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POLARIZED ENDMORPHISMS ON NORMAL PROJECTIVE VARIETIES

DE-QI ZHANG

ABSTRACT. This is the summary of the paper [14]. We show that polarized endomorphisms of rationally connected threefolds with at worst terminal singularities are equivariantly built up from those on Q-Fano threefolds, Gorenstein log del Pezzo surfaces and \( P^1 \). Similar results are obtained for polarized endomorphisms of uniruled threefolds and fourfolds. As a consequence, we show conceptually that every smooth Fano threefold with a polarized endomorphism of degree \( > 1 \), is rational.

1. INTRODUCTION

We work over the field \( \mathbb{C} \) of complex numbers. We study polarized endomorphisms \( f : X \to X \) of varieties \( X \), i.e., those \( f \) with \( f^*H \sim qH \) for some \( q > 0 \) and some ample line bundle \( H \). Every surjective endomorphism of a projective variety of Picard number one, is polarized. If \( f = [F_0 : F_1 : \cdots : F_n] : \mathbb{P}^n \to \mathbb{P}^n \) is a surjective morphism and \( X \subset \mathbb{P}^n \) a \( f \)-stable subvariety, then \( f^*H \sim qH \) and hence \( f|_X : X \to X \) is polarized; here \( H \subset X \) is a hyperplane and \( q = \deg(F_i) \). If \( A \) is an abelian variety and \( m_A : A \to A \) the multiplication map by an integer \( m \neq 0 \), then \( m_A^*H \sim m^2H \) and hence \( m_A \) is polarized; here \( H = L + (-1)^*L \) with \( L \) an ample divisor, or \( H \) is any ample divisor with \( (-1)^*H \sim H \). One can also construct polarized endomorphisms on quotients of \( \mathbb{P}^n \) or \( A \). So there are many examples of polarized endomorphisms \( f \). See [16] for the many conjectures on such \( f \).

In [11], it is proved that a normal variety \( X \) with a non-isomorphic polarized endomorphism \( f \) either has only canonical singularities with \( K_X \sim Q 0 \) (and further is a quotient of an abelian variety when \( \dim X \leq 3 \)), or is uniruled so that \( f \) descends to a polarized endomorphism \( f_Y \) of the non-uniruled base variety \( Y \) (so \( K_Y \sim Q 0 \)) of a specially chosen maximal rationally connected fibration \( X \to Y \). By the induction on dimension and since \( Y \) has a dense set of \( f_Y \)-periodic points \( y_0, y_1, \ldots \) (cf. [2, Theorem 5.1]), the study of polarized endomorphisms is then reduced to that of rationally connected varieties \( \Gamma_{y_0} \) as fibres of the graph \( \Gamma = \Gamma(X/Y) \) (cf. [11, Remark 4.3]).

The study of non-isomorphic endomorphisms of singular varieties (like \( \Gamma_{y_0} \) above) is very important from the dynamics point of view, but is very hard
even in dimension two and especially for rational surfaces; see [9] (about 150 pages).

We consider polarized endomorphisms of rationally connected varieties (or more generally of uniruled varieties) of dimension \( \geq 3 \). Theorem 1.1 - 1.8 below are our main results.

**Theorem 1.1.** Let \( X \) be a \( \mathbb{Q} \)-factorial threefold having only terminal singularities and a polarized endomorphism of degree \( q^3 > 1 \). Suppose that \( X \) is rationally connected. Then we have:

1. There is an \( s > 0 \) such that \( (f^*)^s|_{N^1(X)} = q^s \text{id.} \)
2. Either \( X \) is rational, or \(-K_X\) is big.
3. There are only finitely many irreducible divisors \( M_i \subset X \) with the Iitaka D-dimension \( \kappa(X, M_i) = 0 \).

Theorem 1.1 (3) apparently does not hold on an abelian variety \( A \) with a subtorus of codimension one, though the multiplication map \( m_A \) is polarized as mentioned above. Neither it holds for \( X = S \sim \mathbb{P}^1 \), where \( S \) is a rational surface with infinitely many \((-1\text{-})\text{curves} \) (the blowup of nine general points of \( \mathbb{P}^2 \) is such \( S \) as observed by Nagata).

