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On the birational unboundedness of higher dimensional $\mathbb{Q}$-Fano varieties

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1 Introduction

By a **$\mathbb{Q}$-Fano variety**, we mean a normal projective $\mathbb{Q}$-factorial variety with only log terminal singularities whose anticanonical divisor is ample.

We say that a class of varieties $\mathfrak{V}$ is **birationally bounded** if there is a morphism $f\colon \mathcal{X} \to \mathcal{S}$ between algebraic schemes such that every variety in $\mathfrak{V}$ is birational to one of the geometric fibres of f. We say that $\mathfrak{V}$ is **birationally unbounded** if it is not birationally bounded.

Main theorem

Theorem 1. If $n \geq 6$ then the family of $\mathbb{Q}$-Fano n-folds defined over $\mathbb{C}$ with Picard number one is birationally unbounded.

This result is known for 3-folds by the work of J. Lin [3].

Conjecture 2 (Borisov-Alexeev-Borisov). Fix $\epsilon > 0$. Then the family of $\mathbb{Q}$-Fano varieties of a given dimension with log discrepancy greater than ϵ is bounded.

Conjecture 2 is solved in the surface case by Alexeev and in the toric case by A. Borisov and V. Borisov. Theorem 1 shows that we cannot remove the restriction on log discrepancies from the hypothesis of Conjecture 2 even if we replace "boundedness" by "birational boundedness".

Followings are some of the classes of $\mathbb{Q}$-Fano varieties which have been known to be bounded:

- Smooth Fano varieties (in arbitrary dimension) (cf. [1]).
- $\mathbb{Q}$-Fano 3-folds with canonical singularities (cf. [2]).
- Log terminal $\mathbb{Q}$-Fano pairs of bounded index (in arbitrary dimension) (cf. [4]).

2 Outline of the proof

Let a, l, m and n be positive integers, where a and l are odd. Put $b = (al-1)/2$. Let $\Bbbk$ be an algebraically closed field of char 2.

Step 1. Non-ruled $\mathbb{Q}$-Fano weighted hypersurfaces.

Let $k[x_0, \ldots, x_n]$ and $k[x_0, \ldots, x_n, y]$ be the graded rings whose gradings are given by $\deg x_i = 1$ for $0 \leq i \leq m$, $\deg x_i = a$ for $m+1 \leq i \leq n$ and $\deg y = b$. We define weighted projective spaces as follows.

- $P_k = \mathbb{P}_k(\overbrace{1, \ldots, 1}^{m+1}, \overbrace{a, \ldots, a}^{n\ m}, b) := \operatorname{Proj} k[x_0, \ldots, x_n, y]$.
- $Q_k = \mathbb{P}_k(\overbrace{1, \ldots, 1}^{m+1}, \overbrace{a, \ldots, a}^{n\ m}) := \operatorname{Proj} k[x_0, \ldots, x_n]$.

For $f = f(x_0, \ldots, x_n) \in k[x_0, \ldots, x_n]_{al}$ (the degree al part), we define

- $X_f := (y^2 x_0 - f(x_0, \ldots, x_n) = 0) \subset P_k$.

Condition 3. (1) l is odd, $4 \leq n$ and $0 < m < n$.

(2) $n - m + 1 < l < 2(n - m)$.

Non-ruled weighted hypersurfaces

Theorem 4 ([5], Theorem 7.3). Assume that l, m and n satisfy Condition 3. Then, the following assertions hold for every odd integer $a > (m+1)/2$.

(1) The weighted hypersurface $X_f \subset P_{\mathbb{C}}$ of degree al defined over $\mathbb{C}$ is a non-ruled $\mathbb{Q}$-Fano n-fold with Picard number one for a very general $f \in \mathbb{C}[x_0, \ldots, x_n]_{al}$.

(2) The weighted hypersurface $X_f \subset P_{\Bbbk}$ of degree al defined over $\Bbbk$ is not separably uniruled for a general $f \in \Bbbk[x_0, \ldots, x_n]_{al}$.

