<table>
<thead>
<tr>
<th>Title</th>
<th>Mathematical and Numerical Approaches for Transport Phenomena in Surface Water Networks (Dissertation_全文)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Author(s)</td>
<td>Yoshioka, Hidekazu</td>
</tr>
<tr>
<td>Citation</td>
<td>Kyoto University (京都大学)</td>
</tr>
<tr>
<td>Issue Date</td>
<td>2016-03-23</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.14989/doctor.r13021">https://doi.org/10.14989/doctor.r13021</a></td>
</tr>
<tr>
<td>Type</td>
<td>Thesis or Dissertation</td>
</tr>
<tr>
<td>Textversion</td>
<td>ETD</td>
</tr>
</tbody>
</table>
Mathematical and Numerical Approaches
for Transport Phenomena
in Surface Water Networks

2016

Hidekazu Yoshioka
Contents

List of Figures .................................................................................................................. 7
List of Tables .................................................................................................................... 10
CHAPTER 1  Introduction .................................................................................................... 12
  1.1 Research Background ............................................................................................... 12
  1.2 Research Issues ......................................................................................................... 14
    1.2.1 Issues on Modeling Shallow Water Flows ......................................................... 14
    1.2.2 Issues on Modeling Solute Transport Phenomena ............................................ 15
  1.3 Research Objectives .................................................................................................. 16
  1.4 Structure of This Thesis ............................................................................................ 16
CHAPTER 2  Literature Review .......................................................................................... 18
  2.1 Introduction ............................................................................................................... 18
  2.2 Literature Review on Mathematical Models of Shallow Water Flows ..................... 18
  2.3 Literature Review on Mathematical Models of Solute Transport Phenomena .......... 20
  2.4 Literature Review on Numerical Methods for Shallow Water Flows ..................... 21
  2.5 Literature Review on Numerical Methods for Solute Transport Phenomena .......... 23
CHAPTER 3  New 1-D Shallow Water Equations for Open Channel Network Flows ....... 25
  3.1 Introduction ............................................................................................................... 25
  3.2 Conventional 1-D Shallow Water Equations ............................................................ 27
  3.3 Extended Continuity Equation .................................................................................. 28
  3.4 Momentum Flux Evaluation Schemes ....................................................................... 28
    3.4.1 Conventional Schemes ....................................................................................... 30
      3.4.1.1 M1 Scheme .................................................................................................... 30
      3.4.1.2 M2 Scheme .................................................................................................... 30
    3.4.2 New Scheme: M3 Scheme .................................................................................. 31
    3.4.3 Mathematical Analysis on the Three Schemes .................................................... 31
  3.5 Conclusions .............................................................................................................. 33
CHAPTER 4  Stochastic Process Models for Solute Transport Phenomena in Surface Water Bodies ................................................................................................................. 34

4.1 Introduction ........................................................................................................ 34

4.2 Stochastic Process Model .................................................................................. 35

4.2.1 Stochastic Differential Equation ........................................................................... 35

4.2.2 Kolmogorov’s Equations ................................................................................ 36

4.3 Transport Equation .................................................................................................. 37

4.4 Statistics Equations .............................................................................................. 40

4.4.1 Statistical Moments of Residence Time ................................................................. 40

4.4.2 Killed Probability .................................................................................................. 41

4.5 Shallow Water Counterparts ................................................................................... 43

4.5.1 1-D Model ........................................................................................................... 43

4.5.2 Horizontally 2-D Model ..................................................................................... 45

4.6 Validation and Parameter Estimation ....................................................................... 46

4.6.1 Local Velocity ..................................................................................................... 46

4.6.2 Model Validation .................................................................................................. 48

4.6.2.1 Flow Velocity Measurement ........................................................................... 48

4.6.2.2 Kolmogorov-Smirnov (KS) Test ..................................................................... 51

4.6.2.3 Turbulent Diffusion Coefficients .................................................................... 52

4.7 Conclusions ............................................................................................................ 53

CHAPTER 5  Numerical Scheme for the Extended 1-D Shallow Water Equations ............ 54

5.1 Introduction ............................................................................................................. 54

5.2 Extended 1-D Shallow Water Equations ................................................................. 56

5.3 Dual Finite Volume (DFVF) Scheme ...................................................................... 57

5.3.1 Computational Mesh .......................................................................................... 57

5.3.2 Extended Continuity Equation .......................................................................... 59

5.3.3 Momentum Equation ......................................................................................... 59

5.3.4 Temporal Integration .......................................................................................... 62

5.4 Applications to Test Cases ..................................................................................... 62
5.4.1 Tidal Wave Problem in an Irregular Channel..............................................63
5.4.2 Dam Break Problems in a Flat, Frictionless, and Rectangular Channel ..........64
5.4.3 Dam Break Problem in a Flat, Frictionless, and Triangular Channel ..............67
5.4.4 Thacker’s Test Case ..................................................................................69
5.4.5 Dam Break Problem in a Non-prismatic Rectangular Channel .......................71
5.4.6 Dam Break Problem in a Non-flat Rectangular Channel ................................73
5.4.7 Dam Break Problems in a Non-prismatic and Non-flat Channel ....................74
5.4.8 Steady Flows in a Bifurcating Channel Network .........................................79
  5.4.8.1 Computational Conditions ..................................................................79
  5.4.8.2 Discharge Ratios ................................................................................80
  5.4.8.3 Water Surface Profiles .......................................................................82
5.4.9 Dam Break Problem in a Multiply Connected Open Channel Network with a Rectangular Cross-section ........................................................................85
5.4.10 Dam Break Problem in a Multiply Connected Open Channel Network with a Triangular Cross-section ........................................................................86
5.5 Application to Dam Break Flash Flood ..........................................................88
  5.5.1 Background .........................................................................................88
  5.5.2 Computational Cases ..............................................................................91
  5.5.3 Computational Mesh .............................................................................91
  5.5.4 Model Parameters ................................................................................94
  5.5.5 Initial and Boundary Conditions ...............................................................94
  5.5.6 Computational Results ..........................................................................95
5.6 Conclusions .................................................................................................100

CHAPTER 6 Numerical Scheme for Conservative Parabolic Partial Differential Equation
and Its Application to Kolmogorov’s Forward Equations ......................................101
6.1 Introduction ................................................................................................101
6.2 Conservative Advection-Dispersion Equation ..............................................103
6.3 Dual-Finite Volume for Solute (DFVS) Scheme ...........................................105
  6.3.1 Computational Meshes ........................................................................105

4
CHAPTER 7 Mathematical Analysis on a Numerical Scheme for Non-conservative Parabolic Partial Differential Equation

7.1 Introduction .............................................................................................................. 120
7.2 Nonconservative Advection-Dispersion Equation .................................................. 122
7.3 Conforming Petrov-Galerkin Finite Element (CPGFE) Scheme ......................... 123
  7.3.1 Computational Mesh ......................................................................................... 123
  7.3.2 Spatial Discretization ..................................................................................... 124
  7.3.3 Temporal Discretization ................................................................................. 125
7.4 Numerical diffusivity and decay of the CPGFE Scheme ..................................... 126
7.5 Stability Analysis on the CPGFE Scheme .......................................................... 128
  7.5.1 Steady Case ...................................................................................................... 128
  7.5.2 Unsteady Case .................................................................................................. 130
7.6 Error Analysis on the Conforming Petrov-Galerkin Finite Element Scheme ...... 131
  7.6.1 Discrete Green’s function .............................................................................. 131
    7.6.1.1 Dirichlet-Dirichlet Case .................................................................... 133
    7.6.1.2 Neumann-Dirichlet Case ................................................................. 134
  7.6.2 Analysis on Steady Case .................................................................................. 134
    7.6.2.1 Pointwise Error Estimate .................................................................... 134
    7.6.2.2 Global Error Estimate ........................................................................... 136
7.7 Conclusions ............................................................................................................. 136

CHAPTER 8 Summary and Conclusions ...................................................................... 138
8.1 Summary..............................................................................................................................................138
8.2 Future perspectives...............................................................................................................................140
APPENDIX A................................................................................................................................................142
APPENDIX B................................................................................................................................................144
Acknowledgements......................................................................................................................................146
References ....................................................................................................................................................147
List of Figures

Figure 3-1 A sketch of $B(y,r)$.

Figure 3-2 A sketch of inflow and outflow reaches meeting at a junction.

Figure 4-1 Sketch of the parameters $h$ and $d$ defining the location of a station in the cross-section of canal.

Figure 4-2 Locations of the four observation stations in U-canal.

Figure 4-3 Locations of the four observation stations in K-canal.

Figure 4-4 The observed time series of the local velocity in U-canal (Station U-1).

Figure 5-1 Sketch of $B(y,r)$ for defining Eq.(79).

Figure 5-2 Sketch of the computational mesh: (a) regular mesh, and (b) dual mesh.

Figure 5-3 Comparison of the analytical and computed discharges per unit width solving the tidal wave problem.

Figure 5-4 Comparison of the exact and computed water surface profiles of the dam break problems in a straight rectangular channel (Case DB-A).

Figure 5-5 Comparison of the exact and computed velocity distributions of the dam break problems in a straight rectangular channel (Case DB-A).

Figure 5-6 Comparison of the exact and computed water surface profiles of the dam break problems in a straight rectangular channel (Case DB-B).

Figure 5-7 Comparison of the exact and computed velocity distributions of the dam break problems in a straight rectangular channel (Case DB-B).

Figure 5-8 Comparison of the exact and computed water surface profiles of the dam break problem in a straight triangular channel.

Figure 5-9 Comparison of the exact and computed velocity distributions of the dam break problem in a straight triangular channel.

Figure 5-10 Comparisons of the exact and computed water surface profiles for the Thacker’s test case at selected time steps.

Figure 5-11 Comparisons of the exact and computed discharges for the Thacker’s test case at selected time steps.

Figure 5-12 Sketch of the dam break problem in a non-prismatic experimental channel.
Figure 5-13 Comparisons of the computed and measured water depths at each observation point for the dam break problem in a non-prismatic experimental channel.

Figure 5-14 Sketch of the dam break problem with a triangular bump.

Figure 5-15 Comparisons of the computed and measured water depths at each observation point for the dam break problem with a triangular bump.

Figure 5-16 Sketch of the experimental channel for the dam break problems in a non-prismatic and non-flat channel.

Figure 5-17 Comparisons of the computed, emulated, and measured water depths at each station with the slope $S = 0.00$.

Figure 5-18 Comparisons of the computed, emulated, and measured water depths at each station with the slope $S = 0.01$.

Figure 5-19 Sketch of the open channel network for the hydraulic experiments.

Figure 5-20 Computed water surface profiles for the cases A-1 through A-3.

Figure 5-21 Computed water surface profiles for the cases B-1 through B-3.

Figure 5-22 Computed water surface profiles for the cases C-1 through C-3.

Figure 5-23 Computed water surface profiles for the cases D-1 through D-3.

Figure 5-24 Computed water surface profiles for the cases E-1 through E-3.

Figure 5-25 Computed water surface profiles for the cases F-1 through F-3.

Figure 5-26 Sketch of the hypothetical multiply connected channel network with key nodes.

Figure 5-27 Comparisons of the coarse and fine water surface profiles in reaches A-B-C-D-E and B-F-D at each time step for the dam break problem in a multiply connected channel network with a rectangular cross-section.

Figure 5-28 Comparisons of the coarse and fine water surface profiles in reaches A-B-C-D-E and B-F-D at each time step for the dam break problem in a multiply connected channel network with a triangular cross-section.

Figure 5-29 Map of the study area.

Figure 5-30 Sketch of Fujinuma Dam.

Figure 5-31 Schematic diagram of the computational domain with key nodes for the flood caused by the Fujinuma Dam failure.
Figure 5-32 Schematic diagram of each channel cross-section in the study area.

Figure 5-33 Hydrographs at nodes D, F, and H for each case.

Figure 5-34 Water surface profiles in the channel network at each time step for Case FDB-C.

Figure 5-35 Inundation time distribution in the channel network for each case.

Figure 5-36 Inundation map for each case.

Figure 6-1 Sketches of the computational mesh: (a) regular mesh, and (b) dual mesh. (same with Figure 5-2)

Figure 6-2 Exact and numerical solutions for Test (a) with \( D_h = 0.5 \).

Figure 6-3 Exact and numerical solutions for Test (a) with \( D_h = 0.001 \).

Figure 6-4 Exact and numerical solutions for Test (b).

Figure 6-5 Sketch of the channel network consisting of six reaches.

Figure 6-6 Channel bed elevation and water surface profiles along the reaches A-B-C-D-E and along the reaches B-F-D.

Figure 6-7 Time series of computed solute concentration at the node C and at the node F.

Figure 6-8 Computed solute concentration distributed over the channel network at \( t = 360 \) (s).

Figure 6-9 Computed solute concentration distributed over the channel network at \( t = 1,800 \) (s).

Figure 6-10 Computed solute concentration distributed over the channel network at \( t = 3,600 \) (s).

Figure 6-11 Computed distributions of the total deposition along the reaches A-B-C-D-E and along the reaches B-F-D.

Figure 7-1 The numerical diffusivity \( e_h \) as a function of the non-dimensional variable \( \text{Pe} \).

Figure 7-2 The numerical decay coefficient \( e_h \) as a function of the non-dimensional variables \( \text{Pe} \) and \( \text{Da} \).

Figure 7-3 Plots of the minimum values of the parameter \( \theta \) for guaranteeing temporal stability of the CPGFE scheme with and without lumping the mass matrix.
List of Tables

Table 4-1 Physical parameters of the canals.
Table 4-2 Cross-sectional locations of the stations.
Table 4-3 Observation parameters.
Table 4-4 Identified deterministic velocities $\bar{V} \text{ (m/s)}$.
Table 4-5 Results of the Kolmogorov-Smirnov test.
Table 4-6 Estimated value of each entry of the dispersivity $D \text{ (m}^2/\text{s})$.
Table 5-1 Experimental conditions for the flows in the open channel network.
Table 5-2 Comparisons of the experimental and computational discharge ratios for the cases A-1 through F-3.
Table 5-3 Dimensions of the main and auxiliary embankments of Fujinuma Reservoir.
Table 6-1 Computational conditions specified for Tests (a) and (b).
CHAPTER 1  Introduction

1.1 Research Background
Transport phenomena, such as dispersion of solute particles and migration of individuals of aquatic species, are ubiquitous in surface water bodies. Assessing transport phenomena in surface water bodies are major research topics in hydraulics, hydrology, and related applied research areas, such as environmental and agricultural engineering and ecological engineering, because of their close links with a wide variety of real problems. Water flows and pollutant dispersion in agricultural drainage systems have been analyzed with the emphasis on their impacts on water quality of the downstream receiving water bodies that can be easily polluted (Chono et al., 2009; Chono et al., 2012). Water flows in urban drainage systems during severe rainfall events have been studied for developing infrastructure systems with higher flood mitigation ability (Kim et al., 2015; Yu and Coulthard, 2015). Water flows and sediment transport phenomena in river networks have been major concern in the floodplain areas, such as some parts of southeastern Asian countries because their lives crucially depend on the hydraulic and hydrological environments that significantly, and in some cases suddenly, vary in both space and time (Arias et al., 2014; Lu et al., 2015). Water flows and water quality dynamics in river systems have been studied in detail for preserving or improving their associated ecological systems that crucially depend on the hydro-environments (Liu et al., 2015; Tsuzuki, 2015).

Most of the water flows occurring in surface water bodies, such as rivers, drainage canals, lakes, and reservoirs, typically present turbulent natures where local hydraulic variables stochastically fluctuate in both space and time (Hoffman and Johnson, 2007; Rebollo and Lawandowski, 2014). Transport phenomena of solute and aquatic species occurring in surface water flows are subject to these fluid dynamical fluctuations; the phenomena are therefore considered to be essentially stochastic. Transport phenomena in surface water bodies have been analyzed with mathematical models, in which the underlying physical, chemical, and biological mechanisms are effectively taken into account. Some of such mathematical models explicitly consider the fluid dynamical fluctuations due to turbulence, utilizing closure relationships in order to mathematically well-pose the problems with physical variables (Rebollo and Lawandowski, 2014).
There exist quite a large number of mathematical models for assessing transport phenomena in surface water bodies, which have different spatial dimensionalities (3-D, 2-D, 1-D, and 0-D) with different physical assumptions, such as hydrostatic or non-hydrostatic pressure distribution and static or dynamic water surface profiles. Different mathematical models would have different advantages and disadvantages, and therefore have different purposes as reviewed in Miller et al. (2013). For simulating transport phenomena in open channel networks as one of the most important components of surface water systems in particular, 1-D mathematical models based on the hydrostatic pressure assumption serve as efficient and practical tools where the turbulent natures of the flows are effectively taken into account in their formulation (Szymkiewicz, 2010).

Water flows along open channels with the hydrostatic pressure assumption can be described with the 1-D shallow water equations (SWEs) (Szymkiewicz, 2010). The 1-D SWEs consist of the continuity equation and the momentum equation, which govern spatio-temporal dynamics of mass and momentum along the channels, respectively. The 1-D SWEs can be theoretically applied to hydrostatic water flows having non-rectangular and non-prismatic cross-sectional shapes (Liu et al., 2015; Navas-Montilla and Murillo, 2015). The reduced shallow water models analytically derived from the 1-D SWEs, such as the diffusion wave models (Blandford and Ormsbee, 1993; Moussa and Bocquillon, 2009; Gasiorowski, 2014a-b; Yoshioka et al., 2015), local inertial models (Bates et al., 2010; Yamazaki et al., 2013), and kinematic wave models (Moramarco and Singh, 2000; Reddy et al., 2011), have also been used in practical analysis as the simpler and convenient alternatives. Numerical methods for effectively approximating their solutions have been proposed and still many researchers are working with development of the methods with higher computational accuracy and efficiency. A problem common to the above-mentioned shallow water models are inability to physically-consistently handle mass and momentum transport phenomena around junctions and bends, which are essential components of channel networks. Modeling water flows at and around these points with multi-dimensional hydrodynamic models does not encounter serious technical problems. However, such models are in general computationally far more complicated and demanding than the 1-D shallow water models because of their larger degree of freedoms.
Deriving physically-consistent and easily implementable mathematical and numerical models of water flows around junctions and bends, which are referred to as the internal boundary conditions (IBCs) in this thesis, can be a key to successful shallow water modelling of the flows in open channel networks.

Transport phenomena of solute and aquatic species in surface water bodies, assuming that the latter are regarded as non-passive solute particles, have been analyzed with the governing equations deduced from the mass conservation laws and constitutive equations (Goodwin et al., 2006; Baetens et al., 2013). In the conventional deterministic mathematical models of the solute transport phenomena, one of the most crucial issues is to specify functional forms of the fluxes (Chatwin and Allen, 1985). Most of the conventional mathematical models assume the Fick’s type laws as the constitutive equations to well-pose the problems, which state that solute particles are passively transported due to the advection of the local fluid velocity and the turbulent diffusion driven by the spatial gradient of solute concentration profiles. These constitutive equations are derived considering a physical analogy between turbulent diffusion and molecular diffusion, the latter has already been extensively validated, but the former would not have been. This problem becomes more serious when dealing with the advection-dispersion phenomena of solute in surface water bodies, which are macroscopic counterparts of the turbulent diffusion phenomena driven by both molecular diffusion and resulting macroscopic fluid dynamics. Considering the stochastic nature of transport phenomena in surface water bodies, stochastic process models would potentially serve as more physically consistent and effective analytical tools for their analysis.

1.2 Research Issues

Current research issues on mathematical and numerical modeling of transport phenomena in open channel networks are presented in this section considering the above-mentioned research backgrounds.

1.2.1 Issues on Modeling Shallow Water Flows

Main research issues on mathematical and numerical modeling of water flows in open channel
networks can be listed up as follows.

- Mathematical and numerical modeling of water flows around junctions and bends has not been well studied in the context of the 1-D shallow water models. In addition, only a few numbers of researches have been addressed on physically-consistent treatment of junctions and bends considering both mass and momentum transport phenomena around these points.

- Many accurate numerical methods for solving the 1-D SWEs in uniformly rectangular open channels are available; however, far less number of methods have been proposed for the flows in non-rectangular and non-prismatic open channels and open channel networks, despite they are ubiquitous in real applications.

1.2.2 Issues on Modeling Solute Transport Phenomena

Main research issues on mathematical and numerical modeling of solute transport phenomena in open channel networks can be listed up as follows.

- Mathematical modeling of solute transport phenomena considering turbulent fluctuations of fluid flows has not been carried out from the viewpoint of stochastic process models that would potentially serve as more physically-consistent and effective alternatives to currently widely used deterministic models based on the Fick’s type laws.

- In principle, the conventional deterministic models cannot assess stochasticity involved in the solute transport phenomena, which on the other hand can be consistently dealt with using stochastic process models. However, only a few numbers of researches have been conducted mathematical modeling of the phenomena.

- There exist no or at most only a few numerical models for simulating solute transport phenomena in open channel networks that can physically- and mathematically-
junctions in a consistent manner where significant exchanges of the mass of solute among
the connected reaches would occur.

1.3 Research Objectives
The objective of this thesis is to provide a mathematical and numerical modeling framework
for assessing transport phenomena in surface water bodies, anticipating their implementation
to the problems in open channel networks. This thesis focuses in particular on fundamental
aspects of the modeling framework, which are derivation of governing equations of the phe-
nomena and development of numerical methods for effectively approximating their solutions.
In order to achieve the research objective, this thesis

1) proposes a shallow water model that can physically and mathematically deal with junctions
   and bends in a consistent manner,

2) proposes a mathematical model based on stochastic processes that can consistently describe
   the advection-dispersion phenomena, and

3) presents numerical methods for simulating transport phenomena based on the
   above-mentioned mathematical models.

1.4 Structure of This Thesis
This thesis consists of eight chapters. Chapter 1 presents the background and objective of this
thesis. Chapter 2 provides literature reviews on mathematical and numerical modelling of
transport phenomena in surface water bodies. Chapter 3 presents a new 1-D shallow water
model for describing water flows in open channel networks. Chapter 4 proposes a new sto-
chastic process model governing transport phenomena in turbulent flows and discusses qual-
itative differences between the proposed and conventional models. The transport equation and
the statistics equations of solute particles are derived in this chapter. Chapter 5 presents and
validates a numerical method for effectively approximating solutions to the new shallow water
model presented in Chapter 3. Chapter 6 presents a numerical method for approximating solutions to the transport equations derived in Chapter 5. Chapter 7 presents a numerical method for approximating solutions to the statistics equations derived in Chapter 5. Finally, Chapter 8 provides summary and future perspectives of this research. Note that characters (parameters and variables) are defined in each Chapter and some characters have different meanings in different Chapters.
CHAPTER 2  Literature Review

2.1 Introduction
This chapter provides literature reviews on mathematical and numerical modelling of water flows and solute transport phenomena in surface water bodies, focusing in particular on the problems with shallow water approximations.

2.2 Literature Review on Mathematical Models of Shallow Water Flows
Surface water flows with relatively small vertical mass and momentum transport as encountered in rivers, canals, and shallow lakes, can be reasonably described with the shallow water equations, which are referred to as the SWEs (Szymkiewicz, 2010). The SWEs are nonlinear hyperbolic systems of partial differential equations (PDEs) having source terms that govern mass and momentum transport phenomena of fluids in surface water bodies where the assumptions of the incompressibility and hydrostatic pressure distribution are valid. The SWEs are broadly categorized into the two types based on the spatial averaging procedures employed: the 1-D SWEs and the horizontally 2-D SWEs. The former are effective for simulating the flows with dominant flow directions, such as water flows along rivers and canals (Helmoio, 2005; Kourgialas and Karatzas, 2014; Liu et al., 2015). On the other hand, the latter are necessary for resolving the flows with horizontally 2-D structures that the 1-D SWEs cannot handle, such as wind-induced circulations in lakes and reservoirs, inundation processes in watersheds, and oblique hydraulic jumps created around hydraulic structures (Zhou and Liu, 2013; Hai et al., 2008; Chertock et al., 2015). Because the focus in this thesis is mathematical and numerical modeling of water flows in open channel networks, hereafter the issues on 1-D SWEs are exclusively discussed.

The 1-D SWEs have been used in a wide range of applications. Miller and Chaudhry (1989) performed a hydraulic experiment of a dam break flow with a curved open channel and validated the 1-D SWEs against the experimental water flow utilizing a simple algebraic compensation technique for estimating the lateral water surface gradient. Haahti et al. (2014) simulated unsteady shallow water flows and numerically assessed erosion risks in a ditch network having a non-rectangular cross-sectional shape in a peat land utilizing the 1-D SWEs. Su
et al. (2001) examined validity of the 1-D SWEs on mathematical modelling of tidal bores in an existing bay. Islam et al. (2008) applied the 1-D SWEs to numerical simulation of water flows and advection-dispersion phenomena of contaminant in drainage systems. Valiani and Begnudelli (2006) applied the 1-D SWEs to numerical modeling of dam break flash floods. Hill and Souza (2006) and Wahid et al. (2007) used the 1-D SWEs in numerical simulation of tidal flows in estuaries. Morphological dynamics of river channels have been analyzed with the 1-D and 2-D SWEs coupled with the Exner-type differential equations that govern longitudinal sediment particles transport phenomena (Minatti, 2015; Volp et al., 2015).

1-D shallow water modeling of the flows in open channel networks has been discussed with particular emphasis on the treatment of junctions and bends. Such mathematical modeling typically deals with the 1-D SWEs defined on the locally 1-D open channel networks, which are defined as connected graphs consisting of 1-D reaches connected via 0-D junctions and bends (Yoshioka et al., 2012a). These points mathematically serve as singular points where the 1-D SWEs cannot be directly defined in the conventional sense because of possible discontinuity of dependent variables, which requires the use of internal boundary conditions (IBCs) for well-posedness. Most of the existing IBCs for the 1-D SWEs assume local conservation of mass and some local balance of momentum around junctions and bends. The former is a physically valid condition because the inflow, outflow, and storage if it is considered, of the water around junctions should be exactly conserved. On the other hand, the latter has long been discussed by many researches because of the difficulty of finding physically and mathematically valid description of the momentum balance around junctions and bends. A number of mathematical models, such as the energy balance models (Akan and Yen, 1981; Shabayek et al., 2002), algebraic models-based on the semi-analytical weir formulae (Kesserwani et al., 2008; Kesserwani et al., 2010), and 1-D and 2-D coupled models (Sanders et al., 2011; Neupane and Dawson, 2015), have been proposed so far; however, each method has its own advantages and disadvantages. Similar mathematical problems arise in simulating biological fluid flows in complex blood vessel networks where momentum balance around junctions critically affects the global fluid dynamics (Perdikaris et al., 2014; Epstein et al., 2015). Conclusive answer to determine the best model has not been obtained.
2.3 Literature Review on Mathematical Models of Solute Transport Phenomena

Solute transport phenomena in surface water bodies have been pivotal research topics in hydro-environmental research areas because of their close links to a wide variety of water environmental and ecological issues in the real world. Assessment of the solute transport phenomena involving diffusion and/or dispersion effects has been typically carried out with parabolic partial differential equations (PDEs) having advection terms. Chatwin and Allen (1985) reviewed the development of such PDEs representing solute transport phenomena in rivers and estuaries, focusing on dispersion models with spatial and temporal averaging techniques. Advection-dispersion equations (ADEs) are the PDEs considering dispersion effects. The major water quality models currently in use are to numerically solve the ADEs (Cox, 2003). Fundamental properties of these ADEs have been studied in detail and they are recognized as powerful tools in the analysis of environmental and hydraulic problems. Deng et al. (2010; 2012) evaluated longitudinal dispersion coefficients in several channels modeled as cross-sectionally averaged 1-D systems. Knock and Ryrie (1994) proposed a parameterization method for one dispersion coefficient employed in 2-D ADEs. They also conducted numerical analysis of dispersion of a solute in an actual shallow water body. Hunt (2006) proposed a one-dimensional dispersion equation with a spatially or temporally varying dispersion coefficient. His model includes the effect of velocity shear near the leading and trailing edges of solute profiles. Zoppou and Knight (1997; 1999) provided exact solutions of ADEs with coefficients linearly dependent on distance from point sources in 1-, 2-, and 3- D problems. Zhou et al. (2011) modeled the water system in a river delta region as a combined domain of 1-D river network and 3-D estuary to apply the ADEs for simulation of solute transport. Yoshioka and Unami (2013) numerically simulated advection-dispersion phenomena of solute in looped open channel networks based on a conservative transport equation.

