) \mid H^0(E \times F) \neq 0\}. \tag{3.2}$$

By the Brill-Noether theory, $\mathcal{D}$ is a divisor if $\mathcal{D}$ is a proper subset of $M_H(w) \times K_H(v)$. Since $\operatorname{Pic}^0(K_H(v)) = 0$, we have a decomposition

$$\mathcal{O}_{M_H(w)\times K_H(v)}(\mathcal{D}) \cong \Theta_F \boxtimes \Theta_w.$$

Hence if $\mathcal{D}$ is a divisor, we have a section

$$\mathbb{C} \to H^0(\mathcal{O}_{M_H(w)\times K_H(v)}(\mathcal{D})) \cong H^0(M_H(w), \Theta_F) \otimes H^0(K_H(v), \Theta_w).$$

Thus we have a homomorphism

$$D : H^0(K_H(v), \Theta_w)^\vee \to H^0(M_H(w), \Theta_F). \tag{3.3}$$

Theorem 3.4 ([2, Theorem 3]). *Let (X, H) be a generic polarized abelian surfaces. Let v, w be the Mukai vectors in Theorem such that $d = d' = 1$ and $a, a' < 0$. Then there is an isomorphism*

$$D : H^0(K_H(v), \Theta_w)^\vee \to H^0(M_H(w), \Theta_F). \tag{3.4}$$

Remark 3.1. For the moduli of semi-stable vector bundles on curves, the space of global sections are the space of conformal blocks, and studied extensively. In these cases, similar dualities are knows as the level rank duality.

For the case of surfaces, we don't know representation theoretic meaning of the space of global sections at this moment. We can only say that the holomorphic Euler characteristics of the moduli spaces encode the information of Donaldson invariants, and hence they are related to modular forms [4].

3.3. **A family of minimal models.** As a final remark of Proposition 3.1, we shall explain there is a family of minimal models such that Θ_w is nef and big. We continue to assume that $v=(r,dH,a)$ is primitive, that is, $\gcd(r,d,a)=1$. For $(sH,tH)\in \mathrm{NS}(X)_{\mathbb{Q}}\times \mathrm{Amp}(X)_{\mathbb{R}}$, we have a stability condition $\sigma_{(sH,tH)}$ of Bridgeland. It consists of an abelian subcategory $A_{(sH,tH)}$ of $\mathbf{D}(X)$ consisting of certain 2-termcomplexes and a stabilily fuction $Z_{(sH,tH)}$ which is an analogue of $\chi(E(nH))/\operatorname{rk}E$. Let $M_{(sH,tH)}(v)$ be the moduli of $\sigma_{(sH,tH)}$-semi-stable objects.

Let $\pi:(\mathcal{X},\mathcal{H})\to Y$ be a family of polarized abelian surfaces with $\mathcal{H}_y=H$ and assume that there is a section ρ of π. Then $\mathbb{Z}\oplus\mathbb{Z}\mathcal{H}\oplus\mathbb{Z}\rho\subset R^*\pi_*\mathbb{Z}$. We consider a family of stability conditions $\sigma_{(s\mathcal{H},t\mathcal{H})}$ over Y. We fix s with $d-rs>0$. Then there are finitely many Mukai vectors u defining walls in $\mathbb{R}_{>0}\mathcal{H}$ with respect to v (see the end of [8, sect. 3.1]). Therefore there are $t_1,t_2,...,t_k\in\mathbb{R}_{>0}$ such that every interval (t_i,t_{i+1}) is contained in a chamber for any stability condition $\sigma_{(s\mathcal{H}_y,t\mathcal{H}_y)}$, $y\in Y$. We take $t\in\mathbb{R}_{>0}\setminus\{t_1,...,t_k\}$. Then there is a primitive and isotropic Mukai vector $u=(p,l\mathcal{H},q)$ such that $Z_{(s\mathcal{H},t\mathcal{H})}(u)\in\mathbb{Q}_{>0}Z_{(s\mathcal{H},t'\mathcal{H})}(v)$, where t' is sufficiently close to t. We set $\mathcal{X}':=M_{\mathcal{H}}(\pm u)$, where we fix the sign of $\pm u$ so that $\operatorname{rk}(\pm u)>0$. Then $\mathcal{X}'$ is isomorphic to the moduli of $\sigma_{(s\mathcal{H},t\mathcal{H})}$-stable objects with the Mukai vector u. Let $\mathcal{H}'$ be the polarization of $\mathcal{X}'$ which is naturally defined by $(0,p\mathcal{H},l(H^2))$. Let $\mathbf{E}\in\mathbf{D}(\mathcal{X}\times_Y\mathcal{X}')$ be the universal family of $\sigma_{(s\mathcal{H},t\mathcal{H})}$-stable objects with Mukai vector u. Let $\mathcal{E}$ be a family of $\sigma_{(s\mathcal{H},t\mathcal{H})}$-stable objects with the Mukai vector v. Then by the relative Fourier-Mukai transform $\Phi_{\mathcal{X}\to\mathcal{X}'}^{\mathbf{E}^{\vee}[1]}$, we have a family of Gieseker semi-stable sheaves $\Phi_{\mathcal{X}\to\mathcal{X}'}^{\mathbf{E}^{\vee}[1]}(\mathcal{E})$, where $\mathcal{H}'$ is the polarization. Thus we have a relative version of [9, Thm. 1.4]. In particular, we have a projective family of moduli spaces $M_{(s\mathcal{H},t\mathcal{H})}(v)\to Y$. We have a surjective morphism $\xi:\mathbb{R}H\times\mathbb{R}_{>0}H\to\{\lambda=(x,yH,z)\mid\langle\lambda,(0,rH,d(H^2))\rangle>0,\langle\lambda^2\rangle>0\}/\mathbb{R}_{>0}$ such that $\{\xi(0,tH)\mid t\in\mathbb{R}_{>0}\}=\mathbb{R}_{>0}(0,rH,d(H^2))+\mathbb{R}_{>0}(-1,\frac{-a}{d(H^2)}H,0)/\mathbb{R}_{>0}$. For $\lambda\in\mathbb{Q}_{>0}(0,rH,d(H^2))+\mathbb{Q}_{>0}(-1,\frac{-a}{d(H^2)}H,0)$, we take $[t_i,t_{i+1}]$ such that $\lambda\in\mathbb{Q}_{>0}\xi(0,tH)$, $t\in[t_i,t_{i+1}]$. Then $\theta_v(\lambda)\in\mathrm{Nef}(M_{(0,t\mathcal{H})}(v)/Y)$.