Theorem 1.1 (1) above strengthens (in our situation) Serre's result [12] on a conjecture of Weil (in the projective \( \mathbb{C} \)-case): (Serre) If \( f \) is a polarized endomorphism of degree \( q^{\dim X} > 1 \) of a smooth variety \( X \) then every eigenvalue of \( f^*|N^1(X) \) has the same modulus \( q \).

The proof of Theorem 1.2 below is conceptually done. In a recent paper [15], we have removed the polarizedness assumption in Theorem 1.2.

**Theorem 1.2.** Let \( X \) be a smooth Fano threefold with a polarized endomorphism of degree \( > 1 \). Then \( X \) is rational.

A klt \( \mathbb{Q} \)-Fano variety has only finitely many extremal rays. A similar phenomenon occurs in the quasi-polarized case.

**Theorem 1.3.** Let \( X \) be a \( \mathbb{Q} \)-factorial rationally connected threefold having only Gorenstein terminal singularities and a quasi-polarized endomorphism of degree \( > 1 \). Then \( X \) has only finitely many \( K_X \)-negative extremal rays.

We expect a possible application of Theorem 1.4 below (see Theorem 1.7 for a more detailed version) to the Dynamic Manin-Mumford conjecture for \( (X, f) \) formulated by S.-W. Zhang in [16, Conjecture 1.2.1]. This conjecture for \( (X, f) \) is essentially equivalent to that for \( (X_r, g_r) \) because \( f^{-1} \), as seen in Theorem 1.7, preserves the maximal subset of \( X \) where the birational map \( X \rightarrow X_r \) is not holomorphic.

Further, \( X_r \) is better to be dealt with because it has a fibration structure preserved by \( g_r \). The existence of such a fibration \( \pi : X_r \rightarrow Y \) is guaranteed when \( X \) is uniruled by the recent development in MMP.

**Theorem 1.4.** Let \( X \) be a \( \mathbb{Q} \)-factorial \( n \)-fold, with \( n \in \{3, 4\} \), having only log terminal singularities and a polarized endomorphism \( f \) of degree \( q^n > 1 \).
Let $X = X_0 \to X_1 \to \cdots \to X_r$ be a composition of divisorial contractions and flips. Replacing $f$ by its positive power, we have:

1. **The dominant rational maps** $g_i : X_i \to X_i$ ($0 \leq i \leq r$) (with $g_0 = f$) **induced from $f$**, are all holomorphic.

2. **Let $\pi : X_r \to Y$ be an extremal contraction with $\dim Y \leq 2$. Then $g_r$ is polarized and it descends to a polarized endomorphism $h : Y \to Y$ of degree $q^{\dim Y}$ with $\pi \circ g_r = h \circ \pi$.

The claim in the abstract about the building blocks of polarized endomorphisms, is justified by the remark below.

**Remark 1.5.**

1. The $Y$ in Theorem 1.4 is $\mathbb{Q}$-factorial and has at worst log terminal singularities.

2. Suppose that the $X$ in Theorem 1.4 is rationally connected. Then $Y$ is also rationally connected. Suppose further that $X$ has at worst terminal singularities and $(\dim X, \dim Y) = (3, 2)$. Then $Y$ has at worst Du Val singularities by [8, Theorem 1.2.7]. So there is a composition $Y \to Y$ of divisorial contractions and an extremal contraction $\tilde{Y} \to B$ such that either $\dim B = 0$ and $\tilde{Y}$ is a Du Val del Pezzo surface of Picard number 1, or $\dim B = 1$ and $Y \to B \cong \mathbb{P}^1$ is a $\mathbb{P}^1$-fibration with all fibres irreducible. After replacing $f$ by its power, $h$ descends to polarized endomorphisms $\hat{h} : \tilde{Y} \to \tilde{Y}$, and $k : B \to B$ (of degree $q^{\dim B}$); see Theorems 1.6.

3. By [2, Theorem 5.1], there are dense subsets $Y_0 \subset Y$ (for the $Y$ in Theorem 1.4) and $B_0 \subset B$ (when $\dim B = 1$) such that for every $y \in Y_0$ (resp. $b \in B_0$) and for some $r(y) > 0$ (resp. $r(b) > 0$), $g^{r(y)}|W_y$ (resp. $\hat{h}^{r(b)}|Y_{\tilde{b}}$) is a well-defined polarized endomorphism of the Fano fibre.