Mathematical models employed in the conventional researches are mainly developed based on deterministic conservation laws of mass and the Fick’s laws representing analogies to gradient-type laws of molecular diffusion and heat condition. However, stochasticity inherent in the transport processes is not properly considered in most of such researches because of the deterministic nature of the models. On the other hand, several researchers have been investi-
gating transport phenomena employing other methods. Researches that model stochasticity contained in the transport phenomena using stochastic differential equations, referred to as the SDEs, have turned out to be effective for comprehending the phenomena. As shown in Øksendal (2000), an SDE is a time-dependent differential equation governing a random process. The Kolmogorov forward equation (KFE) and Kolmogorov backward equation (KBE) have also been shown to be equally useful tools for the analysis of randomized phenomena. Bodo et al. (1987) reviewed fundamental properties of SDE, KBE, and KFE. They mentioned the applicability of these equations to hydrological problems. Su (2004) indicated that KFE is closely related to many transport equations employed in environmental engineering problems. Unami et al. (2010) analyzed the behavior of fish ascending an actual agricultural drainage system. They modeled ascending behavior of fishes using a SDE and the associated KBE with the coefficients determined based on swimming and leaping ability of the fishes. Yoshioka et al. (2012b) presented an SDE governing solute transport phenomena in turbulent flows. They also validated the model using time series data sets of flow velocity in real free surface turbulent flows.

Solving parabolic PDEs on connected graphs, such as the locally 1-D open channel networks, requires the use of appropriate IBCs at junctions, so that the singularities at the junctions are rigorously described in mathematical modelling and efficiently handled in numerical computation (Lumer, 1980; Pokorny and Borovskikh, 2004). Practical analysis on the longitudinal dispersion phenomena of solute in open channel networks have commonly been carried out using numerical models coupled with IBCs; however, impacts of the IBCs on the analysis results have overlooked in the past hydraulic research areas except for a few numbers of researches (Basha and Malaeb, 2007; Islam and Chaudhry, 1998).

2.4 Literature Review on Numerical Methods for Shallow Water Flows
Because the analytical solutions to the SWEs are available only for simplified cases such as the flows in friction-less straight open channels (Stoker, 1957), numerical schemes are in general used to approximate their solutions in applications. Development of accurate, stable, and efficient numerical schemes for the SWEs has therefore been a key hydro-environmental research topic. Most of the numerical schemes for the SWEs are based on the finite volume method.
(FVM) equipped with the Riemann-solvers, which are numerical schemes based on exact or linearized solutions to the Riemann problems (Toro and Garcia-Navarro, 2007). Mathematical and numerical aspects of the Riemann solvers have extensively been reviewed in Toro (2009). Related numerical schemes, such as the finite element method (FEM) (Hanert et al., 2005; Atallah and Hazzab, 2013), the discontinuous Galerkin FEM (Xing et al., 2010; Dumbser and Casulli, 2013), and the residual distribution method (Sármány et al., 2013; Ricchiuto, 2015), have also been presented. Some researchers recently found that simpler but sufficiently accurate numerical schemes could be developed without the Riemann-solvers and high-resolution algorithms as demonstrated in the literatures (Chen et al., 2007; Catella et al., 2008; Magdalena et al., 2015; Cea and Blade, 2015; Yoshioka et al., 2015).

Focusing on the 1-D SWEs defined on locally 1-D open channel networks, it is desirable to use computationally efficient numerical methods that can couple the 1-D SWEs with IBCs around junctions and bends. Sanders et al. (2011) simulated contaminant transport in tidal river networks by nesting a 2-D model at junctions, so that the converging and diverging behaviors of the flows are accurately resolved. Capart et al. (2002) developed a characteristic-based FVM scheme for the 1-D SWEs in natural channels and applied the scheme to flow routing of a typhoon-induced flood event in an existing river network. Kesserwani et al. (2008) investigated several algebraic IBCs for subcritical flows at junctions and compared them with a series of experimental data. Kesserwani et al. (2010) performed numerical analysis of free surface water flows at a T-shaped junction in the context of the 1-D SWEs with source terms for discharge and momentum losses by the flow divisions. Trancoso et al. (2009) developed a staggered FVM scheme for open-channel network flows and coupled this scheme with distributed hydrological models. Van Thang et al. (2010) applied a lattice Boltzmann method to shallow water flows in an irrigation canal network equipped with hydraulic structures. Zhu et al. (2011) established an efficient numerical method to deal with backwater effects at junctions based on the method of characteristics. Until now, only a limited number of studies have investigated flows in multiply connected (looped) channel networks with steep, non-rectangular, and non-prismatic channels. Recently, some authors developed simple but physically-based IBCs describing the momentum balances around junctions and bends based on the classical
momentum conservation principle, which have been validated through experimental test cases (Ishida et al., 2011; Ishida et al., 2012; Unami and Alam, 2012; Yoshioka et al., 2014a). Finaud-Guyot et al. (2011) have also proposed an analogous IBC. These IBCs have been considered to potentially serve as a basis of physically consistent numerical models for computationally efficiently simulating shallow water flows.

2.5 Literature Review on Numerical Methods for Solute Transport Phenomena

As with the case of the SWEs, analytical solutions to the PDEs of solute dispersion, such as the conventional ADEs, KFEs, and KBEs, are not available except for simplified cases (Pérez-Guerrero et al., 2009; Pérez-Guerrero and Skaggs, 2010). Their solutions have therefore been approximated numerically in applications. In the view of engineering applications, different numerical methods for the ADEs have been vigorously investigated. Di Paola and Sofi (2002) proposed a weighted residual method to obtain approximate solutions of steady 1-D KFEs. Spencer and Bergman (1993) applied the standard Galerkin finite element method (FEM) for numerical resolution of 2-D KFEs. Kumar and Narayanan (2006) numerically solved a 4-D KFE using the high order finite difference methods (FDMs) as well as the standard Galerkin FEM. Their FDM scheme has been applied to the analysis of a nonlinear stochastic system excited by stochastic perturbations (Narayanan and Kumar, 2012). Although the numerical methods based on the FDM or the FEM have been widely used for the analysis of KFEs, most of these methods are not equipped with conservative property that mass transport problems require. On the other hand, the finite volume method (FVM) based on local conservation laws has the advantage of achieving the conservative property when applied to equations in conservative form such as the KFEs.

Numerically solving the ADEs in locally 1-D open channel networks requires special treatment at junctions where appropriate conservation laws of mass have to be specified for well-posing the problems (Zhang and Aral, 2004; Zhang et al., 2008; Yoshioka et al., 2014a). Most of the conventional schemes solve a PDE separately in reaches and around junctions (Szymkiewicz, 2010; Zhang et al., 2010; Sanders et al., 2011; Tumanova and Čiegis, 2012), which may result in the loss of computational efficiency. Development of computationally
efficient and sufficiently accurate numerical methods to solve the ADEs on connected graphs is required for facilitating numerical modeling of the solute transport phenomena.
CHAPTER 3  New 1-D Shallow Water Equations for Open Channel Network Flows

3.1 Introduction

1-D shallow water modeling of fluid flows in open channel networks plays crucial roles in assessing transport phenomena occurring in surface water bodies. Mathematical modeling of such water flows has typically been carried out with the 1-D shallow water equations, which are referred to as the 1-D SWEs (Stoker, 1957; Szymkiewicz, 2010). The 1-D SWEs are a hyperbolic system of nonlinear partial differential equations (PDEs) having source terms that govern mass and momentum variation along each open channel. The governing equations of the conservation of mass and momentum consisting of the 1-D SWEs are referred to as the continuity equation and the momentum equation, respectively. Simulating water flows in open channel networks can effectively reduce to approximating solutions to the 1-D SWEs on the locally 1-D open channel networks, which are connected graphs that consist of 1-D reaches and 0-D junctions, on which hydraulic quantities, such as the water surface elevation and discharge, are appropriately distributed (Yoshioka et al., 2012a). Hereafter, a bend is identified as a junction connecting exactly the ends of two reaches meeting with a non-zero crossing angle.

Junctions mathematically serve as singular points where the 1-D SWEs cannot be directly defined in the conventional sense because of possible discontinuity and regularity deficits of some dependent variables such as discharges, which requires using internal boundary conditions (IBCs) for well-posing the problems. The IBCs in general cannot be derived directly from the 1-D SWEs and additional phenomenological considerations have to be made for their derivations. In practice, the IBCs should be easily implemented into numerical schemes for efficient computations. For the continuity equation, an appropriate IBC from the viewpoint of classical physics is the conservation of discharges around the junctions, which is inspired from the fact that the total inflow and total out flow must exactly balance around junctions and bends. On the other hand, for the momentum equation, many researches have been carried out for finding out appropriate IBCs to be equipped with; however, no conclusive IBC has been developed so far. A possible cause of this fact is that actual fluid dynamics in and around junctions involves very complicated inherently multi-dimensional phenomena that the 1-D models
cannot resolve, such as recirculating vortices and oblique shocks (Rivière et al., 2011; 2014; Sun et al., 2014).

Most of the conventional mathematical models of the momentum balance at junctions and bends can be broadly categorized into the several types, which are energy balance models (Akan and Yen, 1981; Shabayek et al., 2002), algebraic models based on the semi-analytical weir formulae, and 1-D and 2-D coupled models (Sanders et al., 2011; Neupane and Dawson, 2015). The energy balance models are very simple and easy to implement in numerical computation, but in principle they are only applicable to gradually varied flows where the solutions are smooth. Their applications to the problems with shocks, which are ubiquitous in real problems, are therefore considered to be inappropriate around junctions and bends. The algebraic models based on the weir formulae assume some static equilibrium conditions on the momentum balance around junctions and bends. Some of these models have turned out to give comparably accurate results with the 1-D and 2-D coupled models described below. The 1-D and 2-D coupled models are the most physically appropriate ones among the above-mentioned models because they can resolve the horizontally 2-D water flows both at and around junctions. A significant drawback common to the 1-D and 2-D coupled models is higher computational costs than the others due to using more sophisticated mathematical models and more complicated numerical algorithms. These IBCs thus have both advantages and disadvantages.

There exists another type of IBCs based on the classical momentum conservation principles, which consider linear input and output systems of the inflow and outflow momentum fluxes at junctions and bends in an upwind manner. These IBCs are referred to as the momentum flux evaluation schemes in this thesis. Ishida et al. (2011), Ishida et al. (2012), and Unami and Alam (2012) proposed different momentum flux evaluation schemes, which are equivalent for the flows in single straight open channels. Although the momentum flux evaluation schemes have been used as physically sound IBCs for the momentum equations and have effectively been employed in real applications, their mathematical properties have not been well studied. Mathematical and numerical analysis on the shallow water flows in this thesis focus only on the problems where hydraulic structures, such as weirs and gates, are not equipped around junctions that can significantly control both local and global hydraulic processes
The purpose of this chapter is to present a new 1-D SWEs that can mathematically and physically deal with open channel network flows in a consistent manner. Section 3.2 presents the conventional 1-D SWEs. Section 3.3 presents an extended continuity equation that can be applied to open channel network flows. Section 3.4 proposes a momentum flux evaluation scheme, which turns out to be a physically more consistent IBC than the previously proposed ones from the viewpoint of a non-increase condition of the momentum along the channels around junctions. Section 3.5 concludes this chapter.

3.2 Conventional 1-D Shallow Water Equations

The 1-D SWEs for shallow water flows along a single reach with arbitrary cross-sectional shapes (Lai et al., 2002; Unami and Alam, 2012) consist of the continuity equation

$$\frac{\partial A(\eta)}{\partial t} + \frac{\partial Q}{\partial x} - q = 0$$

(1)

and the momentum equation

$$\frac{\partial Q}{\partial t} + \frac{\partial F}{\partial x} + gA\left(\frac{\partial \eta}{\partial x} + S_{f}\right) = 0$$

(2)

with the momentum flux

$$F = \frac{\beta Q^2}{A}$$

(3)

where \( t \) is the time, \( x \) indicates the 1-D abscissa taken along the reach, \( A = A(t, x) \) is the wetted cross-sectional area of the reach as a smooth function of the water elevation \( \eta = \eta(t, x) \), \( Q = Q(t, x) \) is the discharge, \( \beta(\geq 1) \) is the momentum correction coefficient, \( g \) is the gravitational acceleration, \( q = q(t, x) \) is the lateral inflow without momentum, and \( S_{f} \) is the friction slope given by the Manning’s formula (Yen, 2002)

$$S_{f} = \frac{n^2 \sigma^{4/3} Q}{A^{0.3}}$$

(4)

where \( n \) is Manning’s coefficient and \( \sigma \) is the wetted perimeter. The wet and dry interfaces are not considered in this paper, thus the area \( A \) is assumed to be strictly positive everywhere in a domain. The channel bed elevation is assumed to be spatially continuous and fixed in time. Eqs.(1) and (2) can be applied to arbitrary single reaches if equipped with appropriate initial and boundary conditions. The momentum flux \( F \) in Eq.(3) is the total momentum of the
flow along the stream direction.

3.3 Extended Continuity Equation

The conventional 1-D SWEs cannot be directly applied to simulating open channel network flows because of the lack of IBCs specified at junctions. Here an alternative continuity equation that can consistently deal with open channel network flows is presented. For such shallow water flows, it is reasonable to rewrite the continuity equation (1) in the integral form

\[ \int_{\Omega \cap B(y,r)} \frac{\partial \eta}{\partial t} \, dx + \sum_{j=1}^{\nu} \sigma_j Q_{r,j} + \sum \sigma_j Q_{u} = \int_{\Omega \cap B(y,r)} q \, dx \]  

(5)

where \( \Omega \) represents the domain of the flow as a locally 1-D open-channel network, \( T = \frac{\partial A}{\partial \eta} \) is the top width of the water surface, \( B(y,r) \) is the horizontally 2-D \( r \)-neighborhood of a point \( y \in \Omega \) with sufficiently small \( r(>0) \), \( \nu \) is the total number of reaches that intersect with the boundary of \( B(y,r) \), \( Q_{r,j} \) is the discharge at the \( j \)th intersection, \( \sum Q_{u} \) represents the sum of the discharges specified at the boundary vertices in \( \Omega \cap B(y,r) \), and \( \sigma_j \) is the sign parameter defining the direction of the abscissa in the \( j \)th reach. A schematic diagram of \( B(y,r) \) is shown in Figure 3-1. Here, \( \sigma_j \) is equal to 1 when the \( x \) abscissa in the \( j \)th reach is facing outward to the boundary of \( B(y,r) \) and it is otherwise equal to -1.

Eq.(5) is referred to as the extended continuity equation. Application of Eq.(5) to single channels recovers the conventional equation (1). The continuity equation serves as an IBC at junctions. In fact, taking the limit \( r \to +0 \) in Eq.(5) yields the local mass conservation law

\[ \sum_{j=1}^{\nu} \sigma_j Q_{r,j} = \int_{\Omega \cap B(y,r)} q \, dx . \]  

(6)

Eqs.(1) and (5) are consistent in this sense.

3.4 Momentum Flux Evaluation Schemes

As with the case of the continuity equation, the momentum equation (2) has to be equipped with an appropriate IBC at each junction to determine the dynamics of momentum around that point. Without the loss of generality, a junction connecting \( m(\geq 1) \) inflow reaches and \( n(\geq 1) \) outflow reaches is considered for formulating the IBC for the momentum equation (2) since the IBC is local in nature. A sketch of the junction and the reaches is presented in Figure 3-2. A
junction with \( m = n = 1 \) is a bend by the definition. The \( x \) abscissae along the inflow reaches end at the junction and those along the outflow reaches start from the junction. Let \( -\pi < \theta_{i,j} \leq \pi \) be the absolute value of the contact angle between the outflow reach \( O_i \) (\( 1 \leq i \leq m \)) and the inflow reach \( I_j \) (\( 1 \leq j \leq n \)). The momentum flux \( F \) at the downstream-end of the inflow reach \( I_j \) is denoted by \( F_{j,ds} \) and that at the upstream-end of the outflow reach \( O_j \) by \( F_{j,us} \). The momentum fluxes \( F_{j,ds} \) and are \( F_{j,us} \) non-negative by the definition. Assuming a linear dependence of \( F_{j,us} \) on \( F_{j,ds} \) leads to the IBC for the momentum equation as

\[
F_{j,us} = \sum_{i=1}^{m} w_{i,j} F_{i,ds}
\]  

(7)

where \( w_{i,j} \) is the non-negative coefficient to be determined. The next subsections present conventional and a new momentum flux evaluation schemes with different choices of the coefficient \( w_{i,j} \).

![Diagram](image1.png)

Figure 3-1 A sketch of \( B(y,r) \).

![Diagram](image2.png)

Figure 3-2 A sketch of inflow and outflow reaches meeting at a junction.
3.4.1 Conventional Schemes

The two conventional momentum flux evaluation schemes, which are referred to as the M1 and M2 schemes, are presented in this sub-section.

3.4.1.1 M1 Scheme

The M1 scheme evaluates the flux \( F_{j,us} \) by simply summing up \( F_{i,ds} \) as (Ishida et al., 2011)

\[
F_{j,us} = \sum_{i=1}^{m} F_{i,ds} \quad (w_{i,j} = 1).
\]  

(8)

As shown in Eq.(8), the M1 scheme does not consider the crossing angles of the reaches at junctions in evaluating the flux \( F_{j,us} \). The M1 scheme has been applied to numerically analyzing rainfall-runoff relationships in an agricultural drainage network associated with paddy fields in a Japanese inland valley. A comparison of simulated and observed outflow discharge at the outlet of the system has been performed, demonstrating that the scheme can accurately reproduce the observation results when the lumped-hydrological model, which outputs the inflow discharge to the system using the rainfall intensity as an input, is specified (Ishida et al., 2011).

3.4.1.2 M2 Scheme

Being different from the M1 scheme, the M2 scheme evaluates the flux \( F_{j,us} \) considering the contact angles of the cells meeting at a junction. The M2 scheme evaluates the flux \( F_{j,us} \) as (Unami and Alam, 2012)

\[
F_{j,us} = \sum_{i=1}^{m} F_{i,ds} \cos \theta_{i,j} \quad (w_{i,j} = \cos \theta_{i,j}).
\]  

(9)

This scheme assumes that the cosine of \( F_{i,ds} \) contributes to \( F_{j,us} \), which is considered to be valid in view of the classical momentum balance principle in continuum mechanics. The M2 scheme has been applied to numerical simulation of shallow water flows in a multiply-connected open channel network in a bottom of a Ghanaian inland valley with a hydro-morphic environment. The main inputs of water to the open channel network are direct rainfall and seepage flows from the soils around it after prolonged rainfalls. The computational results of water flows in the open channel network demonstrated that the M2 scheme reasonably simulates flow transitions and very shallow water depths at the order of millimeters without com-
putational instability (Unami and Alam, 2012).

3.4.2 New Scheme: M3 Scheme

A new momentum flux evaluation scheme, which seems to be more physically appropriate than the conventional schemes, is presented in this sub-section. The new scheme is referred to as the M3 scheme, which is formulated as

$$F_{j,um} = \sum_{i=1}^{m} F_{j,di} \xi_j \cos \theta_{i,j} \quad (w_{i,j} = \xi_j \cos \theta_{i,j})$$

(10)

with the discharge ratios $\xi_j$ for the outflow reaches defined as

$$\xi_j = \frac{Q_j}{\sum_{k=1}^{n} Q_k} \quad (1 \leq j \leq n),$$

(11)

which is replaced by 0 when its denominator vanishes. The discharge ratio $\xi_j$ is bounded as $0 \leq \xi_j \leq 1$. Summing up each $\xi_j$ yields

$$\sum_{j=1}^{n} \xi_j = 1.$$  

(12)

As presented in Eq.(10), the M3 scheme assumes that the cosine of $F_{j,di}$ with multiplied by the discharge ratio $\xi_j$ contributes to $F_{j,um}$. This assumption is heuristic; however, it is reasonable to consider that the net distributed flux for an outflow reach decreases as the number of the outflow reaches increases. As shown in Eqs.(9) and (10), the M2 and M3 schemes are equivalent for the flows around a channel bend ($m = n = 1$ and $\theta_{i,j} \neq 0$).

3.4.3 Mathematical Analysis on the Three Schemes

This subsection performs a mathematical analysis of the M1 through M3 schemes focusing on the momentum variation, which is defined below, at the junction J. The three schemes are represented in a unified manner with Eq.(7) using the weight coefficient $w_{i,j}$ as

$$w_{i,j} = \begin{cases} 
1 & \text{(M1 scheme)} \\
\cos \theta_{i,j} & \text{(M2 scheme)} \\
\xi_j \cos \theta_{i,j} & \text{(M3 scheme)} 
\end{cases}.$$  

(13)

By Eqs.(7) and (13), the momentum variation $\Delta F$ at the junction J is defined as
\[\Delta F = \sum_{j=1}^{n} F_{j,us} - \sum_{j=1}^{m} F_{j,ds}\]
\[= \sum_{j=1}^{n} \sum_{i=1}^{m} w_{i,j} F_{i,ds} - \sum_{j=1}^{m} F_{i,ds}\]
\[= \sum_{j=1}^{m} \left( \sum_{i=1}^{n} w_{i,j} - 1 \right) F_{j,ds}, \tag{14}\]

The right hand-side of Eq.(14) is the subtraction of the total momentum fluxes at the downstream-ends of the upstream reaches \((\sum_{i=1}^{n} F_{i,ds})\) from the total momentum fluxes at the upstream-ends of the downstream reaches \((\sum_{j=1}^{m} F_{j,ds})\). The momentum difference \(\Delta F\) should not be positive at each junction because its positivity means the existence of external momentum source at that point, which is clearly an unphysical situation. A momentum flux evaluation scheme should therefore guarantee non-positivity of the \(\Delta F\). This condition is referred to as the momentum condition and is mathematically expressed as
\[\Delta F \leq 0. \tag{15}\]

Substituting the weight \(w_{i,j}\) for the M1, M2 and M3 schemes into Eq.(7) yields
\[\Delta F_{M1} = \sum_{i=1}^{n} \left( \sum_{j=1}^{m} 1 - 1 \right) F_{i,ds}\]
\[= (n - 1) \sum_{i=1}^{m} F_{i,ds}\]
\[\geq 0, \tag{16}\]
\[\Delta F_{M2} = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} \cos \theta_{i,j} - 1 \right) F_{i,ds}\]
\[\leq (n - 1) \sum_{i=1}^{m} F_{i,ds}\]
\[= \Delta F_{M1}, \tag{17}\]

and
\[\Delta F_{M3} = \sum_{i=1}^{m} \left( \sum_{j=1}^{n} \xi_j \cos \theta_{i,j} - 1 \right) F_{i,ds}\]
\[\leq \sum_{i=1}^{m} \left( \sum_{j=1}^{n} \xi_j - 1 \right) F_{i,ds}\]
\[= \sum_{i=1}^{m} (1 - 1) F_{i,ds}\]
\[= 0, \tag{18}\]

respectively, indicating that only the M3 scheme unconditionally guarantees the momentum
condition but the others do not always satisfy the condition. The M1 scheme does not satisfy the condition in particular. The M2 scheme complies with the momentum condition if
\[
\sum_{j=1}^{n} \cos \theta_{ij} \leq 1.
\] (19)

The mathematical analysis results show that the M3 scheme is the most relevant choice among the three schemes in the sense that it does not make \( \Delta F \) positive at each junction.

3.5 Conclusions

The extended 1-D SWEs were presented as a physically more consistent mathematical model for the conventional equations for open channel network flows. The extended continuity equation can consistently describes the flows around junctions and bends. The momentum equation used in the extended 1-D SWEs were identical to those in the conventional 1-D SWEs, but the IBCs to be equipped with is different from those of them. The physical drawback of the conventional momentum flux evaluations schemes were pointed out considering the momentum condition and a new scheme (M3 scheme) that does not encounter this drawback is presented. The new 1-D SWEs can be straightforwardly cast into the numerical models by extending the existing numerical algorithms, which will be presented in chapter 5 of this thesis. Numerical verification of the momentum flux evaluation schemes is also performed in that chapter.
CHAPTER 4  Stochastic Process Models for Solute Transport Phenomena in Surface Water Bodies

4.1 Introduction

Transport phenomena occurring in surface water bodies are pivotal research subjects in hydro-environmental research areas because of their close links to a variety of water environmental and ecological issues in the real world. Analyzing the transport phenomena ultimately reduces to tracking movements of particles driven by both deterministic drifts and stochastic fluctuations. Assuming the continuity and the Markov property of the paths of particles leads to a stochastic differential equation (SDE) driven by some stochastic processes, such as the Brownian motions (Øksendal, 2000). Dimou and Adams (1993) analyzed advection-dispersion phenomena of solute in the context of the cross-sectionally averaged 1-D shallow water theory utilizing an SDE. Its horizontally 2-D counterpart has been proposed in Heemink (1990), followed up on by the subsequent researches focusing on its physically more reasonable extension (Charles et al., 2009) and efficient numerical computation methods (Charles et al., 2008).

An SDE associates a couple of linear partial differential equations (PDEs) governing the conditional probability density function (PDF) of the probability law of the particle movements in the Eulerian coordinates, which are the Kolmogorov’s forward equation (KFE) and the Kolmogorov’s backward equation (KBE) (Risken, 1989; Øksendal, 2000). These PDEs serve as effective mathematical tools for analyzing transport phenomena consistently considering their statistical dynamics (Bodo et al., 1987). Brannan et al. (2001) used the KBEs for evaluating the mean residence time and the escape probability from geophysical vortices. Szurley and Duan (2001) who discussed impacts of the Coriolis parameter on the dynamics of the spatially-distributed statistics have performed similar numerical analysis. Transport phenomena of conservative solute in saturated groundwater bodies have been analyzed in terms of the escape probability (Cai et al., 1996; Fang et al., 2005).