To be more precise, for a point $y \in Y$, we take a small $\beta_y \in \mathrm{NS}(\mathcal{X}_y)_{\mathbb{Q}}$ which gives a resolution $M_{(\beta_y, t\mathcal{H}_y)}(v) \to M_{(0,t\mathcal{H}_y)}(v)$. Then $\theta_v(\lambda) \in \mathrm{Nef}(M_{(\beta_y,t\mathcal{H}_y)}(v))$. Since $\langle \lambda^2 \rangle > 0$, the Kawamata-Viehweg vanishing theorem implies that $H^i(M_{(0,t\mathcal{H}_y)}(v), \theta_v(\lambda)) = 0$ for $i > 0$ and $H^0(M_{(0,t\mathcal{H}_y)}(v), \theta_v(\lambda))$ $(y \in Y)$ forms a locally free sheaf on Y.

Remark 3.2. If $\gcd(r, d(H^2), a) \neq 1$, then there may be abelian surfaces $\mathcal{X}_y$ and Mukai vectors $u = (r', \eta, a')$ such that $0 < r' < r$, $(1, \frac{(\eta,H)}{r'}, \frac{a'}{r'}) = (1, \frac{d(H^2)}{r}, \frac{a}{r})$. Then $M_{(s\mathcal{H}_y,t\mathcal{H}_y)}(v)$ is singular.

References

[1] Beauville, A., *Variétés Kähleriennes dont la première classe de Chern est nulle,* J. Diff. Geom. **18** (1983), 755–782

[2] Bolognese, B., Marian, A., Oprea, D., Yoshioka, K., *On the strange duality conjecture for abelian surfaces II,* arXiv:1402.6715

[3] Bridgeland, T., *Stability conditions on K3 surfaces,* math.AG/0307164, Duke Math. J. **141** (2008), 241–291

[4] L. Göttsche, H. Nakajima and K. Yoshioka: *K-theoretic Donaldson invariants via Instanton Counting,* Pure and Applied Mathematics Quarterly, **5** (2009), 1029–1111.

[5] Marian, A., Oprea, D., *Sheaves on abelian surfaces and srange duality,* Math. Ann. **343** (2009), 1–33.

[6] Markman, E., *The movable cone of hyperkahler manifolds deformation equivalent to a generalized Kummer,* A letter to the author.

[7] Matsuki, K., Wentworth R., *Mumford-Thaddeus principle on the moduli space of vector bundles on an algebraic surface,* Internat. J. Math. **8** (1997), no. 1, 97–148.

[8] Minamide, H., Yanagida, S., Yoshioka, K., *Fourier-Mukai transforms and the wall-crossing behavior for Bridgeland's stability conditions*, arXiv:1106.5217.

[9] Minamide, H., Yanagida, S., Yoshioka, K., *Some moduli spaces of Bridgeland's stability conditions*, International Mathematics Research Notices 2013; doi:10.1093/imrn/rnt126.

[10] Mukai, S., *Duality between $D(X)$ and $D(\widehat{X})$ with its application to Picard sheaves*, Nagoya Math. J. **81** (1981), 153–175.

[11] Yoshioka, K., *Moduli spaces of stable sheaves on abelian surfaces,* Math. Ann. **321** (2001), 817–884, math.AG/0009001

[12] Yoshioka, K., *Bridgeland's stability and the positive cone of the moduli spaces of stable objects on an abelian surface,* arXiv:1206.4838, Adv. Stud. Pure Math. to appear.

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