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Generalized Koszul duality and its application

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0. Motivation

Let k be a field and V be a finite dimensional vector space over k. Then we have the following famous diagram.

$$\begin{array}{ccc} \mathcal{D}^b(\operatorname{coh}\mathbb{P}(V)) & \xrightarrow[\sim]{\text{Beilinson}} & \mathcal{D}^b(\text{mod-}B_n) \\ \text{BGG corresp.}\ \big| \wr & & \\ \underline{\text{grmod}}\,\bigwedge(V^*) & & \end{array}$$

where

$$\begin{cases} B_n \ : \ (\text{the Beilinson algebra}) \\ \bigwedge(V^**) \ : \ (\text{the exterior algebra of } V^*) \\ \underline{\text{grmod}} \ : \ (\text{the stable category}) \end{cases}$$

The BGG correspondence was developed into the theory of Koszul duality.

Let Q be a quiver whose path algebra kQ is finite dimensional and has infinite representation type.

Then we have the following diagram similar to the above diagram.

$$\begin{array}{ccc} \mathcal{D}^b(\operatorname{qcoh}\Pi(Q)) & \xleftarrow[\sim]{M.} & \mathcal{D}^b(\text{mod-}kQ) \\ & \swarrow_{\sim}^{\text{Happel}} & \\ \underline{\text{grmod}}\ T(Q) & & \end{array}$$

where

$$\begin{cases} D(kQ) := \operatorname{Hom}_{k\text{-vect}}(kQ, k) \\ \rho := \mathbb{R}\operatorname{Hom}_{kQ}(D(kQ), kQ)[1] \\ \Pi(Q) := \bigoplus_{n\geq 0}\rho^{\otimes n} \quad (\text{the preprojective algebra }) \\ T(Q) := kQ \oplus D(kQ) \quad (\text{the trivial extension algebra}) \end{cases}$$

Comparing these diagrams, it is natural to consider that $T(Q)$ and $\Pi(Q)$ are "Koszul dual" to each other. But they are not Koszul algebra in the classical sense. To get a framework including this case, we work with graded dg-algebras.

1. Generalized Koszul duality

Let $\mathcal{A}$ be a dg-algebra over k and

$$\mathcal{R} = \mathcal{A} \oplus P_1\langle -1\rangle \oplus P_2\langle -2\rangle \oplus P_3\langle -3\rangle \oplus \cdots$$

be a connected positively graded dg-algebra over $\mathcal{A}$. (where $\langle n\rangle$ is the graded-degree shift operator by n.) We assume that each term P_i is obtained from $\mathcal{A}\otimes_k \mathcal{A}^{op}$ by taking finite number of cones,shifts and direct summand.

We construct its "Koszul dual" $\mathcal{R}^!$.

Definition 1.

$$Q_n := \left(\bigoplus_{\lambda\models n}\bigotimes_{\mathcal{A}}(P_{\lambda_i}[-\lambda_i+1]), \text{"one-sided-twist"}\right)$$

where the "one-sided-twist" is obtained from the totalization of the bar resolution of $\mathcal{R}$ over $\mathcal{A}$.

$$\mathcal{R}^! := \mathcal{A}\oplus Q_1^\vee[-1]\langle 1\rangle \oplus Q_2^\vee[-2]\langle 2\rangle \oplus Q_3^\vee[-3]\langle 3\rangle \oplus \cdots$$

where we set $(-)^\vee := \operatorname{Hom}_{\mathcal{A}\text{-bimod}}(-, \mathcal{A})$.