The main purpose of this chapter is to present a stochastic process model for transport phenomena in surface water bodies, which can be used for comprehending the phenomena with consistently evaluating their stochasticity. The stochastic process model is based on the SDE,
KFE, and KBE for Lagrangian particle movements in turbulent flows. In this chapter, the governing equations of particle movements in turbulent flows are presented in Section 4.2. The deduction procedures for the governing equations of the particle concentration and those of the spatially-distributed statistics are presented in Sections 4.3 and 4.4, respectively. Analogous and advanced stochastic process models for the problems in shallow water bodies are then presented in Section 4.5, which will be used in numerical analysis in later chapters of this thesis. The stochastic process model is validated through measured data of velocity time series in real open channels in Section 4.6. Section 4.7 concludes this chapter.

4.2 Stochastic Process Model

4.2.1 Stochastic Differential Equation

Let $\Omega$ be a spatially $n$-dimensional domain of the water flow. The boundary $\partial \Omega$ of the domain $\Omega$ consists of the two types of boundaries: the open boundary $\partial \Omega_o$ where solute particles are immediately removed and the closed boundary $\partial \Omega_c$ where the particles cannot go through, which satisfy the relationships $\partial \Omega_o \cup \partial \Omega_c = \partial \Omega$ and $\partial \Omega_o \cap \partial \Omega_c = \emptyset$. The position of a particle, which represents either a solute particle or an aquatic organism such as a fish, at the time $t$ is an $n$-dimensional vector denoted by $X_t = [X_{t,i}]$. The vector $X_t$ is considered as a continuous time stochastic process whose evolution in $\Omega$ is governed by the Itô’s SDE

$$dX_t = V(t, X_t)dt + \sqrt{2D(t, X_t)}dB_t$$

(20)

where $B_t$ is the $n$-dimensional standard Brownian motion (Oksendal, 2000), $V = [V_i]$ is the $n$-dimensional drift coefficient, and $D = [D_{i,j}]$ is the $n \times n$-dimensional dispersivity matrix, which is assumed to be positive-definite. The drift $V$ represents the deterministic velocity for a solute particle and the deterministic swimming velocity for an aquatic organism. SDE(20) assumes that the stochasticity involved in the transport phenomena can be modelled with the Brownian motion, which is a continuous time Gaussian process. The dispersivity $D$ modulates the amplitude and correlation of the stochasticity, which is identical to the turbulent diffusion coefficient for the solute transport phenomena in turbulent flows.
4.2.2 Kolmogorov’s Equations

The Kolmogorov’s equations, which are the KFE and KBE associated with the SDE(20), are presented in this sub-section. Let \( P(s,y,t,G) \) be the transient probability such that \( X_i = x \) is observed in a sub-domain \( G \subseteq \Omega \) conditioned on that \( X_j = y \) at the past time \( s(< t) \). The conditional PDF \( p = p(s, y, t, x) \) such that \( X_i = x \) conditioned on that \( X_j = y \) is defined to satisfy the integral relationship

\[
P(s,y,t,G) = \int_G p(s, y, t, x) \, dx
\]

and the Chapman-Kolmogorov’s equation

\[
p(s, y, t, x) = \int_\Omega p(s, y, u, z) \, p(u, z, t, x) \, dz
\]

where \( u \) is arbitrary time such that \( s < u < t \). The generator \( L \) associated with the stochastic process \( X_i \), which is defined as a linear elliptic partial differential operator, is given by

\[
Lf = \sum_{i=1}^n V_i \frac{\partial f}{\partial y_i} + \sum_{i,j=1}^n D_{i,j} \frac{\partial^2 f}{\partial y_i \partial y_j} - Rf
\]

for generic sufficiently regular function \( f = f(y) \) where \( R \geq 0 \) is the decay coefficient defined through the probabilistic equality

\[
R = R(t, x) = \lim_{h \to 0} \frac{1}{h} P_k(t, x, h)
\]

where \( P_k(t, x, h) \) is the probability that the particle with \( X_i = x \) is killed during the time interval \( (t, t+h] \) (Oksendal, 2000). The adjoint operator \( L^* \) of the generator \( L \) is given by

\[
L^* f = \sum_{i=1}^n \frac{\partial}{\partial x_i} (V_i f) - \sum_{i,j=1}^n \frac{\partial^2}{\partial x_i \partial x_j} (D_{i,j} f) + Rf
\]

for generic sufficiently regular function \( f = f(x) \). The KFE governs the time-forward evolution of the conditional PDF \( p \) in terms of the forward variables \( (t, x) \), which is expressed with the adjoint \( L^* \) of the generator as

\[
\left( \frac{\partial}{\partial t} + L^* \right) p = 0
\]

in the domain \( \Omega \). On the other hand, the KBE governs the time-backward evolution of the conditional PDF \( p \) in terms of the forward variables \( (s, y) \), which is expressed as

\[
\left( \frac{\partial}{\partial s} + L \right) p = 0
\]

in the domain \( \Omega \). Eqs.(26) and (27) have to be equipped with initial and terminal conditions for their well-posedness. The initial condition for the KFE at the time \( t = s \) is expressed as
\[ p(s,y,s,x) = \delta(x-y) \]  
(28) 

where \( \delta(\cdot) \) represents the Dirac’s Delta. Similarly, the terminal condition for the KBE at the time \( s = t \) is expressed as 

\[ p(t,y,t,x) = \delta(y-x). \]  
(29) 

Eqs.(26) and (27) also have to be equipped with the boundary conditions. Eqs.(28) and (29) state that one particle cannot occupy different positions at an instance. For the open boundary \( \partial\Omega_0 \), the boundary conditions are given by the homogenous Dirichlet condition 

\[ p = 0. \]  
(30) 

On the other hand, for the closed boundary \( \partial\Omega_c \), the zero-flux condition 

\[ \sum_{j=1}^{n} n_j \int_{\Omega} V_x \frac{\partial}{\partial x_j} (D_{ij} p) = 0 \]  
(31) 

is applied to the KFE and the homogenous Neumann condition 

\[ -\sum_{j=1}^{n} n_j \int_{\Omega} \frac{\partial}{\partial y_j} p = 0 \]  
(32) 

to the KBE where \( \mathbf{n} = [n_j] \) is the unit outward normal vector on the boundary \( \partial\Omega \). 

### 4.3 Transport Equation 

The transport equation governing the particle concentration \( C = C(t,x) \), which is a macroscopic quantity that characterizes spatio-temporal particle dynamics, is deduced on the basis of the linearity of the KFE(26). The total mass \( M(t,G) \) of the particles contained in the sub-domain \( G \subset \Omega \) at the time \( t \) is expressed with an integral relationship as 

\[ M(t,G) = \int_{\Omega} C(t,x) \, dx \]  
(33) 

by Eq.(21). Considering the mass balance of the particles during the course of time \( (s,t] \), another expression of the total mass \( M(t,G) \) is deduced as 

\[ M(t,G) = \int_{\Omega} C(s,y) P(s,y,t,G) \, dy + \int_{s}^{t} \int_{\Omega} q(\tau,y) P(\tau,y,t,G) \, dy \, d\tau \]  
(34) 

where \( q = q(t,x) \) represents source and sink terms. Since the sub-domain \( G \) can be chosen arbitrarily, comparing Eqs.(33) and (34) leads to the integral equation 

\[ C(t,x) = \int_{\Omega} C(s,y) p(s,y,t,x) \, dy + \int_{s}^{t} \int_{\Omega} q(\tau,y) p(\tau,y,t,x) \, dy \, d\tau. \]  
(35)
The basic assumption on the integral equation (35) is the Markov property of the process \( X \),
and it is well-defined if the two integrals in its right hand-side are well-defined. The source and sink \( q \) in the second term of the right hand-side of Eq.(35) possibly depends on the concentration \( C \) as in the case for some ecological dynamics of aquatic species in surface water bodies (Kang and Castillo-Chavez, 2012; Kang and Udiani, 2014). Well-posedness of the problem in such a case should be carefully analyzed, but which is not the focus of this thesis. Hereafter, the term \( q \) is assumed not to depend on \( C \) for the sake of simplicity.

The linear operator \( \frac{\partial}{\partial t} + L' \) is applied to the both-hand sides of Eq.(35) as

\[
\frac{\partial C}{\partial t} + L' C = \left( \frac{\partial}{\partial t} + L' \right) \int_{\alpha} C(s,y) p(s,y,t,x) dy \\
+ \left( \frac{\partial}{\partial t} + L' \right) \int_{\alpha} q(r,y) p(r,y,t,x) dy dr \\
= \int_{\alpha} C(s,y) \left( \frac{\partial p}{\partial t} + L' p \right) dy \\
+ \frac{\partial}{\partial t} \int_{\alpha} q(r,y) p(r,y,t,x) dy dr + \int_{\alpha} L' \left( \int_{\alpha} q(r,y) p(r,y,t,x) dy \right) dr \\
= \frac{\partial}{\partial t} \int_{\alpha} q(r,y) p(r,y,t,x) dy dr + \int_{\alpha} q(r,y) L' p(r,y,t,x) dy dr. 
\]

Application of the conventional Leibnitz’s rule

\[
\frac{\partial}{\partial t} \int_{\alpha} q(r,y) p(r,y,t,x) dy dr = \int_{\alpha} q(r,y) p(r,y,t,x) dy \\
+ \int_{\alpha} \frac{\partial}{\partial t} \left( \int_{\alpha} q(r,y) p(r,y,t,x) dy \right) dr \\
= q(t,x) + \int_{\alpha} \int_{\alpha} q(r,y) \frac{\partial}{\partial t} p(r,y,t,x) dy dr
\]

to Eq.(28) with Eq.(26) yields

\[
\frac{\partial}{\partial t} \int_{\alpha} q(r,y) p(r,y,t,x) dy dr = q(t,x) - \int_{\alpha} \int_{\alpha} q(r,y) L' p(r,y,t,x) dy dr.
\]

Substituting Eq.(38) to Eq.(36) leads to the transport equation

\[
\left( \frac{\partial}{\partial t} + L' \right) C = 0
\]

in \( \Omega \), in which the variables \( V \), \( D \), \( R \), and \( q \) are considered as known functions and the concentration \( C \) as the unknown function. Spatio-temporal evolution of the concentration \( C \) can be fully tracked if the known functions are completely determined over the domain \( \Omega \), which are in general given analytically or computed from the hydrodynamic models such
as the Navier-Stokes equations.

Appropriate initial and boundary conditions are necessary to be equipped with Eq.(39) in order to well-define the problem. A possible approach is to assume the boundary condition of the form

\[ \sum_{i=1}^{n} n_i \left( V_i C - \sum_{j=1}^{n} \frac{\partial}{\partial x_j} \left( D_{i,j} C \right) \right) = \sum_{i=1}^{n} n_i V_{i,B} C_B \]  

(40)

where the subscript B represents the value specified on the boundary. On the closed boundary \( \partial \Omega_c \), it is sometimes reasonable to assume that the right-hand-side of Eq.(40) vanishes assuming that the flow does not cross \( \partial \Omega_c \). Arbitrary non-negative function with sufficiently high spatial regularity can be used as an initial condition, which would be determined on the basis of some physical assumptions or measured data.

For the problems with a spatially inhomogeneous dispersivity, Eq.(39) is apparently different from the conventional Fickian model

\[ \frac{\partial C}{\partial t} + \nabla \cdot (V C) - \nabla \cdot \left( D \frac{\partial C}{\partial x} \right) + R C = q. \]  

(41)

Eqs.(39) and (41) are equivalent for the problems with spatially homogenous turbulent diffusion coefficient. According to Tupper and Yang (2012), the diffusion term, which is the third term in the left-hand side of the conventional model (41), is derived if the fluctuation term \( \sqrt{2} D d \mathbf{B} \) of the SDE(20) is understood as a limit of the time-backward discretization, whereas the Itô’s counterpart assumed in Eq.(39) corresponds to the limit of the time-forward discretization. Understanding the fluctuation term as the limit of the time-backward discretization is unphysical because it means that the current position of a particle depends on the information in the future, which clearly violates the causality principle in physics. On the other hand, the present stochastic process model based on the Itô’s counterpart is physically more consistent from the viewpoint of the causality principle. Another difference between the present and conventional transport equations are their derivation processes. The conventional model a priori assumes the Fick’s law, which provides a semi-empirical closure relationship. The present model is free from the Fick’s law but leads to the similar transport equation based on the probabilistic consideration. In addition, as Verwoerd (2009) pointed out for advection-dispersion phenomena of solute in randomly perturbed porous media, a transport equation

39
of solute formally analogous to Eqs.(39) and (41) can be deduced with the application of the Dynkin’s formula to a stochastic PDE governing particle trajectories, which also does not assume the Fick’s law. These analytical results suggest that the Fick’s law is not always necessary for dealing with the transport phenomena.

4.4 Statistics Equations

The governing equations of the spatially-distributed statistics for Lagrangian particle movements, which are referred to as the statistics equations in this thesis, are deduced on the basis of the linearity of the KBE(27). The statistics focused on in this chapter are the statistical moments of residence time, the probability of killing, and the escape probability, all of which can potentially serve as indicators for assessing the transport phenomena.

4.4.1 Statistical Moments of Residence Time

The residence time, which is a stopping time (Øksendal, 2000), is the total length of time that a particle spent in the domain \( \Omega \) before reaching the open boundary or being killed in the domain \( \Omega \). The residence time \( \tau^{s,y} \) of a particle conditioned on \( X_s = y \) is expressed as

\[
\tau^{s,y} = \inf \{t-s | t > s, X_t = y \in \Omega, X_s \in \Gamma \cup \partial \}
\]

(42)

where \( \partial \) represents the virtual coffin state where killed particles are put on (Øksendal, 2000). The killing is a mathematical concept corresponding to the physical, chemical, and biological phenomena, such as deposition of SS particles from water column to the bottom, decay of reactive solute in water bodies, and deaths of fishes. By Eq.(42), the \( k \) th statistical moment of the residence time, which is denoted by \( M_k(s,y) \) \( (k \geq 1) \), is expressed with the conditional PDF \( p \) as

\[
M_k(s,y) = E^{s,y} \left[ \left( \tau^{s,y} \right)^k \right]
= \int_t^\infty (t-s)^k \Pr \{ t-s < \tau^{s,y} < t-s+dt \} \, dt
= \int_t^\infty (t-s)^k \frac{\partial}{\partial t} \Pr \{ 0 < \tau^{s,y} < t-s \} \, dt
= \int_t^\infty (t-s)^k \frac{\partial}{\partial t} \left[ 1 - \int_{t-s}^t p(s,y,t,x) \, dx \right] \, dt
= -\int_t^\infty (t-s)^k \frac{\partial}{\partial t} \int_{t-s}^t p(s,y,t,x) \, dx \, dt
\]

(43)
where $E^{\tau_y}[\cdot]$ represents the expectation conditioned on $X_s = y$. Assuming $p \to 0$ for the infinitely long time $t \to +\infty$, application of a partial integration to the right hand-side of Eq.(43) yields
\[
M_k(s, y) = -\int_s^{+\infty} (t-s)^{\frac{k}{2}} \frac{\partial}{\partial t} \int_\Omega p(s, y, t, x) \, dx \, dt
= \left[ -(t-s)^{\frac{k}{2}} \int_\Omega p(s, y, t, x) \, dx \right]_s^{+\infty} + \int_s^{+\infty} \int_\Omega k(t-s)^{\frac{k-1}{2}} p(s, y, t, x) \, dx \, dt
= k \int_s^{+\infty} \int_\Omega (t-s)^{\frac{k-1}{2}} p(s, y, t, x) \, dx \, dt
\] (44)
where the condition
\[
\lim_{t \to +\infty} (t-s)^{\frac{k}{2}} \int_\Omega p(s, y, t, x) \, dx = 0
\] (45)
is assumed to be satisfied for all $k \geq 0$. The operator $\frac{\partial}{\partial s} + L$ is then applied to the both-hand sides of Eq.(45) to obtain
\[
\frac{\partial M_k}{\partial s} + LM_k = \left( \frac{\partial}{\partial s} + L \right) k \int_s^{+\infty} \int_\Omega (t-s)^{\frac{k-1}{2}} p(s, y, t, x) \, dx \, dt
= -k \int_s^{+\infty} \int_\Omega (t-s)^{\frac{k-1}{2}} p(s, y, t, x) \, dx \, dt + k \int_s^{+\infty} \int_\Omega (t-s)^{\frac{k-1}{2}} p \, dx \, dt
+ k \int_s^{+\infty} \int_\Omega (t-s)^{\frac{k-1}{2}} \left( \frac{\partial p}{\partial s} + Lp \right) \, dx \, dt
= -kM_{k-1}.
\] (46)
Consequently, the system of PDEs governing $M_k$ is derived as
\[
\frac{\partial M_k}{\partial s} + LM_k = -kM_{k-1} \quad (k \geq 1)
\] (47)
in the domain $\Omega$ subject to the terminal condition
\[
\lim_{s \to +\infty} M_k(s, y) = 0 \quad (k \geq 1)
\] (48)
and also to the boundary conditions analogous to Eqs.(30) and (32). Eq.(47) is a cascade linear system of PDEs whose solutions are obtained in an ascending order starting from the trivial solution $M_0 = 1$ over the domain $\Omega$.

### 4.4.2 The Probability of Killing

The probability of killing $K_G = K_G(s, y)$, which is the probability that a particle conditioned on that $X_s = y$ is killed in the sub-domain $G \subset \Omega$, is defined as
\[ K_G(s, y) = \Pr\{X_{s, y} \in G \mid X_s = y\} . \]  

(49)

According to its definition, the probability of killing \( K_G \) is expressed with the conditional PDF \( p \) and the decay coefficient \( R \) as

\[ K_G(s, y) = \int_s^\infty \int_G R(t, x) p(s, y, t, x) \, dx \, dt \]

(50)

because the PDF that a particle is killed during the time interval \((t, t + dt)\) in the infinitesimal hypercube \( \prod_{i=1}^n (x_i, x_i + dx_i) \) equals to \( R(t, x) p(s, y, t, x) \, dx \, dt \). By the KBE\((27)\), application of the partial differential operator \( \frac{\partial}{\partial s} \) to the both-hand sides of Eq.\((50)\) yields

\[
\frac{\partial K_G}{\partial s} = \frac{\partial}{\partial s} \int_t^\infty \int_G R(t, x) p(s, y, t, x) \, dx \, dt - \frac{\partial}{\partial s} \int_0^t \int_G R(t, x) p(s, y, t, x) \, dx \, dt
\]

\[
= \int_t^\infty \int_G R(t, x) \frac{\partial p}{\partial s} \, dx \, dt - \int_0^t \int_G R(s, x) p(s, y, s, x) \, dx
\]

\[
= -L \left( \int_0^\infty \int_G R(t, x) \, dx \, dt \right) - \chi_G R(s, y)
\]

\[
= -LK_G(s, y) - \chi_G R(s, y)
\]

(51)

where \( \chi_G \) is the indicator function defined by

\[
\chi_G = \begin{cases} 1 & (y \in G) \\ 0 & (y \notin G) \end{cases} .
\]

(52)

From Eq.\((51)\), the governing equation of the probability of killing \( K_G \) is derived as

\[
\frac{\partial K_G}{\partial s} + LK_G + R\chi_G = 0
\]

(53)

in the domain \( \Omega \) subject to the terminal condition

\[
\lim_{s \to \infty} K_G(s, y) = \chi_G
\]

(54)

and also to the boundary conditions analogous to Eqs.\((30)\) and \((32)\). The probability of killing serves as an indicator for assessing purification ability of a receiving water body such as constructed wetlands of surface water type having vegetation areas (Crohn et al., 2006; Shucksmith et al., 2011). Specifying the domain \( \Omega \) as the constructed wetland and the sub-domain \( G \) as a vegetated area in it can evaluate settlement efficiency for SS particles of the vegetated area; the higher the value of the probability \( K_G \) means the higher efficiency of the vegetation area serving as a filter to capture the particles (Takagi et al., 2014; Yoshioka et al., 2015c).

The presented spatially-distributed statistics can potentially serve as useful indicators in applications not only due to consistently characterizing the stochasticity involved in the
transport phenomena but also because their governing equations share the linear operator $L$. The latter fact is advantageous for engineering applications since their numerical resolution can be performed with a same numerical scheme, in which only the source terms, the terminal conditions, and the boundary conditions require different treatments depending on situations.

4.5 Shallow Water Counterparts

The presented stochastic process model, if it is subject to appropriate modifications, can also be applied to transport phenomena in shallow water bodies where the variables are defined through spatial averaging procedures. The shallow water models have served as more practical than the 3-D models because of their simplicity in describing the transport phenomena. They can be coupled with the low-dimensional hydrodynamic equations, such as the shallow water equations, which help develop efficient numerical methods for assessing transport phenomena. The following sub-sections present the 1-D and the horizontally 2-D counterparts of the stochastic process model.

4.5.1 1-D Model

For the 1-D case, a domain of water flows is given as a locally 1-D open channel network, which is a connected graph consisting of finite number of 0-D junctions and 1-D reaches. Hydraulic quantities, such as the wetted cross-sectional area of the reach and the cross-sectionally averaged fluid velocity, are accordingly distributed on it. A key in the 1-D case is to appropriately specify the internal boundary conditions (IBCs) at junctions to well-pose the problems. The SDE in each reach is defined analogously to the conventional 1-D frameworks; however, behaviour of its solution around junctions has to be understood in a weak sense via the local time measure governing the probabilistic law at these points (Freidlin and Sheu, 2000). Treatment of the PDEs, such as the KFE, KBE, and their related differential equations, should also be taken care at junctions. Functional analysis results on the PDEs defined on connected graphs, such as regularity and uniqueness results of solutions, are found in Berkolaiko and Kuchment (2013) and the references therein.

Based on Eq.(20), the SDE governing the Lagrangian particle movements in a 1-D open
channel domain can be deduced as
\[
dX_i = V_{iD}(t,X_i)dt + \sqrt{2D_{iD}(t,X_i)}dB_i
\] (55)
where the particle position \( X_i \) is understood as the 1-D projection of the 3-D model along the channel, \( B_i \) is the 1-D standard Brownian motion, \( V_{iD} \) is the drift coefficient, and \( D_{iD}(\geq 0) \) is the dispersion coefficient. The dispersion coefficient \( D_{iD} \) considers stochastic fluctuations of the Lagrangian particle movements in the longitudinal, lateral, and vertical directions in the channels. Its physical meaning is therefore different from \( D \) in Eq.(20).

The deduction procedures for the PDEs governing the macroscopic quantities, which are the particle concentration and the spatially-distributed statistics, can also be applied to the shallow water models. The governing equation of the cross-sectionally averaged particle concentration \( C_{iD} = C_{iD}(t,x) \) is deduced as
\[
\frac{\partial (AC_{iD})}{\partial t} + L_{iD}(AC_{iD}) = q_{iD}
\] (56)
with the operator \( L_{iD} \) defined for generic sufficiently regular function \( f = f(t,x) \) as
\[
L_{iD}f = \frac{\partial F(f)}{\partial x} + R_{iD}f
\] (57)
and the flux
\[
F = F(f) = V_{iD}f - \frac{\partial}{\partial x}(D_{iD}f)
\] (58)
where \( x \) is the 1-D abscissa taken along the reach, \( A \) is the wetted cross-sectional area of the channel, \( R_{iD}(\geq 0) \) is the decay coefficient, and \( q_{iD} \) represents source and sink.

The statistics equations in a reach are deduced following the procedure presented in the previous section without technical difficulties. For example, the governing equation of the escape probability \( E \) in a reach is given by
\[
\frac{\partial E}{\partial s} + L_{iD}E = 0
\] (59)
with the linear operator \( L_{iD} \) defined for generic sufficiently regular function \( f = f(s,y) \) as
\[
L_{iD} = V_{iD} \frac{\partial f}{\partial y} + D_{iD} \frac{\partial^2 f}{\partial y^2} - R_{iD}f
\] (60)
where \( y \) is the 1-D abscissa taken along the reach.

For the problems in a locally 1-D open channel network \( \Omega \), the governing equations should be understood in a weak sense so that the IBCs at junctions are consistently dealt with.
The governing equations and the IBCs are consistently formulated if they are defined through the weak formulations with appropriate test functions. For the transport equation (56), its weak formulation is given by

\[ \int_{\Omega} w \left( \frac{\partial (AC_{1D})}{\partial t} + R_{1D} AC_{1D} \right) \, dx - \int_{\Omega} \frac{\partial w}{\partial x} F(AC_{1D}) \, dx = q_{1D}^{*} \]  

with the IBC at each junction

\[ \sum_{j} F(AC_{1D}) \bigg|_{j} + \delta_{j} q_{1D} = 0 \]  

where \( w = w(x) \) is the test function in the Hilbert space \( H^{1}(\Omega) \), \( \delta_{j} \) is the Dirac’s Delta concentrated at the junction, \( J \) represents each junction in the domain \( \Omega \), and \( q_{j} \) represents the source and sink concentrated on \( J \). The integrals in the left-hand side of Eq.(61) are calculated in the reaches in \( \Omega \). The summation of the fluxes in the left-hand side of the IBC (62) is taken appropriately considering their signs. Similarly, the statistics equations can be defined in the weak form as

\[ \int_{\Omega} w \left( \frac{\partial \varphi}{\partial x} - R_{1D} \varphi + V_{1D} \varphi - q_{\varphi} \right) \, dy - \int_{\Omega} \frac{\partial (D_{1D} w)}{\partial y} \frac{\partial \varphi}{\partial y} \, dy = 0 \]  

with the IBC

\[ \sum_{j} \left[ D_{1D} \frac{\partial \varphi}{\partial y} \right]_{j} = 0 \]  

where \( \varphi \) represents a spatially-distributed statistics and \( q_{\varphi} \) is the corresponding source and sink: \( q_{\varphi} = -R \chi \) for the probability of killing \( \varphi = K \), \( q_{\varphi} = 0 \) for the escape probability \( \varphi = E \), and \( q_{\varphi} = -kM_{k-1} \) for the statistical moment of the residence time \( \varphi = M_{k} \).

4.5.2 Horizontally 2-D Model

Application of the presented modeling framework to the horizontally 2-D case is straightforward if the known and unknown functions are accordingly defined through depth-averaging procedures. Based on Eq.(20), the SDE governing the Lagrangian particle movements in a horizontally 2-D domain \( \Omega \) is deduced as

\[ dX_{t} = V_{2D}(t, X_{t}) \, dt + \sqrt{2D_{2D}(t, X_{t})} \, dB_{t} \]  

where the notations involved in Eq.(65) are analogous to those in Eq.(20) but the particle position \( X_{t} \) is understood as the horizontally 2-D projection of the 3-D model and the spatial
dimension \( n \) equals to 2. The standard Brownian motion \( \mathbf{B}_t \) in Eq.(65) is now two-dimensional. The dispersivity \( \mathbf{D}_{2D} \) quantifies the stochasticity involved in the Lagrangian particle movements in both the horizontal and vertical directions.

