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Homogeneous isoparametric hypersurfaces with four distinct principal curvatures and moment maps

FUJII, Shinobu (Hiroshima University)

Main Results

Homog. Herm. isoparametric hypersurf. ($\subsetneq \mathbb{S}^n$, $g = 4$) is obtained from some moment maps!!

Our Goal

Isoparametric hypersurf. $\subsetneq \mathbb{S}^n$, $g = 4$

$\Updownarrow$ Is there any relation?

Moment maps

What is an isoparametric hypersurface?

$M \subsetneq \mathbb{S}^n$: a hypersurf. in sphere,

- M: isoparametric

  $\overset{\text{def.}}{\Longleftrightarrow}$ ${}^{\exists}\varphi : \mathbb{S}^n \to \mathbb{R}$: isoparametric & ${}^{\exists}c \in \mathbb{R}$ s.t. $M = \varphi^{-1}(c)$.

  - $\varphi : \mathbb{S}^n \to \mathbb{R}$: isoparametric $\overset{\text{def.}}{\Longleftrightarrow}$ ${}^{\exists}A(t), {}^{\exists}B(t) \in C^{\infty}(\mathbb{R})$ s.t. $\begin{cases} \|\operatorname{grad}\varphi\|^2 = A \circ \varphi, \\ \Delta\varphi = B \circ \varphi. \end{cases}$

Isotropy representations

G/K: Herm. symm. sp. of rank 2,

$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$: the Cartan decomp. of $\mathfrak{g} = \mathrm{Lie}(G)$,

1. the principal orbit of $K \overset{\mathrm{Ad}}{\curvearrowright} \mathfrak{p}$ = homog. isoparametric hypersurf.

   $\Rightarrow {}^{\exists}\varphi : \mathfrak{p} \to \mathbb{R}$: isoparametric,

   $\Rightarrow \varphi$: K-invariant,

2. $K \overset{\mathrm{Ad}}{\curvearrowright} \mathfrak{p}$: Hamiltonian K-action,

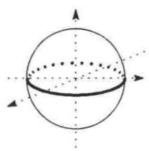
   $\Rightarrow {}^{\exists}\mu : \mathfrak{p} \to \mathfrak{k}^*$: the moment map,

   $\Rightarrow \mathfrak{p} \xrightarrow[K\text{-equivariant}]{\text{moment map } \mu} \mathfrak{k}^* \xrightarrow{K\text{-invariant norm } \|\cdot\|} \mathbb{R}$

Example of isoparametric hypersurf.

$\mathbb{S}^2 \supsetneq M = \mathbb{S}^1(r)$, $0 < r \le 1$.

$\Longrightarrow M$: isoparametric.



$\varphi : \mathbb{S}^2 \to \mathbb{R}$,

$\varphi(x, y, z) := z$.

Facts

$M \subsetneq \mathbb{S}^n$: hypersurf.

- M: homog. $\Rightarrow$ M has constant principal curvatures,
  $\Leftrightarrow$ M: isoparametric,
- M: homog. $\Leftrightarrow$ M: the principal orbit of the isotropy rep. of ${}^{\exists}G/K$: symmetric sp. of rank 2,
- M: isoparametric,
  $g := \#\{\text{distinct principal curvatures of } M\}$,
  - g must be 1, 2, 3, 4, or 6,
  - $g \le 3 \Rightarrow$ "M: homog. $\Longleftrightarrow$ M: isoparametric",
  - $g = 4 \Rightarrow {}^{\exists}$non-homog., classification is still open.