We remark that Noboru Nakayama has produced many examples of polarized $f$ on abelian surfaces which are not scalar. The result below shows that this happens only on abelian surfaces and their quotients.

**Theorem 1.6.** Let $X$ be a normal projective surface. Suppose that $f : X \to X$ is an endomorphism such that $f^*P \equiv qP$ for some $q > 1$ and some big Weil $\mathbb{Q}$-divisor $P$. Then we have:

1. $f$ is polarized of degree $q^n$.

2. There is an $s > 0$ such that $(f^s)^*|\text{Weil}(X) = q^s \text{id}$ unless $X$ is $\mathbb{Q}$-abelian with $\text{rank Weil}(X) \in \{3, 4\}$.

More generally, we prove the two theorems below. Theorem 1.7 below includes Theorem 1.4 as a special case.

**Theorem 1.7.** Let $X$ be a $\mathbb{Q}$-factorial $n$-fold, with $n \in \{3, 4\}$, having only log terminal singularities and a polarized endomorphism $f$ of degree $q^n > 1$. Let $X = X_0 \to X_1 \to \cdots \to X_r$ be a composition of divisorial contractions and flips. Replacing $f$ by its positive power, (I) and (II) hold:

1. **The dominant rational maps** $g_i : X_i \to X_i$ ($0 \leq i \leq r$) (with $g_0 = f$) **induced from $f$**, are all holomorphic. Further, $g_i^{-1}$ preserves
each irreducible component of the exceptional locus of $X_i \to X_{i+1}$ (when it is divisorial) or of the flipping contraction $X_i \to Z_i$ (when $X_i \to X_{i+1} = X_i^+$ is a flip).

(II) Let $\pi : W = X_r \to Y$ be the contraction of a $K_W$-negative extremal ray $\mathbb{R}_{\geq 0}[C]$, with $\dim Y \leq n-1$. Then $g := g_r$ descends to a surjective endomorphism $h : Y \to Y$ of degree $q^{\dim Y}$ such that

$$\pi \circ g = h \circ \pi.$$ 

For all $0 \leq i \leq r$, all eigenvalues of $g_i^*|N^1(X_i)$ and $h^*|N^1(Y)$ are of modulus $q$; there are big line bundles $H_{X_i}$ and $H_Y$ satisfying

$$g_i^*H_{X_i} \sim qH_{X_i}, \quad h^*H_Y \sim qH_Y.$$ 

Suppose further that either $\dim Y \leq 2$ or $\rho(Y) = 1$. Then $H_W$ and $H_Y$ can be chosen to be ample and $g$ and $h$ are polarized.

The contraction $\pi$ below exists by the MMP for threefolds.

**Theorem 1.8.** Let $X$ be a $\mathbb{Q}$-factorial rationally connected threefold having at worst terminal singularities and a polarized endomorphism of degree $> 1$. Let $X \to W$ be a composition of divisorial contractions and flips, and $\pi : W \to Y$ an extremal contraction of non-birational type. Suppose either $\dim Y \geq 1$, or $\dim Y = 0$ and $W$ is smooth. Then $X$ is rational.

**The difficulty 1.9.** In Theorem 1.4, if $X \to X_1$ is a divisorial contraction, one can descend a polarized endomorphism $f$ on $X$ to an one on $X_1$, but the latter may not be polarized any more because the pushforward of a nef divisor may not be nef in dimension $\geq 3$ (the first difficulty). If $X \to X_1$ is a flip, then in order to descend $f$ on $X$ to some holomorphic $f_1$ on $X_1$, one has to show that a power of $f$ preserves the centre of the flipping contraction (the second difficulty). The second difficulty is taken care by a key lemma where the polarizedness is essentially used.

The question below is the generalization of Theorem 1.2 and the famous conjecture: every smooth Fano $n$-fold of Picard number one with a non-isomorphic surjective endomorphism, is $\mathbb{P}^n$ (for its affirmative solution when $n = 3$, see Amerik-Rovinsky-Van de Ven [1] and Hwang-Mok [4]).

**Question 1.10.** Let $X$ be a smooth Fano $n$-fold with a non-isomorphic polarized endomorphism. Is $X$ rational?

For the recent development on endomorphisms of algebraic varieties, we refer to Amerik-Rovinsky-Van de Ven [1], Fujimoto-Nakayama [3], Hwang-Mok [4], S.-W. Zhang [16], as well as [10], [13].

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