Analogous to the 1-D case, the governing equation of the depth-averaged particle concentration \( C_{2D} = C_{2D}(t, x_1, x_2) \) is deduced as

\[
\frac{\partial}{\partial t}(hC_{2D}) + L^*_{2D}(hC_{2D}) = q_{2D},
\]

(66)

with the linear operator \( L^*_{2D} \) defined for generic sufficiently regular \( f = f(t, x_1, x_2) \) as

\[
L^*_{2D} = \frac{\partial}{\partial x_j}(V_{i,2D}f) - \frac{\partial^2}{\partial x_i \partial x_j}(D_{i,j,2D}f) + R_{2D}f
\]

(67)

where the conserved variable in Eq.(66) is the product \( hC_{2D} \). The governing equations of the spatially distributed statistics can also be deduced in an essentially similar manner with the 3-D and 1-D models.

4.6 Validation and Parameter Estimation

This section presents validation results of the stochastic process model and identification results of its parameters using obtained data in real open channel turbulence.

4.6.1 Local Velocity

For a solute particle whose trajectory evolves according to the SDE(20), its local velocity becomes temporally singular due to the embedded irregularity of the Brownian motion. This peculiarity of the stochastic process model is addressed in this sub-section focusing on the case of passive particles. Firstly, assume that the local speed of the particle is identified with the local velocity of the water flows at each point in the domain \( \Omega \). For a deterministic dynamical system that governs Lagrangian movements of a particle where the position \( \mathbf{X}_t \) is temporally differentiable, the local velocity denoted by \( \mathbf{V} \) is defined through the classical differential form

\[
\mathbf{V} = \frac{d\mathbf{X}_t}{dt}.
\]

(68)

The ODE(68) is the definitional equation of the local velocity \( \mathbf{V} \); however, the present stochastic process model describes the position \( \mathbf{X}_t \) as a continuous but not smooth function due to
the existence of the fluctuation term, namely the second term in the right hand side of Eq.(20). Eq.(68) cannot therefore be directly used in the present case. As an alternative to Eq.(68), a small variation of \( \mathbf{X}_t \) over a finite time increment \( \Delta t \) with the temporally integrated SDE

\[
\mathbf{X}_t = \mathbf{X}_0 + \int_0^t V(s, \mathbf{X}_s)ds + \int_0^t \sqrt{2D(s, \mathbf{X}_s)}dB_s
\]

(69)
is considered, which gives the definitional equation of the observed velocity \( \mathbf{V}^{obs,\Delta t}_t \) with \( \Delta t \) as

\[
\mathbf{V}^{obs,\Delta t}_t = \frac{\mathbf{X}^{t+\Delta t}_t - \mathbf{X}^t_t}{\Delta t} = \frac{1}{\Delta t} \int_t^{t+\Delta t} V(s, \mathbf{X}_s)ds + \frac{1}{\Delta t} \int_t^{t+\Delta t} \sqrt{2D(s, \mathbf{X}_s)}dB_s.
\]

(70)

In case that the flow regime is temporally homogeneous and turbulent in particular, the observed velocity \( \mathbf{V}^{obs,\Delta t}_t \) at a fixed point can be approximated as

\[
\mathbf{V}^{obs,\Delta t}_t = \mathbf{V} + \frac{1}{\Delta t} \sqrt{2D} \mathbf{B}_t
\]

(71)

where \( \Delta \mathbf{B}_t = \mathbf{B}_{t+\Delta t} - \mathbf{B}_t \) is the increment of the standard Brownian motion. According to Eq.(71), the process \( \mathbf{V}^{obs,\Delta t}_t - \mathbf{V} \) is governed by the \( n \) -dimensional normal distribution \( \mathcal{N}(\mathbf{0}_n, \Sigma) \) where \( \mathbf{0}_n \) is the \( n \) -dimensional zero vector and

\[
\Sigma = \frac{1}{2\Delta t} D
\]

(72)
is the \( n \times n \) -dimensional covariance matrix. The distribution of the stochastic process \( \mathbf{V}^{obs,\Delta t}_t - \mathbf{V} \) thus depends explicitly on the time increment \( \Delta t \) and has a PDF given as a Gaussian kernel. The assumption that the process \( \mathbf{V}^{obs,\Delta t}_t - \mathbf{V} \) is governed by a Gaussian PDF is appropriate in fully-developed turbulent flows (Nikora et al., 2007). If the covariance matrix of the process \( \mathbf{V}^{obs,\Delta t}_t - \mathbf{V} \) is derived with a sufficient long time series data of local velocity using a regular interval \( \Delta t \), the value turbulent diffusion coefficient \( D \) can be estimated at each fixed point in the flow as

\[
D = \frac{\Delta t}{2} \left( \mathbf{E} \left[ \mathbf{V}^{obs,\Delta t}_t \left( \mathbf{V}^{obs,\Delta t}_t \right)^T - \mathbf{VV}^T \right] \right)
\]

(73)

where \( \mathbf{E}[\phi] \) represents the expectation of a function \( \phi \) related to the variable \( \mathbf{V}^{obs,\Delta t}_t - \mathbf{V} \). For a temporally homogeneous turbulent flow, \( D \) should be a spatially distributed finite value. In order to keep \( D \) finite, the process \( \mathbf{V}^{obs,\Delta t}_t \) has to satisfy the conditions

\[
\lim_{\Delta t \to 0} \Delta t \mathbf{E} \left[ \mathbf{V}^{obs,\Delta t}_t \left( \mathbf{V}^{obs,\Delta t}_t \right)^T \right] > 0
\]

(74)

and

\[
\lim_{\Delta t \to 0} \mathbf{E} \left[ \mathbf{V}^{obs,\Delta t}_t \left( \mathbf{V}^{obs,\Delta t}_t \right)^T \right] = +\infty.
\]

(75)
Using time series data obtained from velocity measurements in agricultural drainage canals at several observation stations, whether the observed local velocity has the Gaussian property is assessed using the KS test. Then, the turbulent diffusion coefficient is estimated using Eq.(73).

### 4.6.2 Model Validation

#### 4.6.2.1 Flow Velocity Measurement

Flow velocity measurements in two agricultural drainage canals in Japan, which are referred to as U-canal and K-canal, were performed in order to obtain time series data of local velocity in actual turbulent flows. Both of the drainage canals are uniformly rectangular-shaped open channels made from of concrete. Flow regimes observed in the canals were steady shallow water flows. The local velocity seen in both canals is thus assumed to be a stationary stochastic process. Incidentally, significant shock waves were observed on the water surface at K-canal arising from a bending of the canal upstream from the observation points of local velocities.

**Table 4-1** provides the water depth $H$, the channel width $B$, and the channel slope $S_0$ measured in each drainage canal. As a spatial coordinate system, an $x\cdot y\cdot z$ three-dimensional orthogonal coordinate system is employed in this sub-section. The $x$-direction is taken parallel to the flow direction, $y$-direction is perpendicular to the flow direction, and $z$-direction is vertical, orthogonal to the bed of canal.

The velocity measurements were conducted at six observation stations. Four of them are located in U-canal (U-1 through U-4) and the other two are in K-canal (K-1 and K-2). **Table 4.2** shows the locations of the stations within their cross-sections ($y\cdot z$ plane), which are represented as the transverse directional distance from the center of the canal $d$, and height from the lowest point of the canal bed $h$. These are also shown graphically in **Figure 4-1. Figures 4-2 and 4-3** graphically show cross-sectional locations of the observation stations at U-canal and K-canal. As indicated in **Table 4-3**, only station U-4 is located at near the sidewall of the canal and the remaining stations are located near the (horizontal) center. The time series of local velocity was measured by three-dimensional acoustic Doppler velocimeters. Different velocimeters were used in the different canals for technical reasons, (Nortek Vectrino and Nortek Vector for U-canal and for K-canal, respectively). **Table 4-3** shows the fixed discrete interval
$\Delta t$, the total number of observations $N$, and the total observation period $t_f$ for each canal.

The observed local velocity is taken as the process $V_{\text{obs}}^t$. Table 4-4 provides the deterministic local velocity observed at each observation station. Subscripts 1, 2, and 3 represent the $x$, $y$ and $z$-directional components, respectively. As shown in the table, the deterministic local velocities at K-canal are faster than those of U-canal. The observed time series of the local velocity at the station U-1 is plotted in Figure 4-4. From a qualitative point of view, the ranges of fluctuation of the local velocity is significantly larger horizontally ($x$- and $y$-directions) than vertically ($z$-direction). Similar tendency was observed in the time series derived at the other stations.

![Figure 4-1 Sketch of the parameters $h$ and $d$ defining the location of a station in the cross-section of canal.](image1)

![Figure 4-2 Locations of the four observation stations in U-canal.](image2)

![Figure 4-3 Locations of the four observation stations in K-canal.](image3)
Figure 4-4 The observed time series of the local velocity in U-canal (Station U-1).

<table>
<thead>
<tr>
<th>Canal</th>
<th>$H$ (m)</th>
<th>$B$ (m)</th>
<th>$S_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>0.14</td>
<td>0.69</td>
<td>$9.0 \times 10^{-3}$</td>
</tr>
<tr>
<td>K</td>
<td>0.32</td>
<td>0.70</td>
<td>$8.0 \times 10^{-4}$</td>
</tr>
</tbody>
</table>

Table 4-2 Cross-sectional locations of the stations.

<table>
<thead>
<tr>
<th>Station</th>
<th>$d$ (cm)</th>
<th>$h$ (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>U-1</td>
<td>2.8</td>
<td>5.9</td>
</tr>
<tr>
<td>U-2</td>
<td>2.8</td>
<td>4.0</td>
</tr>
<tr>
<td>U-3</td>
<td>2.8</td>
<td>0.55</td>
</tr>
<tr>
<td>U-4</td>
<td>33</td>
<td>2.5</td>
</tr>
<tr>
<td>K-1</td>
<td>1.2</td>
<td>5.3</td>
</tr>
<tr>
<td>K-2</td>
<td>0.0</td>
<td>5.3</td>
</tr>
</tbody>
</table>

Table 4-3 Observation parameters.

<table>
<thead>
<tr>
<th>Canal</th>
<th>$\Delta t$ (s)</th>
<th>$N$</th>
<th>$t_f$ (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>U</td>
<td>0.04</td>
<td>1,500</td>
<td>60</td>
</tr>
<tr>
<td>K</td>
<td>0.015625</td>
<td>7,680</td>
<td>120</td>
</tr>
</tbody>
</table>
Table 4-4 Identified deterministic velocities $\vec{V}$ (m/s).

<table>
<thead>
<tr>
<th>Station</th>
<th>$\vec{V}_1$</th>
<th>$\vec{V}_2$</th>
<th>$\vec{V}_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>U-1</td>
<td>0.29</td>
<td>-0.01</td>
<td>0.00</td>
</tr>
<tr>
<td>U-2</td>
<td>0.28</td>
<td>-0.01</td>
<td>0.00</td>
</tr>
<tr>
<td>U-3</td>
<td>0.24</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>U-4</td>
<td>0.12</td>
<td>-0.02</td>
<td>0.02</td>
</tr>
<tr>
<td>K-1</td>
<td>0.93</td>
<td>0.01</td>
<td>0.01</td>
</tr>
<tr>
<td>K-2</td>
<td>0.95</td>
<td>-0.01</td>
<td>-0.01</td>
</tr>
</tbody>
</table>

4.6.2.2 Kolmogorov-Smirnov (KS) Test

The fundamental assumption in the stochastic process model that the time series of the observed local velocity are consistent with the Gaussian distributions should be assessed with a statistical test. Taking this assumption as the null hypothesis, the KS test is applied to the observed data. The KS test has been used when judging whether obtained samples are governed by a certain PDF (Kloeden and Platen, 1999). Since the process $V_i^{\text{obs}} - \vec{V}$ is assumed to be Gaussian, the KS test is suitable for this validation. The KS test is applied to the three-dimensional discrete time series of the local velocity observed at each station.

The results of the KS test with regard to 1% and 5% significance levels are given in Table 4-5. Only the case U-4- $y$ at the 5% significance level is rejected for the time series observed in U-canal. This rejection may be because station U-4 is located near a sidewall of the canal, where the nonlinearity of the turbulence fluctuations cannot be neglected. However, U-4- $y$ at the 1% significance level is accepted, and so there is no discrepancy with the null hypothesis for the time series observed in U-canal. By contrast, for the results of the KS test applied to the data obtained in K-canal, the two cases K-1- $y$ and K-2- $y$ are rejected at both significance levels. This can be attributed to the existence of shock waves on the water surface and the lack of resolution power of the velocimeter employed in K-canal. In summary, with a few exceptions, based on the flow regimes observed at the two canals, it can be concluded that the stochastic process $V_i^{\text{obs}} - \vec{V}$ approximately follows a Gaussian distribution.
Table 4-5 Results of the Kolmogorov-Smirnov test.

<table>
<thead>
<tr>
<th>Case</th>
<th>Significance level 1 %</th>
<th>Significance level 5 %</th>
</tr>
</thead>
<tbody>
<tr>
<td>U-1- x</td>
<td>Accepted</td>
<td>Accepted</td>
</tr>
<tr>
<td>U-1- y</td>
<td>Accepted</td>
<td>Accepted</td>
</tr>
<tr>
<td>U-1- z</td>
<td>Accepted</td>
<td>Accepted</td>
</tr>
<tr>
<td>U-2- x</td>
<td>Accepted</td>
<td>Accepted</td>
</tr>
<tr>
<td>U-2- y</td>
<td>Accepted</td>
<td>Accepted</td>
</tr>
<tr>
<td>U-2- z</td>
<td>Accepted</td>
<td>Accepted</td>
</tr>
<tr>
<td>U-3- x</td>
<td>Accepted</td>
<td>Accepted</td>
</tr>
<tr>
<td>U-3- y</td>
<td>Accepted</td>
<td>Accepted</td>
</tr>
<tr>
<td>U-3- z</td>
<td>Accepted</td>
<td>Accepted</td>
</tr>
<tr>
<td>U-4- x</td>
<td>Accepted</td>
<td>Accepted</td>
</tr>
<tr>
<td>U-4- y</td>
<td>Rejected</td>
<td>Accepted</td>
</tr>
<tr>
<td>U-4- z</td>
<td>Accepted</td>
<td>Accepted</td>
</tr>
<tr>
<td>K-1- x</td>
<td>Accepted</td>
<td>Accepted</td>
</tr>
<tr>
<td>K-1- y</td>
<td>Rejected</td>
<td>Rejected</td>
</tr>
<tr>
<td>K-1- z</td>
<td>Accepted</td>
<td>Accepted</td>
</tr>
<tr>
<td>K-2- x</td>
<td>Accepted</td>
<td>Accepted</td>
</tr>
<tr>
<td>K-2- y</td>
<td>Rejected</td>
<td>Rejected</td>
</tr>
<tr>
<td>K-2- z</td>
<td>Accepted</td>
<td>Accepted</td>
</tr>
</tbody>
</table>

4.6.2.3 Turbulent Diffusion Coefficients

Using the time series of local velocity observed at each station, estimated values of the turbulent diffusion coefficient $D$ are calculated based on Eq.(73). Table 4-6 shows the calculation results of the coefficient $D$. As shown in the table, the diagonal elements of $D$ are approximately 10-fold larger than the non-diagonal elements. Among the diagonal elements, the horizontal elements are considerably larger than the vertical elements. This follows from the fact that the flow regimes observed in the drainage canals are horizontally dominant turbulent flows. The absolute values of the elements of $D$ are smaller than the previous estimates (Singh and Beck, 2003; Suh et al., 2009). This difference is considered to be due to $D$ being treated as a local value in the present study. In addition, the resolution capability of velocimeters is considered to have significant influenced on the results. Further investigations will be necessary for
clarifying the reason for the disparity.

Table 4-6 Estimated value of each entry of the dispersivity \( \mathbf{D} \) \( (m^2/s) \).

<table>
<thead>
<tr>
<th>Station</th>
<th>( D_{11} )</th>
<th>( D_{22} )</th>
<th>( D_{33} )</th>
<th>( D_{12} )</th>
<th>( D_{23} )</th>
<th>( D_{31} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>U-1</td>
<td>( 2.58 \times 10^{-5} )</td>
<td>( 1.63 \times 10^{-5} )</td>
<td>( 8.57 \times 10^{-6} )</td>
<td>( -1.92 \times 10^{-6} )</td>
<td>( -1.35 \times 10^{-6} )</td>
<td>( -2.45 \times 10^{-6} )</td>
</tr>
<tr>
<td>U-2</td>
<td>( 2.30 \times 10^{-5} )</td>
<td>( 1.66 \times 10^{-5} )</td>
<td>( 5.65 \times 10^{-6} )</td>
<td>( -8.00 \times 10^{-7} )</td>
<td>( -7.81 \times 10^{-7} )</td>
<td>( -2.09 \times 10^{-6} )</td>
</tr>
<tr>
<td>U-3</td>
<td>( 3.38 \times 10^{-5} )</td>
<td>( 2.06 \times 10^{-5} )</td>
<td>( 4.86 \times 10^{-6} )</td>
<td>( -2.45 \times 10^{-6} )</td>
<td>( -9.29 \times 10^{-7} )</td>
<td>( -4.60 \times 10^{-6} )</td>
</tr>
<tr>
<td>U-4</td>
<td>( 2.28 \times 10^{-5} )</td>
<td>( 1.45 \times 10^{-5} )</td>
<td>( 8.17 \times 10^{-6} )</td>
<td>( -5.60 \times 10^{-6} )</td>
<td>( -3.63 \times 10^{-7} )</td>
<td>( -3.14 \times 10^{-6} )</td>
</tr>
<tr>
<td>K-1</td>
<td>( 1.97 \times 10^{-4} )</td>
<td>( 1.51 \times 10^{-4} )</td>
<td>( 6.10 \times 10^{-5} )</td>
<td>( 2.03 \times 10^{-5} )</td>
<td>( -7.40 \times 10^{-7} )</td>
<td>( -3.20 \times 10^{-5} )</td>
</tr>
<tr>
<td>K-2</td>
<td>( 1.88 \times 10^{-4} )</td>
<td>( 1.36 \times 10^{-4} )</td>
<td>( 5.38 \times 10^{-5} )</td>
<td>( 1.84 \times 10^{-5} )</td>
<td>( 2.48 \times 10^{-7} )</td>
<td>( -2.42 \times 10^{-5} )</td>
</tr>
</tbody>
</table>

4.7 Conclusions

A stochastic process model for transport phenomena in turbulent surface water bodies was presented. The governing equations of the macroscopic quantities deduced from the KFE and KBE associated with the SDE serve as analytical tools for the transport phenomena with effectively and consistently evaluating the stochasticity involved in the phenomena that the deterministic models cannot deal with. In addition, the presented stochastic process model gets rid of the empirical Fick’s law. Extensions of the present stochastic process model to the shallow water models were finally presented, which would potentially lead to efficient tools in applications to a variety of hydro-environmental problems.
CHAPTER 5  Numerical Scheme for the Extended 1-D Shallow Water Equations

5.1 Introduction

The 1-D shallow water equations (1-D SWEs) are hyperbolic systems of nonlinear partial differential equations (PDEs) having source terms, which govern mass and momentum transport phenomena in open channels where the assumptions of the incompressibility and hydrostatic pressure distribution are well established. The 1-D SWEs are effective mathematical models for simulating the flows with dominant flow directions as encountered in river reaches and agricultural drainage canals. Kocaman and Ozmen-Cagatay (2015) showed that, although not as accurate as the 3-D hydrodynamic model, the 1-D SWEs reasonably simulate dam break wave propagations with reflections occurring in closed open channels. Analytical solutions to the 1-D SWEs are available only for simplified cases such as the flows in frictionless straight open channels with simple cross-sectional shapes (Stoker, 1957; Chen, 2011; Delestre et al., 2012). Numerical schemes have therefore been used to approximate their solutions in applications.

Developing accurate, stable, and efficient numerical schemes for the 1-D SWEs has been one of the key hydro-environmental research topics. Most of the existing numerical schemes for the 1-D SWEs use the finite volume method (FVM) equipped with the Riemann-solvers, which are numerical schemes based on exact or linearized solutions to the Riemann problems (Toro and García-Navarro, 2007). Related numerical schemes, such as the finite element method (FEM) (Hanert et al., 2005) and the discontinuous Galerkin FEM (Dumbser and Casulli, 2013), have also been developed. Some researchers recently found that simpler but sufficiently accurate numerical schemes could be developed without the Riemann-solvers and high-resolution algorithms (Chen et al., 2007; Catella et al., 2008; Magdalena et al., 2015; Cea and Blade, 2015). Wood and Wang (2015) simulated dam break problems in an experimental open channel having bends using an alternating-direction implicit FD scheme with curvilinear coordinates. Appadu (2012a, 2012b) proposed linear FD schemes for solving nonlinear conservation laws with minimized dispersion and dissipation based on the Lax-Friedrichs scheme.

For simulating shallow water flows in mountainous areas in particular, the 1-D mathe-
matical models more effectively perform than the 2-D models due to the narrow and steep topographies of watercourses that justify using the 1-D physical descriptions (Petaccia and Natale 2013; Petaccia et al., 2013). However, numerically solving the 1-D SWEs under such cases would encounter many challenging computational issues, such as irregular cross-sectional shape, steep channel slope, bend, junction, wet and dry interface, shock, and depression. Versatility is therefore a necessary property to be equipped with numerical schemes for solving the 1-D SWEs. Among these computational issues, the most critical ones are bends and junctions where appropriate internal boundary conditions (IBCs) have to be specified for to well-posing the problem. In addition, the IBCs should be physically consistent such that momentum losses at bends and junctions are appropriately handled. In addition, they should be efficiently implemented in numerical computation. So far, only a few researches have focused on the mathematical development and numerical application of such IBCs.

The main purposes of this chapter are to present a new numerical scheme for approximating the solutions to the extended 1-D SWEs in open channel networks, which is referred to as the Dual-Finite Volume for Flow (DFVF) scheme. Spatial discretization procedure of the DFVF scheme is based on the 1-D counterpart of Voronoi diagram (Mishev, 1998). Application of the Voronoi diagram to the extended 1-D SWEs has not been presented in the literatures. Water surface elevation and discharge are taken as the unknowns arranged in a staggered manner. A non-upwind, vertex-centered FVM and an upwind cell-centered FVM with a semi-implicit treatment of the friction slope term are applied to the continuity equation and to the momentum equation, respectively. The DFVF scheme is verified with a number of numerical tests in order to determine its computational accuracy. Finally, the DFVF scheme is applied to numerical simulation of catastrophic flash flooding that result from a recent earthquake-induced dam failure in Japan.

The remainder of this chapter is organized as follows. The extended 1-D SWEs are presented in section 5.2. The numerical formulation of the DFVF scheme is presented in section 5.3. The DFVF scheme is verified through a number of test problems in section 5.4. The DFVF scheme is applied to the numerical simulation of a recent flash flood caused by an earthquake-induced complete dam failure in Japan in section 5.5. Section 5.6 provides conclu-
sions of this chapter.

5.2 Extended 1-D Shallow Water Equations

The conventional 1-D SWEs for free surface water flows along single channels with arbitrary cross-sectional shapes consist of the continuity equation (Unami and Alam, 2012)

\[
\frac{\partial A(\eta)}{\partial t} + \frac{\partial Q}{\partial x} - q = 0
\]  \hspace{1cm} (76)

and the momentum equation

\[
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial x}\left(\frac{\rho Q^2}{A}\right) = -gA\frac{\partial \eta}{\partial x} - gAS_t
\]  \hspace{1cm} (77)

where \( t \) is the time, \( x \) indicates the abscissa, \( A \) is the wetted cross-sectional area as a function of the water surface elevation \( \eta \), \( Q \) is the discharge, \( q \) is the lateral inflow per unit length of the channel with negligible momentum, \( \beta(\geq 0) \) is the momentum correction coefficient, \( g \) is the gravitational acceleration, and \( S_t \) is the friction slope given by Manning’s formula

\[
S_t = \frac{n^2 \sigma^{4/3} Q |Q|}{A^{1/3}}
\]  \hspace{1cm} (78)

where \( n \) is Manning’s coefficient and \( \sigma \) is the wetted perimeter of cross-section. The channel bed elevation is assumed to be fixed.

The conventional SWEs, which are given by Eqs.(76) and (77), can be applied to arbitrary single channels. However, for the flows in channel networks, Eq.(76) shall be understood as the integral form

\[
\int_{\Omega \cap B(y,r)} T \frac{\partial \eta}{\partial t} \, dx + \sum_{j=1}^{\nu} \sigma_j Q_{r,j} + \sum Q_{b} = \int_{\Omega \cap B(y,r)} q \, dx
\]  \hspace{1cm} (79)

where \( \Omega \) represents the domain of the flow as a locally one-dimensional open-channel network, \( T = \frac{\partial A}{\partial \eta} \) is the top width of the water surface, \( B(y,r) \) is the horizontally two-dimensional \( r \)-neighborhood of a point \( y \in \Omega \) with \( r > 0 \) such that no vertex falls on the boundary of \( B(y,r) \), \( \nu \) is the total number of intersections of the reaches and the boundary of \( B(y,r) \), \( Q_{r,j} \) is the discharge at the \( j \)th intersection, \( \sum Q_{b} \) represents the sum of the discharges specified at the boundary vertices in \( \Omega \cap B(y,r) \), and \( \sigma_j \) is the sign parameter defining the direction of the abscissa in the \( j \)th reach. Here, \( \sigma_j \) is equal to 1.
when the abscissa in the $j$th reach is facing outward to the boundary of $B(y,r)$ and is otherwise equal to $-1$. A sketch of $B(y,r)$ is shown in Figure 5-1. Note that $B(y,r)$ in (79) can be replaced by a sufficiently small compact 2-D set contained in $B(y,r')$ for arbitrary $0 < r' < r$. Eq.(79) is referred to as the extended continuity equation. Application of the integral form (79) in single channels recovers the conventional form (76) and the system of PDEs consisting of Eqs.(79) and (77) as the extended 1-D SWEs. The momentum equation is equipped with one of the momentum flux evaluation schemes, which are presented in the previous chapter (Chapter 3).

![Diagram](image)

Figure 5-1 A sketch of $B(y,r)$ for defining Eq.(79). (same with Figure 3-1)

### 5.3 Dual Finite Volume (DFVF) Scheme

Spatial and temporal discretization procedures of a numerical scheme for approximating solutions to the extended 1-D SWEs is presented in this section.

#### 5.3.1 Computational Mesh

The extended 1-D SWEs comprising the extended continuity equation (79) and the momentum equation (77) are numerically solved using the DFVF scheme, utilizing the water surface elevation $\eta$ and the discharge $Q$ as the unknowns to be computed. A couple of staggered computational meshes, which are referred to as a regular mesh and a dual mesh, are used in the DFVF scheme. The extended continuity equation (79) is numerically solved on the dual mesh, while the momentum equation (77) on the regular mesh to avoid conflicting numbers of
equations and unknowns.