Step 2. Construction of "large" birationally trivial families.

For fixed l, m and n satisfying Condition 3, let $\mathcal{X}_a \to \mathcal{S}_a$ be the family of weighted hypersurfaces $X_f \subset P_{\Bbbk}$ of degree al defined over $\Bbbk$.

We say that a family of varieties is **birationally trivial** if every two members of the family are birational.

"Large" birationally trivial families

Lemma 5. Suppose that the family of $\mathbb{Q}$-Fano n-folds defined over $\mathbb{C}$ with Picard number one is birationally bounded. Then, there exists a constant R such that, for every odd integer $a > m+1$ and a general point $s_a \in \mathcal{S}_a$, there is a closed subvariety $\mathcal{B}_a$ of $\mathcal{S}_a$ with the following properties:

(1) $\mathcal{B}_a$ parametrizes a birationally trivial family.

(2) $\mathcal{B}_a$ passes through s_a.

(3) $\dim \mathcal{S}_a - \dim \mathcal{B}_a \leq R$.

Remark 6. Suppose that l, m, n satisfy Condition 3. Let $a_i > m+1$ be an odd integer and $f_i \in \mathbb{C}[x_0, \ldots, x_n]_{a_i l}$ be a very general element for $i = 1, 2$. We can prove that if X_{f_1} and X_{f_2} are birational (over $\mathbb{C}$) then their reduction mod 2 models are also birational (over $\Bbbk$). This observation is crucial in the proof of Lemma 5.

Step 3. Bounding birationally trivial families in char 2.

Let $f \in \Bbbk[x_0, \ldots, x_n]_{al}$ be a general element. We denote by ${}_f\colon X := X_f \dashrightarrow Q_{\Bbbk}$ the restriction of the natural projection $P_{\Bbbk} \dashrightarrow Q_{\Bbbk}$.

If we are over $\Bbbk$, l, m, n satisfy Condition 3 and $a > m+1$, then there is a big line bundle $\mathcal{L} \subset \Omega_Y^{n\ 1}$ on a smooth model Y of X. By analyzing the rational map associated to $\mathcal{L}$, we obtain the following.

Birational invariance of the map ${}_f$

Lemma 7. Suppose that l, m and n satisfy $l < 2(n-m) - 1$ in addition to Condition 3. Let $a > m+1$ be an odd integer and $f \in \Bbbk[x_0, \ldots, x_n]_{al}$ a general element. Then, the map ${}_f\colon X_f \dashrightarrow Q_{\Bbbk}$ is a birational invariant.

This means that, if $g \in \Bbbk[x_0, \ldots, x_n]_{a'l}$ is also general for some $a' > m+1$ and there is a birational map $\colon X_f \dashrightarrow X_g$, then $a = a'$, is an isomorphism and there is an automorphism of $Q_{\Bbbk}$ such that the diagram

$$\begin{array}{ccc} X_f & \longrightarrow & X_g \\ {\scriptstyle \pi_f} \vdots & & \vdots {\scriptstyle \pi_g} \\ Q_{\Bbbk} & \xrightarrow[\sigma]{} & Q_{\Bbbk} \end{array}$$

commutes.

By Lemma 7, we can bound the dimension of birationally trivial subfamilies of $\mathcal{X}_a/\mathcal{S}_a$.

Bounding birationally trivial families

Lemma 8. Suppose that l, m and n satisfy $l < 2(n-m) - 1$ in addition to Condition 3. Then, for every odd integer $a > m+1$ and a general point $s_a \in \mathcal{S}_a$, there is a closed subvariety $\mathcal{W}_a$ of $\mathcal{S}_a$ with the following properties:

(1) $\mathcal{W}_a$ parametrizes the members which are birational to the member corresponds to s_a.

(2) $\dim \mathcal{S}_a - \dim \mathcal{W}_a \to \infty$ as $a \to \infty$.

If $n \geq 6$, then we can find l, m and n satisfying $l < 2(n-m) - 1$ in addition to Condition 3. Now Theorem 1 follows from Lemma 5 and 8.