This $\mathcal{R}^!$ is a connected negatively graded dg-algebra over $\mathcal{A}$. $\mathcal{R}$ and $\mathcal{R}^!$ satisfy the following properties similar to the classical case

Theorem 2.

$$\mathbb{R}\operatorname{Hom}_{\mathcal{R}\text{-gr}}(\mathcal{A},\mathcal{A}) \simeq \mathcal{R}^!$$

$$\mathbb{R}\operatorname{Hom}_{\mathcal{R}^!\text{-gr}}(\mathcal{A},\mathcal{A}) \simeq \mathcal{R}.$$

We can also construct the "Koszul complex " $\mathcal{K}$ which is graded $\mathcal{R}\otimes(\mathcal{R}^!)^{op}$ module. For any graded $\mathcal{R}$ module $\mathcal{M}$, we have

$$\operatorname{Hom}_{\text{gr-}\mathcal{R}}(\mathcal{K},\mathcal{M}) \simeq \mathbb{R}\operatorname{Hom}_{\text{gr-}\mathcal{R}}(\mathcal{A},\mathcal{M}).$$

From the functor

$$F := \operatorname{Hom}_{\text{gr-}\mathcal{R}}(\mathcal{K}, -) \ : \ \mathcal{D}(\text{gr-}\mathcal{R}) \longrightarrow \mathcal{D}(\text{gr-}\mathcal{R}^!)$$

we obtain the following equivalence

Theorem 3.

$$\operatorname{fg}_{/\mathcal{A}}(\text{gr-}\mathcal{R}) \simeq \operatorname{Perf}(\text{gr-}\mathcal{R}^!) \qquad (1)$$

where $\operatorname{fg}_{/\mathcal{A}}(\text{gr-}\mathcal{R})$ (resp. $\operatorname{Perf}(\text{gr-}\mathcal{R}^!)$) is the full sub triangulated category of $\mathcal{D}(\text{gr-}\mathcal{R})$ (resp. $\mathcal{D}(\text{gr-}\mathcal{R}^!)$) generated by $\{\mathcal{A}\langle n\rangle \mid n\in\mathbb{Z}\}$ (resp.$\{\mathcal{R}^!\langle n\rangle \mid n\in\mathbb{Z}\}$).

2. Application

2-1 Let $\mathcal{A}$ be a homologically smooth dg-algebra and P be an invertible $\mathcal{A}$ bi-module and $R = T_{\mathcal{A}}(P)$ be the tensor algebra of P over $\mathcal{A}$.

Theorem 4. *we have the following equivalences by the functor F*

$$\operatorname{Perf}(\text{gr-}\mathcal{R}) \simeq \operatorname{fg}_{/\mathcal{A}}(\text{gr-}\mathcal{R}^!) \qquad (2)$$

From the equivalences 1 and 2, we get the following corollary which generalize the Happel's theorem.

Corollary 5.

$$\operatorname{Perf}(\text{gr-}\mathcal{R})/\operatorname{fg}_{/\mathcal{A}}(\text{gr-}\mathcal{R}) \simeq \operatorname{fg}_{/\mathcal{A}}(\text{gr-}\mathcal{R}^!)/\operatorname{Perf}(\text{gr-}\mathcal{R}^!)$$
$$\simeq \operatorname{Perf}(\mathcal{A})$$

2-2 Let A be a finite dimensional algebra of finite global dimension and P be a tilting A bi-module.

Reversing Serre's vanishing theorem, we define a pair $(\mathcal{D}^{P,\geq 0}, \mathcal{D}^{P,\leq 0})$ of full sub categories of $\mathcal{D}^b(\text{mod-}A)$ as follows.

Definition 6.

$$M^\cdot \in \mathcal{D}^{P,\geq 0} \iff M\otimes^L P^{\otimes^L n} \in \mathcal{D}^{\geq 0}(\text{mod-}A)$$

$$\textit{resp. } M^\cdot \in \mathcal{D}^{P,\leq 0} \iff M\otimes^L P^{\otimes^L n} \in \mathcal{D}^{\leq 0}(\text{mod-}A)$$

for $n >> 0$

Corollary 7. *Assume that $P^{\otimes^L n}$ is pure for every $n\geq 0$, then the following conditions are equivalent*

(i) $(\mathcal{D}^{P,\geq 0}, \mathcal{D}^{P,\leq 0})$ *is a t-structure in* $\mathcal{D}^b(\text{mod-}A)$

(ii) the tensor algebra $T_A(P)$ is a graded coherent ring.