Firstly, a computational domain \( \Omega \), which is a connected graph, is specified. The domain \( \Omega \) is then divided into a regular mesh that consists of regular cells bounded by two nodes, so that any vertex, such as a junction, falls on one of the nodes. The regular cells and the nodes are indexed with the natural numbers. The total numbers of regular cells and nodes are denoted by \( N_c \) and \( N_n \), respectively. The \( i \)th node is denoted by \( P_i \) with its \( x \) abscissa \( x_i \).

The \( k \)th regular cell is denoted by \( \Omega_k \). The length of \( \Omega_k \) is represented by \( l_k \). The two nodes bounding both sides of \( \Omega_k \) are denoted by the \( \varphi(k,1) \)th node \( P_{\varphi(k,1)} \) and the \( \varphi(k,2) \)th node \( P_{\varphi(k,2)} \). The \( x \) abscissa in the cell \( \Omega_k \) is directed from the node \( P_{\varphi(k,1)} \) to the node \( P_{\varphi(k,2)} \). The number of regular cells connected to the node \( P_i \) is denoted by \( \nu(i) \). The \( j \)th regular cell connected to the node \( P_i \) is referred to as the \( \kappa(i,j) \)th regular cell \( \Omega_{\kappa(i,j)} \).

There are two nodes that bound the cell \( \Omega_{\kappa(i,j)} \); one is the \( i \)th node \( P_i \) and the other is referred to as the \( \mu(i,j) \)th node \( P_{\mu(i,j)} \). In the regular cell \( \Omega_{\kappa(i,j)} \), the direction of the abscissa is identified with the parameter \( \sigma_{i,j} \), which is equal to 1 when \( x \) is directed to the node \( P_i \), and is otherwise equal to \(-1\). A dual mesh is generated from the regular mesh. Following the multi-dimensional analogue (Mishev, 1998), the \( i \)th dual cell \( S_i \) is associated with the node \( P_i \) and is defined as

\[
S_i = \left\{ x \mid |x_i - x| < |x_{\mu(i,j)} - x| \text{ for } 1 \leq j \leq \nu(i) \right\}.
\]

The dual mesh consists of \( N_n \) dual cells. The cell interface between \( S_i \) and \( S_{\mu(i,j)} \) is denoted by \( \Gamma_{i,j} \). Figure 5-2 shows a sketch of the computational mesh. The water surface elevation \( \eta \) is attributed to the dual cells, and the discharge \( Q \) is attributed to the regular cells.

The discretized \( \eta \) in \( S_i \) is denoted as \( \eta_i \), and the discretized \( Q \) in \( \Omega_k \) is denoted as \( Q_k \).

The model parameters \( n \) and \( \beta \) are attributed to the regular cells, and model parameters in \( \Omega_k \) are denoted as \( n_k \) and \( \beta_k \).

![Figure 5-2 Sketch of the computational mesh: (a) regular mesh, and (b) dual mesh.](image-url)
5.3.2 Extended Continuity Equation

The extended continuity equation (79) is discretized on a dual mesh. Replacing \( B(y,r) \) with \( S_i \) yields the cell-vertex finite volume formulation of the continuity equation in \( S_i \), as follows:

\[
\int_{S_i} T \frac{\partial \eta}{\partial t} \, dx + \sum_{j=a}^{v(i)} \sigma_{i,j} Q_{t,i,j} = \int_{S_i} q \, dx . \tag{81}
\]

Substituting \( Q_{t,i,j} = Q_{s(i,j)} \) into Eq.(81) yields

\[
\int_{S_i} T \frac{\partial \eta}{\partial t} \, dx + \sum_{j=a}^{v(i)} \sigma_{i,j} Q_{s(i,j)} = \int_{S_i} q \, dx . \tag{82}
\]

Assuming a linear variation of the top width \( T \) in the cell \( \Omega_{s(i,j)} \) yields

\[
T = T_{s(j,0)} + \frac{x-x_i}{\sigma_{s(j)}} \bigg( T_{s(j,1)} - T_{s(j,0)} \bigg) \tag{83}
\]

where \( T_{s(j,0)} \) and \( T_{s(j,1)} \) are the values of \( T \) at the nodes \( P_i \) and \( P_{a(i,j)} \), respectively. According to Eq.(83), the first term on the left-hand side of Eq.(82) is evaluated as

\[
\int_{S_i} T \frac{\partial \eta}{\partial t} \, dx = \frac{d\eta}{dr} \int_{S_i} T \, dx \\
= \frac{d\eta}{dr} \int_{S_i} \bigg( T_{s(j,0)} + \frac{x-x_i}{\sigma_{s(j)}} \bigg( T_{s(j,1)} - T_{s(j,0)} \bigg) \bigg) \, dx \\
= \left( \frac{1}{8} \sum_{j=a}^{v(i)} I_{s(i,j)} \big( 3T_{s(j,0)} + T_{s(j,1)} \big) \right) \frac{d\eta}{dr} . \tag{84}
\]

Each \( \frac{d\eta}{dr} \) is thus computed as

\[
\frac{d\eta}{dr} = \left( -\sum_{j=a}^{v(i)} \sigma_{i,j} Q_{s(i,j)} + \sum q \right) \left( \frac{1}{8} \sum_{j=a}^{v(i)} I_{s(i,j)} \big( 3T_{s(j,0)} + T_{s(j,1)} \big) \right)^{-1} \tag{85}
\]

where \( \sum q \) represents the discharge contributed from the lateral inflows and the boundary conditions.

5.3.3 Momentum Equation

The upwind cell-centered FVM (Unami and Alam, 2012) equipped with a momentum flux evaluation scheme is applied to the momentum equation (77). A semi-implicit discretization method is used to evaluate the friction slope term for avoiding the numerical instability due to the extremely shallow water depth. The cell-centered finite volume formulation of the momentum equation in the cell \( \Omega_i \) leads to
\[ I_k \frac{d\Omega_k}{dt} + [F]_{\partial \Omega_k} = \int_{\Omega_k} gA \left( -\frac{\partial \eta}{\partial x} - S_i \right) dx \]  

(86)

where \( \partial \Omega_k \) represents the interface of \( \Omega_k \) and \( F = \frac{\beta Q^2}{A} \) is the momentum flux to be evaluated on \( \partial \Omega_k \). Each term of Eq.(86) is discretized in the following subsections.

Flux evaluation for Eq.(86) is performed by an upwind spatial discretization. For each generic regular cell \( \Omega_k \) at any time \( t \), the node of the two nodes bounding \( \Omega_k \), to which the flow is directed is referred to as the downstream node, and the other is referred to as the upstream node. The vector starting from the upstream node and ending at the downstream node is denoted by \( \chi_k \). The cell flux \( F_k \) for the regular cell \( \Omega_k \) is determined using the local Froude number as a weight. When the downstream node in \( \Omega_k \) is wet, \( A_{k,\text{DS}} \geq \varepsilon \) for a small threshold value \( \varepsilon \), and the cell cross-sectional area \( A_k \) and the cell cross-sectionally averaged velocity \( V_k \) are calculated as

\[ A_k = (1 - \omega_k) A_{k,\text{DS}} + \omega_k A_{k,\text{US}} \]  

(87)

and

\[ V_k = \frac{Q}{A_k} \]  

(88)

with the weight

\[ \omega_k = \max \left( 1 - \frac{1}{F_{\chi_k}^2}, 0 \right) \]  

(89)

where \( F_{\chi_k}^2 \) is the square of local Froude number defined as

\[ F_{\chi_k}^2 = \frac{\beta_k T_{k,\text{DS}} Q_k^2}{g A_{k,\text{DS}}^2} \]  

(90)

and the subscripts US and DS indicate values at the upstream node and the downstream node, respectively, in \( \Omega_k \). The cell velocity \( V_k \) is taken to be 0 when \( A_{k,\text{DS}} < \varepsilon \). Then, \( \varepsilon \) is taken to be sufficiently small because of its influence on wet and dry interfaces, as discussed in Xia et al. (2010). Finally, the cell flux \( F_k \) is determined as follows:

\[ F_k = \beta_k Q_k V_k. \]  

(91)

The momentum flux \( F \) on the cell interface \( \partial \Omega_k \) is evaluated with one of the momentum evaluation schemes presented in chapter 3 of this thesis. The set of indices of the regular cells, the downstream nodes of which fall on the upstream node of \( \Omega_k \), is denoted as \( U_k \). The flux at the upstream cell interface \( \partial \Omega_{k,\text{US}} \subset \partial \Omega_k \) is prescribed as
\[ F_{\Omega_{k,ls}} = \sum_{i \in \Omega_{k}} w_i F_i \]  

(92)

where \( w_i \) is the coefficient determined according to Eq.(13). On the other hand, the momentum flux at the downstream cell interface \( \partial \Omega_{k,ds} \subset \partial \Omega_{k} \) is identical to the cell flux

\[ F_{\Omega_{k,ds}} = F_k \]  

(93)

with the exception that

\[ F_{\Omega_{k,ds}} = F_k - w_c F_k \]  

(94)

when the downstream node is connected to exactly two regular cells. Finally, the second term of the left-hand side of Eq.(86) is evaluated based on \( F_{\Omega_{k,ls}} \) and \( F_{\Omega_{k,ds}} \) while considering the directions of the flows as well as that of the \( x \) abscissa. The above flux evaluation method has been verified with experimental dam breaks in bend channels (Ishida et al., 2012) and numerical simulations of very shallow surface water flows in an existing open channel network having a number of loops and bends (Unami and Alam, 2012).

The source terms are discretized with an upwind method. The first term of the right-hand side of Eq.(86) is evaluated as

\[ \int_{\Omega_{k}} gA \left( -\frac{\partial \eta}{\partial x} \right) dx = -g l_k \bar{A}_k \left( \eta_{\rho(i,2)} - \eta_{\rho(i,1)} \right) \]  

(95)

where

\[ \bar{A}_k = \begin{cases} A_{k,us} & \left( F_k^2 \geq 1 \right) \\
\frac{A_{k,\rho(i,2)} + A_{k,\rho(i,1)}}{2} & \left( F_k^2 < 1 \right) \end{cases} \]  

(96)

Next, the friction slope term is discretized in a semi-implicit manner (Tseng, 1999) as

\[ \int_{\Omega_{k}} gA (-S_i) dx = -g l_k \bar{A}_k \left( S_{i,k} + \frac{1}{2} J_{S_{i,k}} M \frac{dQ_i}{dt} \right) \]  

(97)

with

\[ S_{i,k} = \frac{n_{i}^2 \sigma_k^{1/3} Q_i}{A_i^{10/3}} \]  

(98)

and

\[ \bar{\sigma}_k = \begin{cases} \sigma_{k,us} & \left( F_k^2 \geq 1 \right) \\
\frac{\sigma_{k,\rho(i,1)} + \sigma_{k,\rho(i,2)}}{2} & \left( F_k^2 < 1 \right) \end{cases} \]  

(99)

where \( J_{S_{i,k}} \) is the Jacobian of \( S_{i,k} \) with respect to \( Q_i \) defined as
\[
J_{s,k} = \frac{2n_{k,1}^2 |Q| \bar{P}_k^{4/3}}{A_k^{10/3}}.
\]

However, \( S_t \) is taken as 0 when \( A_k < \varepsilon \). Finally, each \( \frac{dQ_k}{dt} \) is explicitly computed as

\[
\frac{dQ_k}{dt} = \frac{1}{l_k \left( 1 + \frac{1}{2} g \bar{A}_k J_{s,k} \Delta t \right)} \left( -[F]_{\alpha_k} - g \bar{A}_k \left( \eta_{\alpha(k+1)} - \eta_{\alpha(k-1)} + l_k S_{t,k} \right) \right). \tag{101}
\]

### 5.3.4 Temporal Integration

Application of the DFVF scheme to the 1-D SWEs finally yields a system of ordinary differential equations (ODEs) for the water surface elevation \( \eta \) attributed to the dual cells and the discharge \( Q \) attributed to the regular cells. Boundary conditions are specified at the boundary vertices when necessary, and the system of ODEs is temporally integrated from a prescribed initial condition using the fourth-order Runge-Kutta method. The time increment \( \Delta t \) is chosen to be sufficiently small, so that the Courant-Friedrichs-Lewy condition (Toro, 2001) is satisfied. The DFVF scheme is fully explicit in time and does not necessitate matrix inversion algorithm, which on the other hand is required in the FEV schemes (Ishida et al., 2011; Ishida et al., 2011; Unami and Alam, 2012). The DFVF scheme is therefore considered to be superior to the FEV schemes in terms of computational efficiency. Indeed, preliminary computations for transient problems showed that the DFVF scheme is two to three times faster than the FEV schemes. Much faster computation time can be achieved if a lower order time integration method, such as the forward Euler method and the second-order Runge-Kutta method, is employed. Moreover, the DFVF scheme is simpler than the conventional ones because it has a compact stencil in spatial discretization and does not depend on any high-resolution algorithms. Note that the DFVF scheme exactly preserves the water at rest level because it does not involve water surface reconstruction.

### 5.4 Applications to Test Cases

Numerical tests in straight open channels containing a tidal wave problem and a series of dam break problems, which have served as important benchmarks (Garcia-Navarro et al., 1999; Vázquez-Cendón, 1999; Zoppou and Robarts, 2003), are carried out to determine accuracy of
the DFVF scheme. Throughout this section, the momentum flux coefficient \( \beta \) is fixed to 1.0 in every cross-section, and the threshold vale \( \varepsilon \) is set to be \( 1.0 \times 10^{-6} \) (m\(^3\)): a significantly smaller value than the characteristic water depths of the problems considered in this sub-section.

### 5.4.1 Tidal Wave Problem in an Irregular Channel

The DFVF scheme is applied to a tidal wave problem for examining its capability of handling flows with non-prismatic cross-sections and a non-flat channel bottom. A 1,500 (m) long non-prismatic rectangular, frictionless channel is considered as the domain \( \Omega = (0,1500) \) (m).

Width and bed elevation of the channel is taken from Vázquez-Cendón (1999), both of which are continuous but highly irregular. Initial conditions are set as \( \eta = 11 \) (m) and \( Q = 0 \) (m\(^3\)/s) in the entire domain. Both the upstream and downstream ends are solid walls, but the water surface elevation at the upstream end is specified as

\[
\eta(t,0) = 15 + 4 \sin \left( \pi \left( \frac{4t}{86,400} - \frac{1}{2} \right) \right). \tag{102}
\]

Asymptotic solution in the limit of zero Froude number is available in the literature (Rebollo et al., 2003). The domain is divided into 300 uniform regular cells. The time increment is \( \Delta t = 0.5 \) (s) and computational period is 10,800 (s). Comparison of the computed and analytical discharges per unit width at the time 10,800 (s) is presented in Figure 5-3. Maximum absolute error between the computed and analytical water surface elevations at the time is 0.03 (%). Therefore, plots for the water surface elevations are not provided.
Figure 5-3 Comparison of the analytical and computed discharges per unit width solving the tidal wave problem.

5.4.2 Dam Break Problems in a Flat, Frictionless, and Rectangular Channel

Dam break problems in a flat and frictionless channel are considered. A 2,000 (m) length channel is considered as the domain $\Omega = (0, 2000)$ (m). The initial water surface elevation $\eta$ (m) is

$$\eta = \begin{cases} h_U & (x \leq 1,000) \\ h_D & (x > 1,000) \end{cases}$$

(103)

where $h_U$ and $h_D$ denote the initial upstream and downstream water depths, respectively, with $h_U > h_D > 0$. The initial discharge is $Q = 0$ (m$^3$/s) for the entire computational domain. The initial upstream water depth $h_U$ is fixed to 1.0 (m), while the initial downstream water depths $h_D$ of 0.1 (m) (Case DB-A) and of 0.0 (m) (Case DB-B) are examined. The boundaries are treated as solid walls. Solution to Case DB-B involves a wet and dry interface, which is difficult to accurately capture even if high-resolutions algorithms are employed (Venutelli, 2010; Ouyang et al., 2013). The two cases of $\Delta x = 10$ (m) and $\Delta x = 1$ (m) are examined. The time increment $\Delta t$ is fixed to 0.001 (s). The analytical and computed water surface profiles and velocity distributions at $t = 50$ (s) are illustrated in Figures 5-4 through 5-7 for each computational case. The computational results are shown to be sufficiently accurate, and
the numerical solutions are shown to approach the exact solutions, as the mesh is refined. Computational results do not contain spurious oscillations as found in those with the FEV scheme (Unami and Alam, 2012), and are not inferior to recently published ones slightly underestimating the velocity of the wet and dry interface.

Figure 5-4 Comparison of the exact and computed water surface profiles of the dam break problems in a straight rectangular channel (Case DB-A).

Figure 5-5 Comparison of the exact and computed velocity distributions of the dam break problems in a straight rectangular channel (Case DB-A).
Figure 5-6 Comparison of the exact and computed water surface profiles of the dam break problems in a straight rectangular channel (Case DB-B).

Figure 5-7 Comparison of the exact and computed velocity distributions of the dam break problems in a straight rectangular channel (Case DB-B).
5.4.3 Dam Break Problem in a Flat, Frictionless, and Triangular Channel

A dam break problem in a triangular channel is considered in order to see computational accuracy of the DFVF scheme for rapidly varying flows in non-rectangular channels. Sanders (2011) investigated dam break problems in non-rectangular channels including triangular and trapezoidal channels. Shallow water flows in triangular channels are in general difficult to compute compared with flows in rectangular channels because the top width of the cross-section varies in direct proportion to the water depth in the former case, which would significantly reduce the accuracy of the wave speed of the flows on a dry bed. The test problem here is carried out in the same setting as the previous dam break problem on a dry bed (Case DB-B) except that each cross-section is triangular and has a side gradient of 1:1 and \( h_u \) is set to 10 (m). The FEV scheme (Unami and Alam, 2012) has been preliminarily applied to this problem under the same computational conditions; however, positivity of the water depth is violated near the wet and dry interface due to spurious oscillations, which do not disappear even if more fine computational mesh and time increments are used. This indicates limited applicability of the FEV scheme to transient flows in non-rectangular channels with wet and dry interfaces. **Figures 5-8 and 5-9** show the computational results for the water surface profile and the velocity distribution for this test problem. Convergence of the computed solutions is slower than those of the high-resolution schemes, but preserve monotonicity and positivity of water surface profiles.
Figure 5-8 Comparison of the exact and computed water surface profiles of the dam break problem in a straight triangular channel.

Figure 5-9 Comparison of the exact and computed velocity distributions of the dam break problem in a straight triangular channel.
5.4.4 Thacker’s Test Case

The DFVF scheme is verified with the Thacker’s test case where the analytical solution is available (Thacker, 1981; Ying et al., 2004). This problem is a severe test case that involves both advancing and receding wet and dry interfaces on non-flat channel bed. The computational domain $\Omega$ is a frictionless rectangular channel $(-4,000,4,000)$ (m) with the parabolic bottom elevation profile $z_n$ given by

$$z_n = h_0 \left( \frac{x^2}{a^2} - 1 \right) \text{ (m)}$$

with $h_0 = 10$ (m) and $a = 2,500$ (m). The channel width is set as 1 (m). The initial condition of the water surface elevation $\eta$ is given by

$$\eta = z + \max \{0, Z(0,x) - z \} \text{ (m)}$$

where $Z = Z(t,x)$ is given by

$$Z(t,x) = \frac{1}{4g} \left( -B^2 - B^2 \cos(2\omega t) - 4B\omega x \cos(\omega t) \right) \text{ (m)}$$

with the parameters $B = 5$ (m/s) and $\omega = 0.0056$ (1/s). The initial condition of the discharge $Q$ is set as 0 (m$^3$/s) over $\Omega$. The analytical solutions of $\eta$ and $Q$ for $t > 0$ (s) are given by

$$\eta = z + \max \{0, Z - z \} \text{ (m)}$$

and

$$Q = B \max \{0, Z - z \} \sin(\omega t) \text{ (m$^3$/s)},$$

respectively. The two cases of $\Delta x = 10$ (m) and $\Delta x = 5$ (m) are examined to see convergence of the scheme. The time increment $\Delta t$ is set as $T/4,000$ (s) in the former case and as $T/8,000$ (s) in the latter where $T = 1,121$ (s). The minimum nodal water depth is set as 0.001 (m) to avoid negative water depth encountered at the receding wet and dry interfaces.

Figures 5-10(a) through 5-10(d) plot comparisons of the analytical and computed water surface profiles at each time step. Figures 5-11(a) through 5-11(d) plot comparisons of the analytical and computed discharges. The numerical solutions agree well with the analytical results, accurately resolving the wet and dry interfaces. The computed $\eta$ and $Q$ converge to the analytical ones as the computational mesh is refined.
Figure 5-10 Comparisons of the exact and computed water surface profiles for the Thacker’s test case at selected time steps.

Figure 5-11 Comparisons of the exact and computed discharges for the Thacker’s test case at selected time steps.
5.4.5 Dam Break Problem in a Non-prismatic Rectangular Channel

The DFVF scheme is validated through a dam break problem in an experimental non-prismatic rectangular channel with a flat and wet bed. Bellos et al. (1992) carried out a series of experimental dam break problems in a converging and diverging rectangular channel. The experimental channel is shown in Figure 5-12. Detailed drawings of the channel are summarized in Bellos et al. (1991). The width of the upstream part of the channel is 1.40 (m). The channel width gradually decreases from 1.40 (m) to 0.60 (m) toward the contraction at \( x = 8.5 \) (m), where a vertical wall serving as a dam is installed, and gradually increases to 1.40 (m) toward the downstream end, at which a weir is located. The initial upstream and downstream water depths of the dam are set to 0.250 (m) and 0.101 (m). The dam is instantaneously removed at the initial time \( t = 0 \) (s). As shown in Figure 5-12, the channel has an asymmetrical shape and the resulting water flow may exhibit a horizontally two-dimensional nature. Nevertheless, as shown in and cross-sectionally averaged modeling has found to be successful in simulating experimental results with satisfactory precision. Tseng (1999) and Tseng and Yen (2004) verified several high-resolution schemes with this test problem. Manning’s coefficient \( n \) of the channel has been set as 0.012 (s/m\(^{1/3}\)), as in the previous researches (Lai and Khan, 2012). Since the dimensions of the downstream weir are not available in the literatures, the channel length is extended to 9.3 (m) following Hicks et al. (1997), and a critical flow condition is specified at the new downstream end. Therefore, reflection waves at the original downstream end cannot be produced. The channel is divided into regular cells of length \( \Delta x = 0.1 \) (m). The time increment \( \Delta t \) is set to be 0.01 (s).

Figure 5-13 shows the comparison of computed and measured water depths at the observation stations at \( x = 4.5 \) (m), \( x = 8.5 \) (m), and \( x = 13.5 \) (m). Computational results adequately capture water depth variations due to a moving shock and a depression wave, and do not show significant differences with those of the high-resolution schemes.
Figure 5-12 Sketch of the dam break problem in a non-prismatic experimental channel.

Figure 5-13 Comparisons of the computed and measured water depths at each observation point for the dam break problem in a non-prismatic experimental channel.
5.4.6 Dam Break Problem in a Non-flat Rectangular Channel

The DFVF scheme is validated with the laboratory dam break experiment that Alcrudo and Frazão (1999) conducted. The experimental setting is presented in Figure 5-14. The channel has a rectangular cross-section of 1.75 (m) wide, and is composed of an upstream reservoir filled with water with the depth of 0.75 (m), and a dry downstream channel with a symmetrical triangular bump with the height of 0.40 (m). The vertical wall installed at $x = 15.5$ (m) is assumed to be instantaneously removed at $t = 0$ (s). The Manning’s coefficient $n$ is set to be 0.0125 (s/m$^{1/3}$) following Liang and Marche (2009). The upstream end is taken as a solid wall and a free outflow condition is specified at the downstream end. This problem involves wet and dry interfaces and receding bores that occur when the discharged water from the reservoir hits the bump. The domain is divided into 190 uniform regular cells. The time increment is $\Delta t = 0.01$ (s).

Figure 5-15 shows comparisons of the computed and measured water depths at $x = 19.5$ (m), $x = 25.5$ (m), and $x = 28.5$ (m). Sudden changes of the water depths are well resolved, though with slight overestimations. The computed water depths in Figures 5-15(a) and 5-15(b) involve numerical oscillations, which attenuate in time and therefore do not limit applicability of the present scheme. High-resolution algorithms will suppress these oscillations, but forfeiting the simplicity of the scheme. Computational results are quantitatively consistent with those using the high-resolution schemes of Liang and Marche (2009), Singh et al. (2011), and Bollermann et al. (2013) where the channel is divided into 760, 3,800, and 200 uniform cells along the flow direction, respectively.

---

**Figure 5-14** Sketch of the dam break problem with a triangular bump.
Figure 5-15 Comparisons of the computed and measured water depths at each observation point for the dam break problem with a triangular bump.

5.4.7 Dam Break Problems in a Non-prismatic and Non-flat Channel

Hydraulic experiments using an experimental rectangular flume were carried out for examining capability of the DFVF scheme handling dam break problems in non-prismatic and non-flat channels. These experiments are more severe than the preceding two test cases because the latter involve only either non-prismatic cross-section or non-flat channel bed. The experimental settings of the presented cases are presented in Figure 5-16. Length and width of the flume were 20.0 (m) and 0.60 (m), respectively. The channel consisted of an upstream reservoir, an upstream reach, a symmetrical broad crested weir, and a downstream reach with a free outfall. A couple of blocks with a thin gate installed just upstream of them served as a dam to fill up water in the
reservoir. A trapezoidal obstacle was installed on the weir to create a 1.20 (m)-long non-prismatic reach having a gradual contraction with the angle of 54.3 (deg) and a 0.25 (m)-wide sudden expansion. Distance between the downstream sides of the obstacle and the top of the weir was 0.06 (m). Height of the obstacle was sufficiently high to serve as a sidewall creating a channel construction on the weir. The two cases of the flume slopes $S = 0.00$ and $S = 0.01$ were examined. Initially the downstream reach and the top of the weir were dry and the reservoir was filled with still water such that the water depth just upstream of the blocks equals to 0.37 (m). The upstream reach was also filled with still water to serve as a pool whose water surface elevation equals to the elevation of the upstream-side of the top of the weir. The gate was suddenly removed to create a 0.30 (m)-wide partial dam breach. The opening of the gate created a flash flood propagating on the pool before flowing over the weir. The flows downstream of the weir involved a series of oblique hydraulic jumps in both of the test cases due to the sudden expansion of the channel cross-section.

Water depths in the channel were measured with hydrostatic head level gauges with the resolution of 1 (s) (HOBO U-20) at the three stations as shown in Figure 5-16. Locations of the stations were $x = 8.0$ (m) (in the upstream reach), $x = 16.0$ (m) (in the downstream reach), and $x = 20.0$ (m) (at the downstream end), respectively. Traveling time of the flows to the downstream end of the channel were estimated from the measured data and video images as 9.0 (s) with $S = 0.00$ and 8.0 (s) with $S = 0.01$, respectively. The Manning’s coefficient $n$ of the channel has been estimated as 0.01 (s/m$^{1/3}$). The gate is assumed to be instantaneously removed at the initial time $t = 0$ (s). The entire channel is uniformly divided into 1,000 regular cells with the length of 0.02 (m). The time increment $\Delta t$ is set as 0.005 (s). Response characteristics of the head level gauges as a convolution operator between the true (from video images; input data) and measured water depths (output data) was preliminary estimated with the inverse analysis method using Tikhonov regularization (Kirsch, 1996). The computed water depths at the observation stations with the DFVF scheme were then emulated with the estimated convolution operator to create another set of water depths data (emulated data) accounting for the response characteristics of the gauges, which are more appropriate to be compared with the measured results.

