代数幾何学シンポジウム記録
2008年度　pp.136-136

The reflexivity of a Segre product of projective varieties

(Joint work with Hajime Kaji, [Math. Ann. **342** (2008), 279–289])
Satoru Fukasawa (Waseda University/JSPS)

Main Theorem (Fukasawa-Kaji)

For a proj var $Y \subset \mathbb{P}^N$ of dim n, with Hessian rank r / char $p \geq 0$,

The Segre product $X = \mathbb{P}^m \times Y$: non-reflexive

$$\Leftrightarrow \begin{cases} (1)\ 0 < n - m < r,\ p = 2,\ m + n : \text{odd, or} \\ (2)\ n - m > r \text{ and } Y : \text{non-reflexive.} \end{cases}$$

Segre product := **Product embedded by Segre embedding**

Main Theorem'

$$\mathbb{P}^m \times Y\text{: reflexive} \Leftrightarrow \begin{cases} \text{(i) } m \geq n, \\ \text{(ii) } 0 \leq n - m < r,\ p \neq 2, \\ \text{(iii) } 0 \leq n - m < r,\ p = 2,\ m + n : \text{even} \\ \text{(iv) } n - m = r, \text{ or} \\ \text{(v) } n - m > r \text{ and } Y : \text{reflexive.} \end{cases}$$

Y	p	r	R/N	$\mathbb{P}^m \times Y$	R/N	cond.
$\mathbb{P}^n$	≥ 0	0	R	$\mathbb{P}^m \times \mathbb{P}^n\ (m \geq n)$	R	(i)
$\mathbb{P}^1 \times \mathbb{P}^1$	$\neq 2$	2	R	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	R	(ii)
$\mathbb{P}^2 \times \mathbb{P}^2$	2	4	R	$\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$	R	(iii)
$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	2	2	N	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	R	(iv)
$\mathbb{P}^1 \times \mathbb{P}^2$	≥ 0	2	R	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^2$	R	
$\mathbb{P}^1 \times \mathbb{P}^3$	≥ 0	2	R	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^3$	R	(v)
$\mathbb{P}^1 \times \mathbb{P}^1$	2	2	R	$\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$	N	(1)
$\mathbb{P}^1 \times \mathbb{P}^2$	2	2	R	$\mathbb{P}^2 \times \mathbb{P}^1 \times \mathbb{P}^2$	N	
$F_d\ (d \equiv 1 \bmod p)$	> 0	0	N	$\mathbb{P}^m \times F_d\quad (m < n)$	N	(2)

Table: The reflexivity of Segre products of projective varieties

R:= Reflexive / N:= Non-Reflexive
F_d: Fermat hypersurf. of deg d in $\mathbb{P}^{n+1}$

Reflexivity

$$Y \text{ is reflexive} \overset{\text{def}}{\Leftrightarrow} C(Y) = C(Y^*)$$

$\mathbb{P}^{N*} := \{H \mid \text{hyperplane in } \mathbb{P}^N\}$ dual proj. space;

$Y \subset \mathbb{P}^N$;

$C(Y) := \overline{\{(y, H) \in Y_{sm} \times \mathbb{P}^{N*} \mid \mathbb{T}_y Y \subset H\}} \subset \mathbb{P}^N \times \mathbb{P}^{N*}$ conormal variety

$\downarrow \pi_Y$

$\mathbb{P}^{N*}$

$Y^* := \pi_Y(C(Y)) \subset \mathbb{P}^{N*}$; dual variety

$C(Y^*) \subset \mathbb{P}^{N**} \times \mathbb{P}^{N*} = \mathbb{P}^N \times \mathbb{P}^{N*}$

Rem. Y: reflexive $\Rightarrow Y^{**} = Y$ (Proj. Duality)

Hessian Rank r

$$r = \operatorname{rank}\left[\frac{\partial^2(h|Y)}{\partial y_i \partial y_j}\right]_{i,j}$$

$(y, H) \in C(Y)$: general pt,
h: rational function defining H,
(y_i): local coord. of Y at y.

Rem. $\operatorname{rank} d\pi_Y = N - 1 - (\dim Y - r) \leq \dim Y^*$

Monge-Segre-Wallace Criterion

Y is reflexive.

$\Leftrightarrow \pi_Y : C(Y) \to Y^*$ is generically smooth.

$\Leftrightarrow \operatorname{rank} d\pi_Y = \dim Y^*$

PROOF OF MAIN THEOREM

- Determine rank $d\pi_X$ and dim X^*.
- Use Monge-Segre-Wallace Criterion.

APPLICATION

Any Non-reflexive example of a Segre product gives a Negative Answer to Kleiman-Piene's question for Gauss maps.

Rem. The Gauss map of any Segre product is gen. smooth onto its image.