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On Mixed Plurigenera of Algebraic Plane Curves

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The purpose is to study algebraic plane curves from the viewpoint of birational geometry of pairs by making use of mixed plurigenera. To introduce the notion of mixed plurigenera, we begin by recalling birational geometry of pairs (S, D) where D is an algebraic curve on an algebraic surface S.

We shall consider algebraic varieties defined over the field of complex numbers. Here by a surface we mean a 2-dimensional projective non-singular variety.

When D is a non-singular curve on S, we consider complete linear systems $|mK_S + aD|$ where K_S is a canonical divisor on S and $m \geq a \geq 0$.

It is not hard to check that $\dim |mK_S + aD|$ is birationally invariant. Hence, $\dim |mK_S + aD| + 1$, denoted by $P_{m,a}[D]$, is called (m, a)-genus of the pair (S, D). Occasionally, it is called (m, a)-genus of D. (m, a)-genera may be called mixed plurigenera. Note that $(m, 0)$-genus becomes m-genus of S and (m, m)-genus turns out to be logarithmic m-genus of $S - D$, written simply as $P_m[D]$. From $P_m[D]$, the Kodaira dimension $\kappa[D]$ is introduced.

In what follows, suppose that S is a rational surface. Thus the study of pairs (S, D) may be understood as birational geometry of plane curves.

Let C be a curve on $\mathbf{P}^2$. Then after successive blowing ups, we obtain a non-singular curve D and a surface S which is obtained from $\mathbf{P}^2$. Then (S, D) is birationally equivalent to $(\mathbf{P}^2, C)$. By making use of (S, D), we define $P_{m,a}[C]$ to be $P_{m,a}[D]$.

Occasinally, (S, D) is said to be a non-singular model of the pair $(\mathbf{P}^2, C)$.

In 1928, Coolidge studied algebraic plane curves C and obtained the remarkable result to the effect that any rational plane curve can be transformed into a straight line on $\mathbf{P}^2$ by a birational transformation of $\mathbf{P}^2$, whenever $P_{2,1}[C] = 0$. In this case, $\kappa[C] = -\infty$.

In 1961, Nagata obtained the following result. If $g = g(D) > 0$, then $D^2 \leq 4g + 5$. Further if $D^2 = 4g + 5$, then $g = 1$ and (S, D) is birationally equivalent to $(\mathbf{P}^2, \Gamma)$, where Γ is a non-singular cubic.

Since 1983, the theory of minimal models (S, D) was introduced and has been extensively studied by Iitaka. He determined the structure of (S, D) when $\kappa[D] = 0$ or 1. Moreover he showed that, if $\kappa[D] = 2$, then any relatively minimal pair (S, D) becomes minimal. Therefore, given a plane curve C, we have a minimal pair (S, D) which is birationally equivalent to $(\mathbf{P}^2, C)$, provided that $\kappa[D] = 2$. Hereafter we suppose $\kappa[D] = 2$.

When $S \neq \mathbf{P}^2$, the minimal model (S, D) is derived from a $\sharp$-minimal pair (Σ_B, C), Σ_B being a Hirzebruch surface, which has type

$$[\sigma * e, B; \nu_1, \nu_2, \cdots, \nu_r]$$

By Riemann-Roch theorem and vanishing theorem due to Kawamata, the following formulas are obtained by Iitaka:

$$\begin{aligned} P_{2,1}[D] &= Z^2 + 2 - g, \\ P_{3,1}[D] &= 3Z^2 + 8 - 7g + D^2, \quad (\sigma \geq 6) \\ P_{4,2}[D] &= (2Z - D)^2 + 2(Z^2 - g + 1) + 1, \end{aligned}$$

where $Z = K_S + D$.

Moreover,

$$P_2[D] = \begin{cases} P_{2,1}[D] = Z^2 + 2 & (g = 0), \\ P_{2,1}[D] + 1 = Z^2 + 2 & (g = 1), \\ P_{2,1}[D] + 3g - 3 = Z^2 + 2g - 1 & (g > 1). \end{cases}$$

Thus, mixed plurigenera (m, a)-genus are computed through g, Z^2 and D^2.

So far, the structures of pairs (S, D) have been studied in the following cases: (1). $P_2[D] = 2g - 1, 2g, 2g + 1$. (2). $P_{2,1}[D] = 1, 2, 3$. (3). $P_{3,1}[D] = 1, 2, 3$.

Here, we shall enumerate the types of pairs (S, D) in the following cases: $P_2[D] = 2g + 2$, $P_{2,1}[D] = 4$, $P_{3,1}[D] = 4, 5, 6$ and $P_{4,2}[D] \leq 12$.

The tables of these types will appear in the bottom of sections.

Finally, we shall give concrete examples which satisfy

$$P_2[D] \leq 2g + 2, P_{2,1}[D] \leq 4, P_{3,1}[D] \leq 6$$

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Table 1: $P_{3,1}[D] = 4$

$k = P_{2,1}[D] - 2$.

p	a	prototype	k	g
1	2	$[7 * 7; 1]$	119	36
		$[7 * 14, 2; 1]$	119	36
	1	$[9 * 13, 1; 4^{10}]$	1	0
0	6	$[6 * 9; 1]^*$	133	40
	2	$[8 * 9; 4^8]^*$	4	8
	0	$[14 * 14; 7^7, 6, 4]^*$	1	1
		$[12 * 12; 6^6, 5^3]^*$	1	1
		$[12 * 12; 6^7, 5]^*$	3	6
		$[10 * 10; 5^7]^*$	5	11
		$[10 * 10; 5^6, 4^3]^*$	2	3
		$[8 * 8; 4^5]^*$	8	19