Figures 5-17(a) through 5-17(c) and Figures 5-18(a) through 5-18(c) plot comparisons of
the computed, emulated, and measured water depths at the three observation stations with the slopes $S = 0.00$ and $S = 0.01$, respectively. These figures show that both the computed and emulated solutions have reasonable phase and amplitude accuracy. They well capture the gradual decreasing of the water depths at all the stations in both of the computational cases. The emulated ones present superior accuracy at the station $x = 8.0$ (m) in particular, demonstrating potential validity of the 1-D SWEs to the dam break problems that cannot be assessed without the emulation technique. Traveling time of the wet and dry interface to the downstream end of the channel with the emulated solutions were 8.7 (s) with $S = 0.00$ and 7.2 (s) with $S = 0.01$, respectively, reproducing the observed values with slight underestimations. Discrepancies between the numerical and measured results are considered due to the inherently 2-D natures of the flows created by the concrete blocks and the obstacle in the channel. Another possible cause of the discrepancies is the dispersivity of the flows resulting from non-hydrostatic effects, which is not appropriately handled by the SWEs. Despite these limitations of the 1-D SWEs, the computational results with the DFVF scheme are satisfactory accurate, indicating its applicability to dam break problems both with non-prismatic cross-section and non-flat channel bed.

Figure 5-16 Sketch of the experimental channel for the dam break problems in a non-prismatic and non-flat channel.

76
Figure 5-17 Comparisons of the computed, emulated, and measured water depths at each station with the slope $S = 0.00$. 

77
Figure 5-18 Comparisons of the computed, emulated, and measured water depths at each station with the slope $S = 0.01$. 
5.4.8 Steady Flows in a Bifurcating Channel Network

The three momentum flux evaluation schemes, which are the M1 (Ishida et al., 2011), M2 (Unami and Alam, 2012), and M3 schemes (Chapter 3), are applied to a series of numerical simulation of steady shallow water flows in a bifurcating open channel network. Comparative numerical analysis on the momentum flux evaluation schemes is carried out in this section. Main purpose of the analysis is to demonstrate superiority of the M3 scheme to the others in accurately predicting water surface profiles and discharge ratios. The analysis is divided into two parts. The first part is verification with the discharge ratios for the experimental data of the flows in an open channel network. The second part is comparisons of the computed water surface profiles.

5.4.8.1 Computational Conditions

Performances of the schemes are evaluated with the hydraulic experimental data of Ashida and Kawai (1979). The hydraulic experiments were performed with an acryl-made Y-shaped open channel network with the uniform bed slope of 0.001. Figure 5-19 gives a schematic sketch of the open channel network serving as a computational domain. The key nodes of the domain are alphabetically labeled from I through L, which are an upstream end I, a junction J, and two downstream ends K and L. The network had the three reaches, which are an upstream reach I-J, a main downstream reach J-K, and a sub downstream reach J-L. The lengths of the reaches I-J, J-K, and J-L were 4 (m), 3 (m), and 3 (m), respectively. The channel width of the reaches I-J, J-K, and J-L were 0.2 (m), 0.2 (m), and 0.1 (m), respectively.

Ashida and Kawai (1979) carried out in total 18 hydraulic experiments with the open channel network. In their experiments, the discharge ratio for the reach J-L, which is defined as

\[ \xi = \frac{Q_{J-L}}{Q_I} \]  

was measured for each combination of the inflow discharge \( Q_I \) and the contact angles \( \theta_{uk} \) and \( \theta_{ul} \) as summarized in Table 5-1. The discharge ratio \( \xi \) serves as a major factor that controls the flow structures downstream of a flow diverging point (Hardy et al., 2011; Van et al., 2012). To theoretically or numerically predict \( \xi \) is a difficult issue even for the flows with two downstream reaches as focused on in this section. The situation may be further complicated for
the flows involving a junction that connects more than three reaches (Mignot et al., 2008; Rivière et al., 2011). The Manning’s coefficient is determined as 0.012 (s/m$^{1/3}$) in the entire domain because the reaches had acryl-made walls whose surfaces are slightly made roughly using small dice-like obstacles. Varying the value of the Manning’s coefficient around the specified value within the range of −10% to 10% does not significantly affect the computational results presented below. The inflow discharge $Q_i$ is prescribed at the upstream end I without specifying the inflow momentum flux $F$. A uniform outflow condition is specified at the downstream ends K and L. A still water with the depth of 0.05 (m) in the entire domain is chosen as the initial guess of each computational case. The domain is divided into a computational mesh with 300 cells and 301 nodes. The time increment for the temporal integration in the DFVF scheme is fixed to 0.01 (s), which is a sufficiently small value to guarantee the CFL condition (Toro, 2010). The terminal time of each computational case is chosen as 3,600 (s) so that a steady numerical solution is obtained.

5.4.8.2 Discharge Ratios

Comparisons of the experimental and computational discharge ratios $\xi$ are summarized in Table 5-2. The computational results with a 2-D shallow water model of Koseki et al. (2013) are also presented in Table 5-2 for more detailed investigations. The 2-D model gives the highest accuracy. The errors of the 1-D models are considered due to their limitation that the junction is treated as a volume-less node, which on the other hand is naturally taken into account as a part of the computational domain in the 2-D model. The computational results of M1 scheme are same for all the cases because of its inability to incorporate the geometrical information of the domain, implying that the momentum flux around a junction should be evaluated with the contact angles of the reaches connected to the junction. The M2 scheme performs the worst except for few cases because of the reason elucidated in the next section. The M3 scheme performs quite well with the relative errors smaller than 10% for the cases where the inflow discharge $Q_i$ is relatively high. The results of the M-3 scheme are comparable to those of the 2-D model in some cases. The obtained computational results suggest the sufficient accuracy of the M3 scheme to predict the discharge ratios.
Figure 5-19 Schematic sketch of the open channel network for the hydraulic experiments

Table 5-1 Experimental conditions for the flows in the open channel network.

<table>
<thead>
<tr>
<th>Case</th>
<th>$\theta_{\text{IK}}$ (deg)</th>
<th>$\theta_{\text{UL}}$ (deg)</th>
<th>$Q_i$ (L/s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-1</td>
<td>30</td>
<td>0</td>
<td>1.0</td>
</tr>
<tr>
<td>A-2</td>
<td>30</td>
<td>0</td>
<td>2.0</td>
</tr>
<tr>
<td>A-3</td>
<td>30</td>
<td>0</td>
<td>3.8</td>
</tr>
<tr>
<td>B-1</td>
<td>30</td>
<td>30</td>
<td>1.0</td>
</tr>
<tr>
<td>B-2</td>
<td>30</td>
<td>30</td>
<td>2.0</td>
</tr>
<tr>
<td>B-3</td>
<td>30</td>
<td>30</td>
<td>3.8</td>
</tr>
<tr>
<td>C-1</td>
<td>30</td>
<td>60</td>
<td>1.0</td>
</tr>
<tr>
<td>C-2</td>
<td>30</td>
<td>60</td>
<td>2.0</td>
</tr>
<tr>
<td>C-3</td>
<td>30</td>
<td>60</td>
<td>3.8</td>
</tr>
<tr>
<td>D-1</td>
<td>60</td>
<td>0</td>
<td>1.0</td>
</tr>
<tr>
<td>D-2</td>
<td>60</td>
<td>0</td>
<td>2.0</td>
</tr>
<tr>
<td>D-3</td>
<td>60</td>
<td>0</td>
<td>3.8</td>
</tr>
<tr>
<td>E-1</td>
<td>60</td>
<td>30</td>
<td>1.0</td>
</tr>
<tr>
<td>E-2</td>
<td>60</td>
<td>30</td>
<td>2.0</td>
</tr>
<tr>
<td>E-3</td>
<td>60</td>
<td>30</td>
<td>3.8</td>
</tr>
<tr>
<td>F-1</td>
<td>60</td>
<td>60</td>
<td>1.0</td>
</tr>
<tr>
<td>F-2</td>
<td>60</td>
<td>60</td>
<td>2.0</td>
</tr>
<tr>
<td>F-3</td>
<td>60</td>
<td>60</td>
<td>3.8</td>
</tr>
</tbody>
</table>
5.4.8.3 Water Surface Profiles

Figures 5-20 through 5-25 present the computed water surface profiles in the entire open channel network for the cases A-1 through F-3 computed with the M2 and M3 schemes. The downstream end K (L) is taken as the vertical elevation control point. The computational profiles for the M1 scheme are not plotted in the figures because they present unphysical high water depths, which are more than three times higher than the observed values in some cases. This is considered due to the inability to appropriately handle the flows having a junction. According to Ashida and Kawai (1979), the water flows in the open channel network were entirely subcritical in all the cases. They also reported that the flows in the reach I-J presented drawdown curves. The flows computed using the M3 scheme are entirely subcritical for all the cases with successfully reproducing the drawdown curves in the reach I-J. On the other hand, in almost all of the cases, the flows computed using the M2 scheme involve a spurious water depth change at the junction that associates a false hydraulic jump as found in Yoshioka et al. (2011), which was not observed in the experiments. The high relative errors of the discharge ratio for the M2 scheme are considered due to this peculiar behaviour.

Ashida and Kawai (1979) also presented measured water surface profiles around the junction J for the cases with the inflow discharge $Q_i$ of 3.8 (L/s). The measured water levels were between 0.048 m from 0.058 m for all these cases, which are consistent with the computational results of the M3 scheme. The scheme also quantitatively reproduces the experimentally observed changes in the water surface profiles around the junction. The obtained computational results clearly indicate the advantages of the scheme over the existing ones.
Table 5-2 Comparisons of the experimental and computational discharge ratios for the cases A-1 through F-3.

<table>
<thead>
<tr>
<th>Case</th>
<th>Exp.</th>
<th>M1</th>
<th>M2</th>
<th>M3</th>
<th>2-D</th>
</tr>
</thead>
<tbody>
<tr>
<td>A-1</td>
<td>0.300</td>
<td>0.335 (11.7)</td>
<td>0.451 (50.3)</td>
<td>0.326 (8.7)</td>
<td>N.A.</td>
</tr>
<tr>
<td>A-2</td>
<td>0.350</td>
<td>0.335 (4.3)</td>
<td>0.444 (26.9)</td>
<td>0.317 (9.4)</td>
<td>N.A.</td>
</tr>
<tr>
<td>A-3</td>
<td>0.389</td>
<td>0.335 (13.9)</td>
<td>0.434 (11.6)</td>
<td>0.305 (21.6)</td>
<td>N.A.</td>
</tr>
<tr>
<td>B-1</td>
<td>0.250</td>
<td>0.335 (34.0)</td>
<td>0.429 (71.6)</td>
<td>0.314 (25.6)</td>
<td>N.A.</td>
</tr>
<tr>
<td>B-2</td>
<td>0.300</td>
<td>0.335 (11.7)</td>
<td>0.422 (40.7)</td>
<td>0.306 (2.0)</td>
<td>N.A.</td>
</tr>
<tr>
<td>B-3</td>
<td>0.324</td>
<td>0.335 (3.4)</td>
<td>0.412 (27.2)</td>
<td>0.297 (8.3)</td>
<td>N.A.</td>
</tr>
<tr>
<td>C-1</td>
<td>0.250</td>
<td>0.335 (34.0)</td>
<td>0.341 (36.4)</td>
<td>0.286 (14.1)</td>
<td>0.260 (4.0)</td>
</tr>
<tr>
<td>C-2</td>
<td>0.285</td>
<td>0.335 (17.5)</td>
<td>0.335 (17.5)</td>
<td>0.281 (1.4)</td>
<td>0.295 (1.4)</td>
</tr>
<tr>
<td>C-3</td>
<td>0.295</td>
<td>0.335 (13.6)</td>
<td>0.325 (10.2)</td>
<td>0.276 (6.4)</td>
<td>0.295 (0.0)</td>
</tr>
<tr>
<td>D-1</td>
<td>0.330</td>
<td>0.335 (1.5)</td>
<td>0.341 (3.3)</td>
<td>0.355 (7.6)</td>
<td>N.A.</td>
</tr>
<tr>
<td>D-2</td>
<td>0.360</td>
<td>0.335 (6.9)</td>
<td>0.335 (6.9)</td>
<td>0.344 (4.4)</td>
<td>N.A.</td>
</tr>
<tr>
<td>D-3</td>
<td>0.421</td>
<td>0.335 (20.4)</td>
<td>0.513 (66.7)</td>
<td>0.329 (21.9)</td>
<td>N.A.</td>
</tr>
<tr>
<td>E-1</td>
<td>0.300</td>
<td>0.335 (11.7)</td>
<td>0.500 (66.7)</td>
<td>0.343 (14.3)</td>
<td>0.307 (2.3)</td>
</tr>
<tr>
<td>E-2</td>
<td>0.335</td>
<td>0.335 (0.0)</td>
<td>0.478 (42.7)</td>
<td>0.334 (0.3)</td>
<td>0.319 (4.8)</td>
</tr>
<tr>
<td>E-3</td>
<td>0.355</td>
<td>0.335 (5.6)</td>
<td>0.435 (22.5)</td>
<td>0.321 (9.6)</td>
<td>0.337 (5.1)</td>
</tr>
<tr>
<td>F-1</td>
<td>0.275</td>
<td>0.335 (21.8)</td>
<td>0.372 (35.3)</td>
<td>0.317 (15.3)</td>
<td>N.A.</td>
</tr>
<tr>
<td>F-2</td>
<td>0.310</td>
<td>0.335 (8.1)</td>
<td>0.361 (16.5)</td>
<td>0.310 (0.0)</td>
<td>N.A.</td>
</tr>
<tr>
<td>F-3</td>
<td>0.335</td>
<td>0.335 (0.0)</td>
<td>0.347 (3.6)</td>
<td>0.301 (10.1)</td>
<td>N.A.</td>
</tr>
</tbody>
</table>

1. N.A. not available
2. The numbers in parenthesis “( )” represent the relative errors from the experimental values
Figure 5-20 Computed water surface profiles for the cases A-1 through A-3.

Figure 5-21 Computed water surface profiles for the cases B-1 through B-3.

Figure 5-22 Computed water surface profiles for the cases C-1 through C-3.

Figure 5-23 Computed water surface profiles for the cases D-1 through D-3.
5.4.9 Dam Break Problem in a Multiply Connected Open Channel Network with a Rectangular Cross-section

In this and the next subsections, additional numerical test cases are performed for assessing convergence and robustness of the DFVF scheme through the application of dam break problems in multiply connected channel networks. These test problems are challenging because they require a stable discretization method for unsteady transcritical flows of extremely shallow depth as well as a consistent junction treatment technique. Only a few attempts at cross-sectionally averaged modeling have been made for dam break problems in multiply connected channel networks with bends. Figure 5-26 illustrates a plane view of the channel network as the computational domain with key nodes labeled alphabetically from A to F. The key nodes are referred to as the upstream end (A), the downstream end (E), the bifurcation point (B), the converging point (D), and the bending points (C and F). The reaches have a rectangular cross-sectional shape and are equal in terms of length, Manning’s coefficient, and bed slope, which are 10 (m), 0.03 (s/m^1/3), and 0.01, respectively. The channel width is set to 0.25 (m) in reaches B-F-D, and 0.5 (m) in the other reaches. A vertical wall serving as a dam is installed at the middle of reach A-B, dividing the computational domain into an upstream res-
ervoir and a downstream channel network. The water depth just upstream of the dam is 2.0 (m) and the channel network downstream of the dam is initially dry. Both the upstream and downstream ends are solid walls. The dam is instantaneously removed at the initial time $t = 0$ (s). Each reach is divided into regular cells of length $\Delta x = 0.25$ (m), and the time increment $\Delta t$ is fixed to 0.001 (s). The numerical solution with $\Delta x = 0.05$ (m) and $\Delta t = 0.00025$ (s) is regarded as the fine solution to be compared with the coarse solution. The M2 scheme has been applied to the DFVF scheme.

Figure 5-27 compares the water surface profiles with coarse and fine meshes in reaches A-B-C-D-E and B-F-D at each time step. The surge from the reservoir is divided at junction B, taking the local minimum water depth at that point. These divided flows converge at junction D and hit the downstream wall, generating a receding bore. There are also other receding bores starting at bending points C and F, as observed in the experimental dam break problems in bend channels (Fražaro and Zech, 1999). The flood arrival time at downstream end E is $t = 24.0$ (s). The coarse solution is slightly diffusive but exhibits only minor numerical oscillations and is sufficiently close to the fine solution.

5.4.10 Dam Break Problem in a Multiply Connected Open Channel Network with a Triangular Cross-section

Numerical simulation of a dam break problem in a multiply connected channel network with a triangular cross-section is carried out for further examining robustness of the DFVF scheme. The computational conditions are identical to those of the previous dam break problem, except that each cross-section is triangular and the water depth upstream of the dam is 1.0 (m). The side gradient of the cross-section is $1: \sqrt{3}$ in reaches B-F-D and is 1:1 in the other reaches. Figure 5-28 compares the coarse and fine computed water surface profiles in reaches A-B-C-D-E and B-F-D at each time step. The flood arrival time at downstream end E is $t = 24.5$ (s). As in the previous test problem, the coarse water surface profiles are reasonably close to the fine profiles with respect to resolving the receding bores. The computational results of this and the previous test problems show a sufficiently high capability of the scheme to handle dam break problems in multiply connected channel networks.
Figure 5-26 Sketch of the hypothetical multiply connected channel network with key nodes.

Figure 5-27 Comparisons of the coarse and fine water surface profiles in reaches A-B-C-D-E and B-F-D at each time step for the dam break problem in a multiply connected channel network with a rectangular cross-section.
Figure 5-28 Comparisons of the coarse and fine water surface profiles in reaches A-B-C-D-E and B-F-D at each time step for the dam break problem in a multiply connected channel network with a triangular cross-section.

5.5 Application to Dam Break Flash Flood

The DFVF scheme is now applied to numerical simulation of a recent flash flood event caused by the complete failure of Fujinuma Dam in Japan.

5.5.1 Background

Fujinuma Dam was an earth fill dam that formed Fujinuma Reservoir, which was located in Sukagawa City, Fukushima Prefecture, Japan (N37.302, E140.195; WGS84). Fujinuma Reservoir had been used as an irrigation tank for a downstream command area of 850 (ha). The maximum storage capacity of the reservoir was 1,500,000 (m$^3$). Figure 5-29 presents a map of the study area including the reservoir. Taki Village is northeast of Fujinuma reservoir, and a paddy field area and Naganuma Village are located southeast of the reservoir. At present, no details on the design and construction history of Fujinuma Dam are available, except for very limited information (EERI, 2011; Matsumoto, 2011). According to these literatures, the construction of Fujinuma Dam began in 1937 and was completed in 1949. Figure 5-30 presents
the main and auxiliary embankments of Fujinuma Dam. The dimensions of the main and auxiliary embankments are summarized in Table 5-3. As depicted in Figures 5-29 and 5-30, Taki Village was located just downstream of the main embankment. According to Kokusyou (2011), Fujinuma Dam was built after the settlement of Taki Village regardless of the potential risk of its failure. Fujinuma Dam was probably not well compacted because modern compaction machines were not available during its construction period. Fujinuma Dam was repaired from 1977 to 1979 and from 1984 to 1992, but the safety factor of the dam at that time has been estimated to be 1.15, which was still less than the required value of 1.20 (Panel to evaluate the seismic stability of dams and small ponds for agricultural purpose of Fukushima Prefecture, 2012).

Fujinuma Reservoir was significantly damaged by the Great East Japanese Earthquake that occurred on March 11, 2011 with the maximum magnitude of 9.0. The epicenter of the earthquake was approximately 240 (km) from the reservoir (USGS, 2011). The seismic intensity at Sukagawa City was reported to be 5 to 6 (Ono et al., 2011). The return period of the earthquake was estimated to be 1,000 years by Yim et al. (2012). In anticipation of the upcoming irrigation season, Fujinuma Reservoir was filled almost to capacity at the time of the earthquake. The main embankment completely failed due to the earthquake, which resulted in the uncontrolled release of the entire content of the reservoir, and Taki Village was critically struck by the surge from the reservoir. According to a local media report, the surge had a water depth of at least several meters when it struck Taki Village. In total 19 houses were consequently washed away or destroyed, 155 houses were flooded below floor level, and eight people were killed (The Asahi Shinbun Digital, 2011). Kokusyou (2011) reported that the flooding in Taki Village lasted at least one hour. EERI (2012) concluded that the main embankment begun breaching within 20 minutes after the earthquake, and that complete overtopping of the dam occurred several minutes later. Harder et al. (2011) and Pradel et al. (2012) inferred that an immediate cause of the complete failure was a large drop in the crest elevation due to an earthquake-induced landslide. Harder et al. (2011) conjectured that the surge from the reservoir traveled downstream along the valley to the northeast until flowing into the Sunoko River and struck Taki Village, before turning 90 degrees to the southeast along the river channel.
Kyokawa et al. (2011) performed a field survey in the valley downstream of the main embankment and estimated the surge speed to be approximately 10 (m/s). The auxiliary embankment had not failed at the time of the earthquake; however, a small arc-shaped deformation of the slope immediately adjacent to the embankment has been reported (Kayen et al., 2011; Tiwari et al., 2012).

Table 5-3 Dimensions of the main and auxiliary embankments of Fujinuma Reservoir.

<table>
<thead>
<tr>
<th>Dam dimension</th>
<th>Main embankment</th>
<th>Auxiliary embankment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Crest height (m)</td>
<td>18.5</td>
<td>15.0</td>
</tr>
<tr>
<td>Crest length (m)</td>
<td>133</td>
<td>60</td>
</tr>
<tr>
<td>Crest width (m)</td>
<td>6</td>
<td>N.A.</td>
</tr>
<tr>
<td>Slope gradient on the upstream side</td>
<td>1:2.8</td>
<td>N.A.</td>
</tr>
<tr>
<td>Slope gradient on the downstream side</td>
<td>1:2.5</td>
<td>N.A.</td>
</tr>
</tbody>
</table>

N.A. Not available

Figure 5-29 Map of the study area.
5.5.2 Computational Cases

As described in the previous subsection, only the main embankment failed due to the earthquake; however, a significant portion of the paddy field area could have been affected by the flood in the case in which the auxiliary embankment failed. In fact, several observation results have indicated that there was a risk of failure of the auxiliary embankment (Kayen et al., 2011; Tiwari et al., 2012). It is therefore worth considering such alternative scenarios as possible cases of more significant damages to the area downstream of the reservoir than that which actually occurred. Hence, the present study considers three possible scenarios:

- Case FDB-A: only the main embankment failed.
- Case FDB-B: only the auxiliary embankment failed.
- Case FDB-C: both the main and auxiliary embankments failed simultaneously.

In Cases FDB-A and FDB-B, the reservoir water is discharged from the main and auxiliary embankments, respectively. On the other hand, in Case FDB-C, the reservoir water is discharged separately from both of the embankments. Case FDB-A corresponds to the event that actually occurred, and the other cases are hypothetical scenarios.

5.5.3 Computational Mesh

The computational mesh for the numerical simulation is determined from a 1:25,000 scale
map, aerial view of the flood flow path (Kayen et al., 2011) and satellite image data and digital elevation data obtained from Google Earth (Google Inc., Mountain View, Calif.). **Figure 5-31** illustrates the computational mesh with key nodes alphabetically labeled A through K. The total numbers of regular cells and dual cells of the computational mesh are $N_r = 327$ and $N_u = 327$, respectively. The M3 scheme has been applied to the DFVF scheme. The channel network in the simulation consists of Fujinuma Reservoir (A-B-C and B-G), the valley downstream of the main embankment (C-D), the valley downstream of the auxiliary embankment (G-H), Sunoko River (J-D-E), Kouka River (K-I-E-F), and the paddy field area (H-I). Nodes A, J, and K are upstream ends and node F is the downstream end. The node E is the conversion point at which the Sunoko River joins the Kouka River. The node D is considered to serve as the representative point of Taki Village, at which the surge from the main embankment hits the village. The node H is the upstream end of the paddy field area. The main and auxiliary embankments, respectively, fall on nodes C and G. Reach H-I is determined so that the flow channel partially varies in accordance with the drainage canals running in the paddy field area. The computational mesh is highly spatially non-uniform, and the maximum and minimum lengths of the regular cells are 201 (m) and 6 (m), respectively. The maximum bed slope of the channel is 0.23 in the valley G-H. Each channel cross-section is approximated as a compound shape, as shown in **Figure 5-32**, where $B$ is the channel width, $H_c$ is the channel height, $M_0$ is the side slope of the channel, and $M_f$ and $M_e$ are the parameters that account for flooding of the channels. The cross-sections of the dam and the valleys in particular are given as rectangular ($M_0 = 0$ and $H_e = +\infty$) and triangular shapes ($B = 0$ and $H_e = +\infty$), respectively.
Figure 5-31 Schematic diagram of the computational domain with key nodes for the flood caused by the Fujinuma Dam failure.

Figure 5-32 Schematic diagram of each channel cross-section in the study area.
5.5.4 Model Parameters
The extended 1-D SWEs have two model parameters: the Manning’s coefficient $n$ and the momentum flux coefficient $\beta$. These parameters vary depending on local flow structures, and their estimation is a particularly difficult problem for natural channels with compound cross-sections (Moussa, 2004). The present study therefore regards them as global constants rather than local variables. Thus, the Manning’s coefficient $n$ is set as a moderate constant value everywhere in the domain. Preliminary computations for several values of $n$ within the range of 0.030 to 0.050 (s/m$^{1/3}$) revealed that the computed flow fields have a qualitatively similar nature, except for the differences in flood attenuation and delay. Since the main focus of the numerical simulation here is the verification of the DFVF scheme, Manning’s coefficient $n$ is fixed to 0.040 (s/m$^{1/3}$). The momentum flux coefficient $\beta$ is set as 1.1 because the cross-sectional shape has little complexity. The threshold value $\varepsilon$ is set as $1.0 \times 10^{-6}$ (m$^3$).

