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Rational Curves on Hypersurfaces

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We study the family of rational curves on hypersurfaces $/k = \bar{k}$, ch $\geqslant 0$.

Family of Rational Curves

Let $e \geqslant 1$, $d \geqslant 1$, and $n \geqslant 3$.

Def. For $X = X_d \subset \mathbb{P}^n$: hypersurf of deg d, set

$$R_e(X) := \left\{ \begin{array}{c} \text{smooth rat curves } C \text{ of deg } e \text{ in } \mathbb{P}^n \\ \text{lying in } X \end{array} \right\} \subset \mathrm{Hilb}^{et+1}(X/k).$$

Expected Dimension

For $\mu := (n+1-d)e + n - 4 = \chi(N_{C/X})$,

$$R_e(X) \neq \emptyset \Rightarrow \dim R_e(X) \geqslant \mu.$$

Starting Point (Lines, $e = 1$)

Thm A (W. Barth-A. Van de Ven '78/79 ($\mathbb{C}$), J. Kollár '96 (ch $\geqslant 0$)).

(a) $R_1(X) = \emptyset$ for gen X if $\mu < 0$.

(b) $R_1(X)$: smooth of dim μ for gen X if $\mu \geqslant 0$.

(c) $R_1(X)$: connected for any X if $\mu \geqslant 1$, except when $X = X_2 \subset \mathbb{P}^3$.

What can we say about the family $R_e(X)$ for deg $e \geqslant 2$?

Thm B (J. Harris-M. Roth-J. Starr '04 ($\mathbb{C}$)).

$e \geqslant 1$, $d < (n+1)/2 \Longrightarrow R_e(X)$: integral, loc comp int sch of dim μ for gen X.

Main Thm (Furukawa '07 (ch $\geqslant 0$)).

(a) $d \geqslant \max(e-2, 1) \Longrightarrow R_e(X) = \emptyset$ for gen X if $\mu < 0$.

(b) $\left.\begin{array}{l} 1 \leqslant e \leqslant 3,\ d \geqslant 1, \text{ or} \\ e \geqslant 4,\ d \geqslant 2e-3 \end{array}\right\} \Longrightarrow R_e(X)$: smooth of dim μ for gen X if $\mu \geqslant 0$.

(c) $R_2(X)$: connected for gen X if $\mu \geqslant 1$, except when $X = X_3 \subset \mathbb{P}^3$.

Exceptional Case of (c)

Prop. $X = X_3 \subset \mathbb{P}^3$: smooth cubic $\Longrightarrow R_2(X)$ has 27 connected comp.

$\because$ For the lines $L_1, \ldots, L_{27} \subset X$,

$$R_2(X) \hookrightarrow \bigcup L_i^* \subset (\mathbb{P}^3)^\vee$$
$$C \mapsto \text{(2-pl spanned by } C),$$

where $L_i^* := \{ H \subset \mathbb{P}^3 : \text{2-pl}, L_i \subset H \}$. □

Key of the Proof

Fix $f : \mathbb{P}^1 \xrightarrow{\sim} C \subset \mathbb{P}^n$: deg e.

$I_C^\circ := \{ X = X_d \subset \mathbb{P}^n : \text{smooth}, C \subset X \}$,

$Z_C^\circ := \{ X \in I_C^\circ \mid h^1(N_{C/X}) \neq 0 \}$.

Tools

- Castelnuovo-Mumford regularity of $\mathcal{I}_C^2$.
- Theory of catalecticant matrices.
- Bounds of a_i for $f^*N_{C/\mathbb{P}^n} \simeq \bigoplus \mathcal{O}_{\mathbb{P}^1}(a_i)$.

Prop. assumption of (b)

$$\Longrightarrow \operatorname{codim}(Z_C^\circ, I_C^\circ) \geqslant \mu + 1.$$

$\because$ $0 \to \mathcal{I}_C^2 \to \mathcal{I}_C \to N_{C/\mathbb{P}^n}^\vee \to 0$ induces

$$\begin{array}{ccc} H^0(\mathbb{P}^n, \mathcal{I}_C(d)) & \overset{\delta_C}{\twoheadrightarrow} & \mathrm{Hom}_{\mathcal{O}_{\mathbb{P}^1}}(f^*N_{C/\mathbb{P}^n}, f^*(\mathcal{O}_{\mathbb{P}^n}(d))) \\ \cup\ \flat & & \cup\ \sharp \\ \hat{Z}_C = \delta_C^{-1}(\mathsf{Z}) & \twoheadrightarrow & \mathsf{Z} := \{ \alpha \mid H^0(\alpha) : \text{not surj} \}. \end{array}$$

(Prop) $\leftsquigarrow$ codim($\flat$) $\leftsquigarrow$ codim($\sharp$). □