5.5.5 Initial and Boundary Conditions
Initial and boundary conditions to be equipped with the extended 1-D SWEs are determined empirically because of the lack of available hydrological data. The paddy field area is assumed to be initially dry because the actual dam failure occurred during the pre-irrigation period in Japan. The valleys are also assumed to be initially dry. The rivers in the domain are initially assumed to be in a steady state, such that the water depth does not exceed $H_c$ and remains moderate. Thus, the inflow discharge 10 (m$^3$/s) is prescribed at the upstream-ends J and K. The reservoir is initially assumed to be at full capacity, with a maximum water depth of 18.5 (m). The main and auxiliary embankments are instantaneously and completely removed at the initial time $t = 0$ (s) in each computational case. Although instantaneous and complete dam removal is not a realistic assumption, it serves as the worst-case failure and allows the earliest bound of the flood arrival time to be assessed (Grimaldi and Poggi, 2010). Upstream end A is treated as a solid wall. A critical flow condition is specified at the downstream-end D where a hydraulic drop is installed. The time increment $\Delta t$ is fixed to 0.001 (s). Flow computation for each case is continued until the terminal time $t = 10,800$ (s).
5.5.6 Computational Results

Figure 5-33 presents computed hydrographs at the nodes D, F, and H for each case. The immediate, sharp rise in the hydrograph at the node D in this case implies that the people in Taki Village did not have sufficient time to evacuate before the flooding. The computed flood arrival time at the node D in Case FDB-A is 95 (s), and the average surge velocity in the valley C-D is calculated to be 12 (m/s), which is close to the estimated result (Kyokawa et al., 2011). The maximum water depth at the node D is more than 9 (m) in Case FDB-A, exceeding the channel depth by more than 4 (m), which also agrees well with eyewitness reports and field survey results. Only slight differences are found between the hydrographs at the node D for Cases FDB-A and FDB-C, despite the reservoir water being discharged separately from the two embankments in the latter case. In Case FDB-C, the ratio of the total discharged water volume from the main embankment to that discharged from the auxiliary embankment is 2.5:1.0. The maximum water depth at the node H is approximately 8 (m) for both Cases FDB-B and FDB-C. Hydrograph attenuation due to the divided discharge of the reservoir water in Case FDB-C is remarkable at the node H, compared to that at the node D. These computational results indicate that a reduction in flood damage is actually expected in Case FDB-C. However, the maximum water depth at Taki Village (node D) and the paddy field area (the node H) do not significantly decrease, even when both of the embankments failed simultaneously.

Figures 5-34(a) through 5-34(d) present the water surface profiles for Case FDB-C at the peak discharge from the reservoir \((t = 28 \ (s))\), the peak water depth at the nodes H \((t = 108 \ (s))\), D \((t = 228 \ (s))\), and F \((t = 1,915 \ (s))\), respectively, where the channel bottom elevation at node F is taken as 0. Flood propagation in the channels is reasonably simulated without numerical failure, which demonstrates the robustness of the developed numerical scheme.

Figure 5-35(a) through 5-35(c) show the distributions of the inundation time, which is defined as the total time that the water depth exceeds the channel depth \(H_c\), for the rivers and the paddy field area for each case. The inundation time along the Sunoko River is approximately an hour for Cases FDB-A, which is compatible with the observed facts. The inundation time for the paddy field area in Case FDB-B is more than two hours. In Case FDB-C, the in-
undation time along the Sunoko River and in the paddy field area is at least half of one hour. In this case, the paddy field area would suffer from flooding for more than an hour. Figures 5-36(a) through 5-36(c) show inundation maps for each case. The inundation areas for Cases FDB-A, FDB-B, and FDB-C are 3.03 (km²), 2.15 (km²), and 3.94 (km²), respectively. Figures 5-35 and 5-36 indicate that partially non-flooded channels exist in the Kouka River in Case FDB-C, whereas the river is completely flooded in Case FDB-B. The Sunoko River is completely flooded in Cases FDB-A and FDB-C, and a large portion of the paddy field area is inundated in Cases FDB-B and FDB-C.

Overall, the computational results are compatible to the actual flood event. The computational results for Case FDB-C in particular indicate a potential risk of serious flooding in the downstream area, even when the reservoir water is released separately from two embankments. At present, Fujinuma Reservoir remains damaged. Restoration of the reservoir will be necessary in order to secure an irrigation water supply for rice farming in the study area. However, the computational results clearly show flood risks for the dam downstream area and indicate that extensive discussion is necessary to carefully manage Fujinuma Reservoir in order to minimize hazard and failure risk.
Figure 5-33 Hydrographs at nodes D, F, and H for each case.

Figure 5-34 Water surface profiles in the channel network at each time step for Case FDB-C.
Figure 5-35 Inundation time distribution in the channel network for each case.
Figure 5-36 Inundation map for each case.
5.6 Conclusions

This chapter developed a new numerical scheme for simulating shallow water flows in open channel networks, which is referred to as the DFVF scheme. The scheme was implemented into solving the extended 1-D SWEs where the continuity equation was formulated in an integral form so that the flows around junctions could be handled physically consistently; namely, so that the momentum variation decreases. The scheme uses a novel staggered discretization method, in which a vertex-centered FVM is applied to the continuity equation, while an upwind cell-centered FVM is applied to the momentum equation, thereby avoiding conflicting numbers of equations and unknowns. A semi-implicit temporal discretization is employed in evaluating the friction slope term for effectively handling the flows with extremely shallow water depths.

Computational accuracy of the DFVF scheme was determined through a series of test problems. The scheme successfully produced numerical solutions for the dam break problems in multiply connected channel networks, which were challenging problems. The scheme was finally applied to numerical simulation of flash floods resulting from the Fujinuma Dam failure considering three scenarios, in which only the main embankment failed, only the auxiliary embankment failed, and both the main and auxiliary embankments failed simultaneously. Flow channels from the reservoir to the downstream rivers were considered as a computational domain, which involves non-prismatic and compound channel cross-sections, steep channel slopes, channel bifurcations and convergences, and sharp bends. The DFVF scheme successfully computed flood propagations in the domain without numerical failure. The computational results indicated a high potential flood risk of the reservoir downstream area and suggested the necessity of careful planning with respect to the restoration of Fujinuma Dam.

The DFVF scheme is simple, robust, and computationally efficient. Rapidly varying flows in channel networks are adequately simulated without numerical failures. The proposed scheme is considered to serve as an effective tool for numerical simulation of open channel network flows.
CHAPTER 6  Numerical Scheme for Conservative Parabolic Partial Differential Equation and Its Application to Kolmogorov’s Forward Equations

6.1 Introduction

Fluid flows in porous media or surface water bodies are inherently stochastic. Solute transport phenomena in such a flow are considered as macroscopic appearance of solute particles behaving as stochastic processes. A stochastic differential equation (SDE) representing a stochastic process is associated with its Kolmogorov’s forward equation (KFE) and Kolmogorov’s backward equation (KBE) (Øksendal, 2000). Bodo et al. (1987) reviewed fundamental properties of SDEs, KFEs, and KBEs, which have turned out to be applicable to some of hydrological problems. Cai et al. (1996) established an evaluation method for groundwater contamination using the analytical solutions of KBEs. Fang et al. (2005) extended their evaluation method to more general problems of groundwater pollution processes, using numerical solutions of KBEs for validation. Heemink (1990) presented an advection-dispersion model derived from the KFE governing the transport of solute particles in horizontally 2-D shallow water bodies. Dimou and Adams (1993) discussed consistency between conventional depth-integrated mass transport models and particle tracking models equipped with KFEs, in the context of well-mixed estuarine water flows. Yoshioka et al. (Chapter 4 of this thesis) found that the governing equation of conservative solute transport in surface water flows is deduced from the KFE associated to the SDE for Lagrangian movements of solute particles.

The KFEs, the KBEs, and the transport equations related with them are categorized as the advection-diffusion equations (ADEs). In the view of engineering applications, different numerical methods for the ADEs have been vigorously investigated. Although the numerical methods based on the finite difference (FD) or the finite element (FE) schemes have been widely used for approximating solutions to KFEs, most of these schemes are not equipped with conservative property that mass transport problems require. On the other hand, the finite volume (FV) schemes based on local conservation laws have the advantage of achieving the conservative property when applied to equations in conservative form such as the KFEs.
The majority of existing FV schemes is cell-centered, employing flux-limiter functions in evaluating numerical fluxes on the cell interfaces. Casulli and Zanolli (2005) developed a cell-centered FV scheme for the ADEs in horizontally two- and three-dimensional surface water flows. Mercier and Delhez (2010a; 2010b) proposed a flux-limiter function for advection equations in spatially and temporally varying one-dimensional flow fields. Liang et al. (2010) developed a total variation diminishing (TVD) Mac-Cormack scheme with an operator splitting technique to exhibit higher conservative property than the conventional schemes. The flux-limiter functions are essential in cell-centered FV schemes to preserve TVD properties in discretized advection-dominant ADEs (Hubbard, 1999).

In contrast to cell-centered FV schemes, cell-vertex FV schemes do not necessarily use any flux-limiter function but employ dual mesh in the spatial discretization (Mishev, 1998). In multi-dimensional problems, dual mesh is also termed Voronoi mesh. In some cell-vertex FV schemes, evaluation of fluxes on the cell interfaces is implemented using the analytical solutions of two-point boundary value problems, yielding numerical solutions with TVD property. This evaluation method is referred to as fitting technique (Miller and Wang, 1994). The cell-vertex FV schemes with the fitting technique have successfully been applied to different ADEs including the Black-Scholes equation governing option pricing (Wang, 2004; Chernogorova and Valkov, 2011; Wang et al., 2014), the Hamilton-Jacobi-Bellman equation appearing in optimization problems (Richardson and Wang, 2006), and the reactive solute transport equation in a steady incompressible flow field (Fuhrmann et al., 2011; Fiebach et al., 2014; Eymard et al., 2014). However, to the authors’ knowledge, the fitting technique has not been applied to the ADEs of KFE type.

This chapter presents a numerical scheme suitable for approximating solutions to the KFEs and their related differential equations such as the solute transport equation, focusing on implementation to locally 1-D open channel networks that require special numerical treatment for junctions (Zhang et al., 2004; Zhang et al., 2008). A cell-vertex FV scheme with a fitting technique is developed to discretize the solute transport equation. Unlike conventional numerical schemes that explicitly specify internal boundary conditions (IBCs) at junctions (Islam and Chaudhry, 1998; Basha and Malaeb, 2007), the FV scheme using dual mesh enables consistent
treatment of junctions as implicit IBCs. Several numerical tests are carried out to evaluate accuracy and conservative property of the present FV scheme.

6.2 Conservative Advection-Dispersion Equation

The conservative ADE, which is a solute transport equation, focused on in this chapter has the form (Chapter 4)

\[
\frac{\partial (AC)}{\partial t} + \frac{\partial (VAC)}{\partial x} - \frac{\partial^2 (DAC)}{\partial x^2} + RAC = f
\]  \hspace{1cm} (110)

where \( t \) is the time, \( x \) is the 1-D abscissa, \( C \) is the cross-sectionally averaged solute concentration, \( A \) is the cross-sectional area of reaches, \( V \) is the drift coefficient, \( D(>0) \) is the dispersivity, \( R(\geq 0) \) is the decay coefficient, and \( f \) represents the source and sink terms, which are independent of \( C \). The extended KFE derived in the previous chapter can be regarded as the ADE where the cross-sectional area \( A \) is formally replaced by 1 and the solute concentration \( C \) by an appropriate conditional PDF. Eq.(110) can be rewritten in a flux form as

\[
\frac{\partial (AC)}{\partial t} + \frac{\partial F}{\partial x} + RAC = f
\]  \hspace{1cm} (111)

with the flux \( F \) defined by

\[
F = VAC - \frac{\partial (DAC)}{\partial x}.
\]  \hspace{1cm} (112)

The solute transport equations have to be equipped with appropriate initial and boundary conditions for its well-posedness.

Eq.(110) is well-posed for the problems in 1-D reaches provided that the known functions are sufficiently regular; however, appropriate IBCs have to be equipped with Eq.(110) for the problems with locally 1-D open channel networks. Because Eq.(110) is a parabolic PDE, it has to be equipped with two IBCs at each junction. The first IBC is the continuity of the concentration \( C \) at each junction, meaning that solute concentration is uniquely determined at each junction. The second IBC is local conservation of mass at each junction. This IBC can be expressed as

\[
\sum_{\text{Junction}} F = \delta_j f
\]  \hspace{1cm} (113)

where \( \delta_j \) represents the Dirac’s Delta concentrated at the junction. The continuity of the so-
lute concentration $C$ reduces Eq.(113) to
\[
\sum_{\text{Junction}} \left( VAC - \frac{\partial (DAC)}{\partial x} \right) = C_j \sum_{\text{Junction}} VA - \sum_{\text{Junction}} \frac{\partial (DAC)}{\partial x},
\]
(114)
where $C_j$ represents the solute concentration $C$ at the junction. Because the discharge $Q$ is defined as $Q = VA$, the conservation of mass of water at the junction yields
\[
\sum_{\text{Junction}} VA = \sum_{\text{Junction}} Q = 0.
\]
(115)
For passively-transported particles, Eq.(115) thus reduces Eq.(114) to
\[
\sum_{\text{Junction}} \left( VAC - \frac{\partial (DAC)}{\partial x} \right) = -\sum_{\text{Junction}} \frac{\partial (DAC)}{\partial x},
\]
(116)
meaning that the IBC in this case actually represents the balance of diffusive fluxes at each junction. It is possible to give consistent mathematical formulation of the solute transport equation on generic locally 1-D open channel network following that of the extended continuity equation presented in chapter 3 of this thesis. The formulation is given by
\[
\int_{\Omega \cap B(y,r)} \frac{\partial (AC)}{\partial t} \, dx + \sum_{j=1}^\nu \sigma_j F_{r,j} + \sum_{j=1}^\nu F_{b,j} + \int_{\Omega \cap B(y,r)} RAC \, dx = \int_{\Omega \cap B(y,r)} f \, dx
\]
(117)
where $\Omega$ represents the domain of the flow as a locally 1-D open-channel network, $B(y,r)$ is the horizontally 2-D $r$-neighborhood of a point $y \in \Omega$ with sufficiently small $r(>0)$, $\nu$ is the total number of reaches that intersect with the boundary of $B(y,r)$, $F_{r,j}$ is the flux at the $j$th intersection, $\sum F_{b,j}$ represents the sum of the fluxes specified at the boundary vertices in $\Omega \cap B(y,r)$, and $\sigma_j$ is the sign parameter defining the direction of the abscissa in the $j$th reach. Here, $\sigma_j$ is equal to 1 when the abscissa in the $j$th reach is facing outward to the boundary of $B(y,r)$ and is otherwise equal to $-1$. Eq.(117) is referred to as the extended solute transport equation. Application of Eq.(117) to single channels recovers Eq.(110). The continuity equation serves as an IBC at junctions. In fact, taking the limit $r \to +0$ in Eq.(5) yields the local mass conservation law
\[
\sum_{j=1}^\nu \sigma_j F_{r,j} = \int_{\Omega \cap B(y,r)} f \, dx = \delta_t f,
\]
(118)
which is identical to the IBC(113). The next section presents an numerical method for numerically approximating solutions to Eq.(117).
6.3 Dual-Finite Volume for Solute (DFVS) Scheme

6.3.1 Computational Meshes

The solute transport equation (117) is numerically solved with a cell-vertex FV scheme, which is referred to as the Dual-Finite Volume for Solute (DFVS) scheme for consistently dealing with junctions without explicitly using IBCs. The present DFVS scheme uses analogous computational meshes those are used in the scheme for numerically solving the 1-D SWEs, which was presented in the previous chapter of this thesis.

A couple of staggered computational meshes, which are referred to as a regular mesh and a dual mesh, are used in the DFVS scheme. Firstly, the domain $\Omega$ is divided into a regular mesh that consists of regular cells bounded by two nodes, so that any vertex, such as a junction, falls on one of the nodes. The regular cells and the nodes are indexed with the natural numbers. The total numbers of regular cells and nodes are denoted by $N_c$ and $N_n$, respectively. The $i$ th node is denoted by $P_i$ with its $x$ abscissa $x_i$. The $k$ th regular cell is denoted by $\Omega_k$. The length of $\Omega_k$ is represented by $l_k$. The two nodes bounding both sides of $\Omega_k$ are denoted by the $\varphi(k,1)$th node $P_{\varphi(k,1)}$ and the $\varphi(k,2)$th node $P_{\varphi(k,2)}$. The $x$ abscissa in the cell $\Omega_k$ is directed from the node $P_{\varphi(k,1)}$ to the node $P_{\varphi(k,2)}$. The number of regular cells connected to the node $P_i$ is denoted by $\nu(i)$. The $j$ th regular cell connected to the node $P_i$ is referred to as the $\kappa(i,j)$th regular cell $\Omega_{\kappa(i,j)}$. There are two nodes that bound the cell $\Omega_{\kappa(i,j)}$; one is the $i$ th node $P_i$ and the other is referred to as the $\mu(i,j)$th node $P_{\mu(i,j)}$. In the regular cell $\Omega_{\kappa(i,j)}$, the direction of the abscissa is identified with the parameter $\sigma_{i,j}$, which is equal to 1 when $x$ is directed to the node $P_i$, and is otherwise equal to $-1$. A dual mesh is generated from the regular mesh. Following the multi-dimensional analogue (Mishev, 1998), the $i$th dual cell $S_i$ is associated with the node $P_i$ and is defined as

$$S_i = \left\{ x \in \mathbb{R}^2 \mid x - x_i \leq x_{\rho(i,j)} - x_i \right\} \text{ for } 1 \leq j \leq \nu(i).$$

(119)

The dual mesh consists of $N_n$ dual cells. The cell interface between $S_i$ and $S_{\rho(i,j)}$ is denoted by $\Gamma_{i,j}$. Figure 6-1 sketches of the computational mesh. The solute concentration $C$ is attributed to the dual cells, and the flux $F$ is attributed to the regular cells. The drift $V$ and the cross-sectional area $A$ is attributed to the dual cells. The dispersivity $D$ is attributed to linear functions in regular cells, which are possibly discontinuous at the nodes. The de-
cay coefficient $R$ and the source and sink $f$ are distributed in either regular cells or dual cells depending on the problems.

Figure 6-1 Sketches of the computational mesh: (a) regular mesh, and (b) dual mesh. (same with Figure 5-2)

6.3.2 Spatial Discretization
Approximating $B(\gamma,r)$ by $S_i$ in Eq.(117) with the application of the Gauss-Green theorem results in

$$\frac{\partial}{\partial t} \int_{S_i} u dx + \sum_{j=1}^{v(i)} \sigma_{i,j} F_{i,j} + \int_{S_i} R dx = \int_{S_i} f dx$$

(120)

where the notation $u = AC$ is used in (120), which is also used in what follows for the sake of brevity and $F_{i,j} = \left[ V u - \frac{\partial (Du)}{\partial x} \right]_{s_{i,j}}$ is the flux to be evaluated on $\Gamma_{i,j}$. The integrals in (120) prescribe an internal boundary condition when $S_i$ contains a junction, overcoming the difficulties that the most numerical schemes such as the conventional numerical schemes encounter (Szymkiewicz, 2008; Zhang and Aral, 2004). The three integrals appearing in (120) are evaluated with the one-point integration as

$$\frac{\partial}{\partial t} \int_{S_i} u dx \approx |S_i| \frac{du_i}{df} ,$$

(121)

$$\int_{S_i} R dx \approx |S_i| R u_i ,$$

(122)

and

$$\int_{S_i} q dx \approx |S_i| q_i$$

(123)
where \( |S_i| \) is the measure of \( S_i \), which is given by
\[
|S_i| = \frac{1}{2} \sum_{j=1}^{v(i)} l_{s(i,j)} .
\] (124)

The flux \( F_{i,j} \) on the cell interface \( \Gamma_{i,j} \) is evaluated with a fitting technique, which utilizes the exact solution \( \tilde{u} \) of the two-point boundary value problem
\[
\frac{\partial}{\partial x} \left( V_{s(i,j)} \tilde{u} - \frac{\partial (D_{i,j} \tilde{u})}{\partial x} \right) = 0 \quad \text{in} \quad \Omega_{s(i,j)}
\] (125)
subject to the boundary conditions
\[
\tilde{u}(x_i) = u_i, \quad \tilde{u}(x_{\mu(i,j)}) = u_{\mu(i,j)}
\] (126)
where \( D_{i,j} \) is the linear interpolation of \( D \) in \( \Omega_{s(i,j)} \), which is represented as
\[
D_{i,j} = D_i + \frac{D_{\mu(i,j)} - D_i}{\sigma_{i,j} l_{s(i,j)}} (x - x_i).
\] (127)

The exact solution \( \tilde{u} \) is explicitly obtained to evaluate \( F_{i,j} \) as
\[
F_{i,j} = V_{s(i,j)} \tilde{u} - \frac{\partial (D_{i,j} \tilde{u})}{\partial x} = W_{i,j} \left( \frac{D_{\mu(i,j)}}{D_i} \right)^{a_{i,j}} u_i - u_{\mu(i,j)} - 1
\] (128)
with
\[
a_{i,j} = \frac{W_{i,j} \sigma_{i,j} l_{s(i,j)}}{D_{\mu(i,j)} - D_i}
\] (129)
and
\[
W_{i,j} = V_{s(i,j)} \frac{D_{\mu(i,j)} - D_i}{\sigma_{i,j} l_{s(i,j)}}.
\] (130)

However, the l’Hospital’s rule is applied to (128) for degenerate cases as
\[
F_{i,j} = \frac{D_{\mu(i,j)} - D_i}{\sigma_{i,j} l_{s(i,j)} \log \left( \frac{D_{\mu(i,j)}}{D_i} \right)} (u_i - u_{\mu(i,j)})
\] (131)
when \( D_i \neq D_{\mu(i,j)}, V_{s(i,j)} \neq 0 \), and \( W_{i,j} = 0 \),
\[
F_{i,j} = V_{s(i,j)} \exp \left( \frac{V_{s(i,j)} \sigma_{i,j} l_{s(i,j)}}{D_i} \right) u_i - u_{\mu(i,j)} - 1.
\] (132)
when \( D_i = D_{\rho(i,j)} \) and \( V_{[i,j]} \neq 0 \), and
\[
F_{i,j} = \frac{D_i u_i - D_{\rho(i,j)} u_{\rho(i,j)}}{\sigma_{i,j} [i,j]}
\]
(133)

when \( V_{[i,j]} = 0 \). The scheme satisfies the TVD condition because the inequalities
\[
\frac{\partial (\sigma_{i,j} F_{i,j})}{\partial u_i} > 0
\]
(134)
and
\[
\frac{\partial (\sigma_{i,j} F_{i,j})}{\partial u_{\rho(i,j)}} < 0
\]
(135)
hold. The FVM scheme approaches the fully upwind scheme when the dispersivity \( D \) becomes infinitely small. Flux evaluation using the fitting technique ensures at least first-order spatial convergence of the scheme even for advection-dominant problems (Roos, 1994; Wang, 2004). One boundary condition should be prescribed on each boundary node, specifying the value of either the unknown \( C \) or the flux \( F \).

6.3.3 Temporal Discretization

The DFVS scheme finally leads to the system of ordinary differential equations (ODEs) of the form
\[
\frac{du}{dt} = \Xi u + d
\]
(136)
where \( u = [u_i] \) is the \( N_u \)-dimensional nodal solution vector, \( \Xi = [\Xi_{i,k}] \) is the \( N_u \times N_u \)-dimensional matrix arising from the spatial discretization, and \( d = [d_i] \) is the \( N_u \)-dimensional vector independent from \( u \). The boundary conditions are included in \( d \). The system of ODEs(136) is temporally integrated from a specified initial condition using the \( \theta \)-method (Knabner and Angermann, 2002).

6.4 Theoretical Stability Analysis

This section gives a stability condition of the DFVS scheme. The vector \( d \) is set as the null vector since it does not contribute to stability. It follows that the \( i \)th column of the system of ODEs(136) is described as
\[
\frac{du_i}{dr} = \sum_{k=1}^{N} \varepsilon_{ijk} u_k
\]
\[
= -R_i u_i - \frac{1}{|S|} \sum_{j=1}^{V} \sigma_{ij} F_{ij}
\]
\[
= - \left( R_i + \frac{1}{|S|} \sum_{j=1}^{V} \partial \left( \sigma_{ij} F_{ij} \right) \frac{u_j}{\partial u_i} \right) u_i - \frac{1}{|S|} \sum_{j=1}^{V} \frac{\partial \left( \sigma_{ij} F_{ij} \right)}{\partial u_{\mu(i,j)}} u_{\mu(i,j)}
\]
\[
= \alpha_i u_i + \sum_{j=1}^{V} \beta_{i,j} u_{\mu(i,j)}
\]

with
\[
\alpha_i = - \left( R_i + \frac{1}{|S|} \sum_{j=1}^{V} \frac{\partial \left( \sigma_{ij} F_{ij} \right)}{\partial u_i} \right) < 0
\]

and
\[
\beta_{i,j} = \frac{1}{|S|} \frac{\partial \left( \sigma_{ij} F_{ij} \right)}{\partial u_{\mu(i,j)}} > 0.
\]

Applying the \( \theta \)-method to Eq.(137) yields
\[
\frac{u_i^{(m+1)} - u_i^{(m)}}{\Delta t} = \left( \theta \alpha_i^{(m+1)} + (1-\theta) \alpha_i^{(m)} \right) \left( \theta u_i^{(m+1)} + (1-\theta) u_i^{(m)} \right)
\]
\[
+ \sum_{j=1}^{V} \left( \theta \beta_{i,j}^{(m+1)} + (1-\theta) \beta_{i,j}^{(m)} \right) \left( \theta u_{\mu(i,j)}^{(m+1)} + (1-\theta) u_{\mu(i,j)}^{(m)} \right)
\]
\[
= \overline{\alpha}_i^{(m)} \left( \theta u_i^{(m+1)} + (1-\theta) u_i^{(m)} \right) + \sum_{j=1}^{V} \overline{\beta}_{i,j}^{(m)} \left( \theta u_{\mu(i,j)}^{(m+1)} + (1-\theta) u_{\mu(i,j)}^{(m)} \right)
\]

with
\[
\overline{\alpha}_i^{(m)} = \theta \alpha_i^{(m+1)} + (1-\theta) \alpha_i^{(m)}
\]

and
\[
\overline{\beta}_{i,j}^{(m)} = \theta \beta_{i,j}^{(m+1)} + (1-\theta) \beta_{i,j}^{(m)}
\]

where \( \Delta t > 0 \) is the time increment, \( m \geq 0 \) is the integer, and the superscript \( (m) \) denotes the value evaluated at the time step \( t = m\Delta t \). Eq.(140) is rewritten as
\[
\left( \frac{1}{\Delta t} - \theta \overline{\alpha}_i^{(m)} \right) u_i^{(m+1)} - \theta \sum_{j=1}^{V} \overline{\beta}_{i,j}^{(m)} u_{\mu(i,j)}^{(m+1)} = \left( \frac{1}{\Delta t} + (1-\theta) \overline{\alpha}_i^{(m)} \right) u_i^{(m)} + (1-\theta) \sum_{j=1}^{V} \overline{\beta}_{i,j}^{(m)} u_{\mu(i,j)}^{(m)}.
\]

Assembling Eq.(143) for every \( i \) yields the linear system
\[
K^{(m)} u^{(m+1)} = L^{(m)} u^{(m)}
\]
where the $N_x \times N_y$-dimensional matrices $K^{(m)} = \left[K_{i,j}^{(m)}\right]$ and $L^{(m)} = \left[L_{i,j}^{(m)}\right]$ are defined as
\[
K_{i,j}^{(m)} = \begin{cases} 
\frac{1}{\Delta t} - \theta \alpha_{i,j}^{(m)} & (i = k) \\
-\theta \beta_{i,j}^{(m)} & (k = \mu(i,j)) \\
0 & \text{(Otherwise)}
\end{cases}
\]
and
\[
L_{i,j}^{(m)} = \begin{cases} 
\frac{1}{\Delta t} + (1-\theta) \alpha_{i,j}^{(m)} & (i = k) \\
(1-\theta) \beta_{i,j}^{(m)} & (k = \mu(i,j)) \\
0 & \text{(Otherwise)}
\end{cases}
\]
respectively. It suffices for stability of the scheme that $K^{(m)}$ is an $M$-matrix and $L^{(m)}$ is positive definite. Here an $M$-matrix is a diagonally dominant matrix with positive diagonal entries and non-positive off-diagonal entries (Hundsdoerfer and Verwer, 2007). Inverse of an $M$-matrix is positive definite. Eq.(145) shows that $K^{(m)}$ is an $M$-matrix if it is diagonally dominant and Eq.(146) shows that $L^{(m)}$ is positive definite if its diagonal entries are positive. It is obtained that $K^{(m)}$ is diagonally dominant for $\Delta t$ such that
\[
\frac{1}{\Delta t} + \left(\theta^2 R_i^{(m+1)} + \theta(1-\theta) R_i^{(m)}\right) + \theta^2 \sum_{j=1}^{M} \left[ \frac{\partial (\sigma_{i,j} F_{i,j})}{\partial u_i} + \frac{\partial (\sigma_{i,j} F_{i,j})}{\partial u_{\mu(i,j)}} \right]^{(m+1)} \\
+ \theta(1-\theta) \sum_{j=1}^{M} \left[ \frac{\partial (\sigma_{i,j} F_{i,j})}{\partial u_i} + \frac{\partial (\sigma_{i,j} F_{i,j})}{\partial u_{\mu(i,j)}} \right]^{(m)} > 0
\]
for all $i$, and $L^{(m)}$ is positive definite for $\Delta t$ such that
\[
\frac{1}{\Delta t} + (1-\theta) \alpha_{i,j}^{(m)} > 0
\]
for all $i$. The scheme is stable if $\Delta t$ is chosen so that both the inequalities Eq.(147) and (148) hold true.

### 6.5 Application to Test Problems

The DFVS scheme is verified with steady and unsteady test cases where analytical solutions are available. The system of ODEs(136) is temporally integrated with the Crank-Nicolson method with second-order convergence ($\theta = 0.5$), and the resulting linear system is solved with
the Gauss-Seidel method, which is an iterative method suitable for inverting diagonally dominant and sparse matrices. The time increment $\Delta t$ is determined so that the conditions (147) and (148) are satisfied.

6.5.1 Verification of Computational Accuracy

The DFVS scheme is applied to one-dimensional test problems for determining its computational accuracy. The unit interval $\Omega = (0,1)$ is taken as the computational domain, which is divided into 100 uniform regular cells with 101 dual cells. The two test problems, which are Tests (a) and (b), are examined in this sub-section. Computational conditions of the solute transport equation (117) are specified in each Test but $q = 0$ is fixed. The known functions $V$, $D$, and $R$, the initial condition (I.C.), and the boundary conditions (B.C.) are specified in Tests (a) and (b) as summarized in Table 6-1, where $0 < D_0 < 1$ is a constant and $U(x)$ is given by

$$U(x) = \frac{(1+x)^{1-D_0}}{1-D_0} - 2 \frac{1-x}{1-D_0}.$$  \hspace{1cm} (149)

Test (a) is the problem with linearly varying dispersivity, and the DFVS scheme is verified for both the diffusion dominant case ($D_0 = 0.5$) and the advection dominant case ($D_0 = 0.001$).

Test (b) is the problem with time-dependent coefficients proposed in Ponsoda et al. (2008) as a severe computational test case. This is a challenging problem because the Peclet number Pe associated with this problem, which is given by

$$Pe = \frac{\cos t}{1 + \cos t}$$  \hspace{1cm} (150)

diverges to $-\infty$ as $t$ approaches $\pi - 0$. The exact solutions of Test (a) and Test (b) are

$$u(t,x) = \exp(-t)U(x)$$  \hspace{1cm} (151)

and

$$u(t,x) = \exp(t + x),$$  \hspace{1cm} (152)

respectively. The time increment $\Delta t$ is set as 0.01. Exact and numerical solutions for Test (a) with $D_0 = 0.5$ and $D_0 = 0.001$ are presented in Figures 6-2 and 6-3, respectively, while those for Test (b) are shown in Figure 6-4. For all the test problems, complete agreement can be seen between the exact and computed solutions.
Table 6-1 Computational conditions specified for Tests (a) and (b).

<table>
<thead>
<tr>
<th></th>
<th>V</th>
<th>D</th>
<th>R</th>
<th>I.C</th>
<th>B.C.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test (a)</td>
<td>1</td>
<td>$D_o(1 + x)$</td>
<td>1</td>
<td>$u(0, x) = U(x)$</td>
<td>$u(t, 0) = \exp(-t)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$u(t, 0) = \exp(t)$</td>
<td>$u(t, 0) = \exp(t + 1)$</td>
</tr>
<tr>
<td>Test (b)</td>
<td>$\cos(t)$</td>
<td>$1 + \cos(t)$</td>
<td>0</td>
<td>$u(0, x) = \exp(x)$</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$u(t, 0) = \exp(t)$</td>
<td>$u(t, 1) = \exp(t + 1)$</td>
</tr>
</tbody>
</table>

Figure 6-2 Exact and numerical solutions for Test (a) with $D_o = 0.5$.

Figure 6-3 Exact and numerical solutions for Test (a) with $D_o = 0.001$.

Figure 6-4 Exact and numerical solutions for Test (b).
6.5.2 Verification of Conservation Property

Conservative property of the DFVS scheme is verified with a numerical simulation of conservative solute transport in a hypothetical multiply connected open channel network. A locally one-dimensional open channel network consisting of six reaches is taken as the computational domain $\Omega$ in the present test case, which is divided into $N_c = 600$ uniform regular cells with $N_u = 600$ dual cells. As shown in Figure 6-5, the key nodes defining the boundaries of the reaches are alphabetically labeled A to F, and the $x$ direction in each reach is identified. The key nodes are referred to as the upstream-end (A), the downstream-end (E), the bending points (C and F), and the junctions (B and D). The reaches, which have uniform rectangular cross-sections, are identical to each other in terms of width, length, and bed slope, which are set as 0.5 (m), 200 (m), and 0.005 (m/m), respectively. However, different values are chosen for the Manning’s coefficient $n$ (s/m$^{1/3}$) representing the channel roughness as 0.04 for the reaches B-F-D and as 0.02 for the other reaches. A steady flow field in the channel network is calculated with the numerical method proposed by the M2 scheme (Unami and Alam, 2012), specifying the discharge at the upstream-end A as 0.02 (m$^3$/s) and imposing the uniform depth condition at the downstream-end E. The solute concentration $C$ is non-dimensionalized without loss of generality, and the solute transport equation (39) governing $u = AC$ is examined under the initial condition

$$C(0, x) = 0 \quad \text{in} \quad \Omega ,$$

(153)

the inflow boundary condition

$$F_A = \begin{cases} \sin \left( \frac{\pi t}{360} \right) & (0 \leq t \leq 360) \\ 0 & (t > 360) \end{cases} ,$$

(154)

and the free outflow boundary condition

$$F_E = V_e A_e C_E$$

(155)

where the subscripts A and E denote the values at the node A and at the node E, respectively. The solute concentration $C$ is inversely calculated as the computed $u$ divided by the wetted cross-sectional area $A$. The cross-sectionally averaged velocity $V$ is directly obtained from the calculated flow field, while the dispersivity $D$ is given by the conventional formula (Unami et al., 2010; Kim, 2012)

$$D = \gamma \frac{\rho}{\sqrt{g s h}}$$

(156)
where $\gamma$ is a non-dimensional positive constant, $\theta$ is the hydraulic radius, $g$ is the gravitational acceleration, and $S_f$ is the friction slope determined by the Manning’s formula

$$S_f = \frac{n^2 Q |Q|}{A^2}$$

(157)

where $Q$ is the discharge where the coefficient $\gamma$ is set as 1.0. Both the decay coefficient $R$ and the generation rate $q$ are fixed to zero in order to represent conservative solute transport. Numerical simulation is performed during the time interval $[0, T]$ with the time increment $\Delta t = 1$ (s). The terminal time $T$ is taken sufficiently large so that $u$ is very close to the steady solution $u = 0$ at the time $t = T$. Here, $T$ is set as 10,000 (s). Then, total amount of the solute flowing into the channel network is calculated from (154) as

$$\int_0^T F \cdot n dt = \int_0^T \sin \left( \frac{\pi t}{360} \right) dt = \int_0^{360} \sin \left( \frac{\pi t}{360} \right) dt = \frac{720}{\pi} \approx 229.183.$$  

(158)

The obtained water elevation profile in the channel network is illustrated in Figure 6-6, showing that backwaters and abrupt water depth changes are observed at upstream regions of the junctions and the bending points. Computed discharges in the reaches B-C-D and in the reaches B-F-D are 0.013 (m$^3$/s) and 0.007 (m$^3$/s), respectively. Figure 6-7 shows time series of the solute concentration at the node C and at the node F. The plume of solute arrives earlier with less diffusion at the node C than at the node F, because the both of $V$ and $D$ are larger in the reaches B-C-D than in the reaches B-F-D. Figures 6-8, 6-9, and 6-10 show distributions of $C$ in the channel network at $t = 360$ (s), at $t = 1,800$ (s), and at $t = 3,600$ (s), respectively. The change of the water depth profile results in slight discontinuity appearing in the distribution of $C$ around the node C, as shown in Figure 6-9. Total amount of the solute flowing out from the channel network during the time interval $[0, T]$ is computed as 229.183, indicating that the scheme correctly conserves mass. Computational results show that the DFVS scheme reasonably simulates conservative solute transport in the open channel network.
Figure 6-5 Sketch of the channel network consisting of six reaches.

Figure 6-6 Channel bed elevation and water surface profiles along the reaches A-B-C-D-E and along the reaches B-F-D.
Figure 6-7 Time series of computed solute concentration at the node C and at the node F.

Figure 6-8 Computed solute concentration distributed over the channel network at \( t = 360 \) (s).

Figure 6-9 Computed solute concentration distributed over the channel network at \( t = 1,800 \) (s).
Figure 6-10 Computed solute concentration distributed over the channel network at \( t = 3,600 \) (s).

### 6.5.3 Numerical Simulation of Sediment Transport

Another numerical simulation is carried out to see applicability of the DFVS scheme to transport of solute that may be deposited. Examples of such solute are suspended sediments deposited on floodplains (Thonon et al., 2007) and floating seeds dispersed in fluvial environment (Groves et al., 2009). Evaluating the deposition rate is important to understand and accurately predict sediment transport in surface water flows. Here, it is assumed that re-suspension of the sediment does not take place. The flow field, the cross-sectionally averaged velocity \( V \) and the dispersivity \( D \) are the same as in the previous numerical simulation. The decay coefficient \( R \), which is interpreted as the deposition coefficient, is set as 0.0001 (1/s) in the entire channel network. The generation rate \( q \) is prescribed as

\[
q(t,x) = \delta(t,x - x_m) \tag{159}
\]

where \( x_m \) represents \( x \) at the midpoint of the nodes A and B, and \( \delta(t,x - x_m) \) is the two-dimensional Dirac measure concentrated on \( (t,x) = (0,x_m) \). The generation rate \( q \) in Eq.(159) represents impulsive injection of the solute. The solute concentration \( C \) is again non-dimensionalized and the solute transport equation (117) is solved under the initial condition (153), the inflow boundary condition

\[
F_{I_A} = 0 \tag{160}
\]

and the free outflow boundary condition (155). The terminal time \( T \) is again set as 10,000 (s).
Figure 6-11 shows distributions of the total deposition \( d \) per unit length of the channel, which is defined as

\[
d = d(x) = \int_0^R R(x) A(x) C(t,x) \, dt,
\]

along the reaches A-B-C-D-E and along the reaches B-F-D. As shown in Figure 6-11, a large amount of the solute is deposited at upstream regions of the junctions and the bending points where the backwaters occur. The total deposition is higher in the reaches B-F-D than in the reaches B-C-D, because of the longer residence time of the solute that the flow field brings about via \( V \) and \( D \).

![Graph](image)

Figure 6-11 Computed distributions of the total deposition along the reaches A-B-C-D-E and along the reaches B-F-D.

6.6 Conclusions

The DFVS scheme for numerically approximating solutions to the solute transport equations, which are conservative parabolic PDEs, was presented. In the DFVS scheme, the dependent variable was chosen as the solute concentration that is assumed to be spatially continuous over the domains. Spatial discretization of the DFVS scheme was based on the fitting technique where each term in the finite volume formulation is evaluated using the analytical solutions to
local two-point boundary value problems. The IBCs specified at junctions were consistently and efficiently incorporated into the discretized equations with the help of concurrently using regular and dual meshes. Theoretical stability condition of the DFVS scheme was presented and it was shown that the scheme gives positivity-preserving numerical solutions. Computational performance of the scheme was finally examined through applications to test problems and hypothetical numerical simulations.
CHAPTER 7  Mathematical Analysis on a Numerical Scheme for Non-conservative Parabolic Partial Differential Equation

7.1 Introduction

Advection-Dispersion Equations (ADEs) are elliptic or parabolic partial differential equations (PDEs) arising in a wide variety of scientific and engineering problems (Salsa, 2009). Longitudinal dispersion phenomena of solute particles in 1-D open channel networks are formulated as initial (or terminal) and boundary value problems of the ADEs on connected graphs equipped with appropriate internal boundary conditions (IBCs) at junctions (Yoshioka et al., 2012a). Other possible application examples of the ADEs are longitudinal movements of aquatic organisms living in river systems (Ramirez, 2012; Sarhad et al., 2014). Although the ADEs are typically simplified versions of 3-D mathematical models (Sinha et al., 2012), their usefulness in practical applications is still in high demand. Since analytical solutions to the ADEs are available only for limited cases (Friedlin and Sheu, 2000; Kumar et al., 2009), numerical schemes are utilized for approximating their solutions in applications.

Accurate and stable numerical resolution of the ADEs has been a challenging task. A major difficulty encountered in numerically solving the ADEs is computational instability caused by the existence of the advection terms (Quarteroni and Valli, 2008). A large reaction term can also be a potential source of numerical instability (Duan et al., 2012). Another difficulty, a mathematical and computational issue particular to the ADEs defined on connected graphs, is mathematically consistent and computationally efficient treatment of the IBCs. Some researches pointed out that an improper treatment of the IBCs would yield inaccurate and/or physically invalid numerical solutions (Basha and Malaeb, 2007). In addition, most of the existing numerical schemes separately discretize an ADE at junctions and in reaches, which would result in the loss of computational efficiency (Sanders et al., 2011; Tumanova and Čiegis, 2012; Zhang et al., 2010).

This chapter presents a finite element scheme, which is referred to as the conforming Petrov-Galerkin Finite Element (CPGFE) scheme, for numerically approximating solutions to the Kolmogorov’s backward equations (KBEs) and the related statistics equations. These
PDEs are essentially non-conservative and applications of finite volume (FV) schemes, although which are effective for solving conservative PDEs such as the Kolmogorov’s forward equations (KFEs), is inappropriate. The differential operators defining the KBEs and the statistics equations comply with the maximum principles, indicating that the numerical models should also be equipped with analogous properties, which are referred to as the discrete maximum principles. Numerical schemes not complying with the discrete maximum principals would compute unphysical numerical solutions to the KBEs with spurious oscillations, and compute negative probability density functions (PDFs) in the worst cases. So far, a number of numerical schemes for the ADEs complying with the discrete maximum principles have been presented. Such examples include the finite difference (FD) schemes (Hernandez-Martinez et al., 2013), the FE schemes (Kuzmin, 2006; Knobloch, 2010 Mincsovics, 2010), the discontinuous Galerkin schemes (Horváth and Mincsovics, 2013; Badia and Hierro, 2015). To the authors’ knowledge, however, no or at most only a few researches focused on numerical schemes for the ADEs on connected graphs.

In the present CPGFE scheme, analytical solutions to local two-point boundary value problems are utilized in determining the test and trial functions in the scheme so that stable numerical computation is achieved with compact computational stencils. This numerical technique is referred to as the fitting technique, and it has been recognized to work well in solving the steady ADEs (De Falco and O’Riordan, 2011), and the non-degenerate (NG-Stynes et al., 1988) and the degenerate (Valkov, 2011) unsteady ADEs in 1-D domains. It is demonstrated in this chapter that an advantageous point of the present CPGFE scheme is its ability of consistently and efficiently specifying the IBCs into its spatial discretization.

The remainder of this chapter is organized as follows. Some preliminaries and a concise introduction for the non-conservative ADEs are given in section 7.2. The CPGFE scheme is presented in section 7.3. The numerical diffusivity and numerical decay coefficient are analytically evaluated is presented in section 7.4. Stability analysis on the scheme is performed in section 7.5. Error analysis on the scheme based on the concept of the discrete Green’s function (DGF) (Miller, 2012) is performed in section 7.6. Section 7.7 gives conclusions of this chapter.

121
7.2 Nonconservative Advection-Dispersion Equation

The non-conservative ADE, which can be viewed as a generalization of the KBEs and the statistics equations, focused on in this chapter has the form

\[
\frac{\partial u}{\partial t} - \nu \frac{\partial u}{\partial x} - D \frac{\partial^2 u}{\partial x^2} + Ru = f \tag{162}
\]

where \( t \) is the time, \( x \) is the 1-D abscissa, \( u = u(t,x) \) is the unknown, \( V = V(t,x) \) is the drift coefficient, \( D = D(t,x)(> 0) \) is the dispersivity, \( R = R(t,x)(\geq 0) \) is the decay coefficient, and \( f = f(t,x) \) represents the source and sink terms, which are independent of \( u \).

Eq.(162) has to be equipped with appropriate initial and boundary conditions for well-posedness. It is assumed in this chapter that the solution can be uniquely determined in the usual \( H^1 \)-sense.

Eq.(162) is well-posed for the problems in 1-D reaches provided that the known functions are sufficiently regular; however, appropriate IBCs have to be equipped with Eq.(162) for the problems with locally 1-D open channel networks. Because Eq.(162) is a parabolic PDE, it has to be equipped with two IBCs at each junction. The first IBC is the continuity of the unknown \( u \) at each junction, meaning that solute concentration is uniquely determined at each junction.

The second IBC assumed is the usual Kirchhoff condition stating the balance of the diffusive fluxes, which is given by

\[
- \sum_{\text{junction}} D \frac{\partial u}{\partial x} = \delta_j f \tag{163}
\]

where \( \delta_j \) represents the Dirac’s Delta concentrated at the junction. It is possible to give consistent mathematical formulation of the solute transport equation on generic locally 1-D open channel network following that of the extended continuity equation presented in chapters 3 and 6 of this thesis. The formulation is given by

\[
\int_{\Omega_1} w \frac{\partial u}{\partial t} \, dx - \int_{\Omega} wV \frac{\partial u}{\partial x} \, dx + \int_{\Omega} \frac{\partial(Dw)}{\partial x} \frac{\partial u}{\partial x} \, dx + \int_{\Omega} wRu \, dx = \int_{\Omega} w f \, dx \tag{164}
\]

with generic \( H^1 \)-class weight function \( w \). Eq.(164) is consistent with the ADE(162) and the IBC(163). This is proven as follows. Let \( P \) represent a junction in the domain \( \Omega \) connecting \( \nu \) reaches and \( x_j \) be the local 1-D abscissa in the \( j \)th reach starting from \( P \). Choosing the weight

\[
w_f = \max \left( \frac{\xi - x_j}{\xi}, 0 \right), \quad j = 1, 2, \ldots, \nu \tag{165}
\]

122
with a sufficiently small positive constant $\zeta$ and substituting it into Eq.(164) yields
\[
\sum_{j=1}^{\nu} \int_{0}^{\xi_j} w_p \left( \frac{\partial u}{\partial t} + Ru - f \right) \, dx_j = \sum_{j=1}^{\nu} \int_{0}^{\xi_j} w_p \left( V - \frac{\partial D}{\partial x_j} \right) \frac{\partial u}{\partial x_j} \, dx_j - \sum_{j=1}^{\nu} \int_{0}^{\xi_j} \frac{\partial w_p}{\partial x_j} D \frac{\partial u}{\partial x_j} \, dx_j. \tag{166}
\]
Letting $\zeta \to +0$ in Eq.(166) leads to
\[
\lim_{\zeta \to +0} \sum_{j=1}^{\nu} \int_{0}^{\xi_j} w_p \left( \frac{\partial u}{\partial t} + Ru - f \right) \, dx_j = \lim_{\zeta \to +0} \sum_{j=1}^{\nu} \int_{0}^{\xi_j} f \, dx_j, \tag{167}
\]
\[
\lim_{\zeta \to +0} \sum_{j=1}^{\nu} \int_{0}^{\xi_j} w_p \left( V - \frac{\partial D}{\partial x_j} \right) \frac{\partial u}{\partial x_j} \, dx_j = 0, \tag{168}
\]
and
\[
\lim_{\zeta \to +0} \sum_{j=1}^{\nu} \int_{0}^{\xi_j} \frac{\partial w_p}{\partial x_j} D \frac{\partial u}{\partial x_j} \, dx_j = \lim_{\zeta \to +0} \sum_{j=1}^{\nu} \int_{0}^{\xi_j} D \frac{\partial u}{\partial x_j} \, dx_j \tag{169}
\]
\[
= \sum_{j=1}^{\nu} \left( D \frac{\partial u}{\partial x_j} \right)_{x_j=0}.
\]
Substituting Eqs.(167), (168), and (169) into Eq.(166) yields
\[
-\sum_{j=1}^{\nu} \left( D \frac{\partial u}{\partial x_j} \right)_{x_j=0} = \lim_{\zeta \to +0} \sum_{j=1}^{\nu} \int_{0}^{\xi_j} f \, dx_j, \tag{170}
\]
which is equivalent to Eq.(163).

### 7.3 Conforming Petrov-Galerkin Finite Element (CPGF) Scheme

#### 7.3.1 Computational Mesh

The CPGFE scheme presented in this chapter uses the compact computational stencils that consistently discretizes Eq.(164). Firstly, the domain $\Omega$ is divided into a regular mesh that consists of regular cells bounded by two nodes, so that any vertex, such as a junction, falls on one of the nodes. The word “regular cell” used in chapters 5 and 6 and the word “element” used in what follows are compatible with each other. The elements and the nodes are indexed with the natural numbers. The total numbers of elements and nodes are denoted by $N_e$ and $N_n$, respectively. The $i$th node is denoted by $P_i$ with its $x$ abscissa $x_i$. The $k$th element is denoted by $\Omega_k$. The length of $\Omega_k$ is represented by $l_k$. The two nodes bounding both sides of $\Omega_k$ are denoted by the $\phi(k,1)$th node $P_{\phi(k,1)}$ and the $\phi(k,2)$th node $P_{\phi(k,2)}$. The $x$ abscissa in the element $\Omega_k$ is directed from the node $P_{\phi(k,3)}$ to the node $P_{\phi(k,2)}$. The number of elements connected to the node $P_i$ is denoted by $\nu(i)$. The $j$th element con-
nected to the node \( P_i \) is referred to as the \( \kappa(i,j) \) th element \( \Omega_{\kappa(i,j)} \). There are two nodes that bound the element \( \Omega_{\kappa(i,j)} \); one is the \( i \) th node \( P_i \) and the other is referred to as the \( \mu(i,j) \) th node \( P_{\mu(i,j)} \). In the element \( \Omega_{\kappa(i,j)} \), the direction of the abscissa is identified with the parameter \( \sigma_{\kappa(i,j)} \), which is equal to 1 when \( x \) is directed to the node \( P_i \), and is otherwise equal to \(-1\). The local abscissa

\[
0 \leq z_{i,j} = \frac{x-x_i}{\sigma_{\kappa(i,j)}L_{\kappa(i,j)}} \leq 1
\]

(171)
is defined in \( \Omega_{\kappa(i,j)} \) for the sake of brevity. The unknown \( u \) is attributed to the nodes. The drift \( V \) is attributed to the elements. The dispersivity \( D \) is attributed to linear functions in elements, which are possibly discontinuous at the nodes. The decay coefficient \( R \) and the source and sink \( f \) are distributed in either elements or nodes depending on the problems.

### 7.3.2 Spatial Discretization

The CPGFE scheme is based on the fitting technique, in which the trial and test functions are determined from the analytical solutions to local two-point boundary value problems. In the CPGFE scheme, the trial function \( \phi \) and the test function \( w \) are determined from the analytical solutions to

\[
a_e \frac{d^2 \phi}{dx^2} + b_e \frac{d \phi}{dx} - c_e \phi = 0
\]

(172)
and

\[
a_e \frac{d^2 w}{dx^2} + b_e \frac{d w}{dx} - c_e w = 0
\]

(173)
in each element, respectively where the subscript “e” denotes each element and the coefficients \( a_e, b_e, \) and \( c_e \) are determined as

\[
a_e = V_e - \frac{\partial D}{\partial x} \bigg|_{x_e},
\]

(174)

\[
b_e = D_e,
\]

(175)
and

\[
c_e = R_e,
\]

(176)
respectively where \( D_e \) in Eq.(175) is the arithmetic mean of the nodal \( D \) in the element. Utilizing the fitting technique, the unknown \( u \) is interpolated in the element \( \Omega_{\kappa(i,j)} \) as

\[
u = u_i \phi_{i,j,0} + u_{\mu(i,j)} \phi_{i,j,\mu}
\]

(177)
with the trial functions \( \phi_{i,j,0} \) and \( \phi_{i,j,3} \) given by
\[
\phi_{i,j,0} = \frac{e^{\lambda_{s(i,j)}^+ z_{i,j}} - e^{\lambda_{s(i,j)}^- z_{i,j}}}{e^{\lambda_{s(i,j)}^+} - e^{\lambda_{s(i,j)}^-}}
\]
and
\[
\phi_{i,j,3} = \frac{e^{\lambda_{s(i,j)}^+ z_{i,j}} - e^{\lambda_{s(i,j)}^- z_{i,j}}}{e^{\lambda_{s(i,j)}^+} - e^{\lambda_{s(i,j)}^-}},
\]
respectively where \( \lambda_{s(i,j)}^\pm \) are the non-dimensional numbers defined by
\[
\lambda_{s(i,j)}^\pm = \frac{1}{2} \left( -\text{Pe}_{s(i,j)} \pm \sqrt{\text{Pe}_{s(i,j)}^2 + 4 \text{Da}_{s(i,j)}} \right)
\]
with the cell Peclet number
\[
\text{Pe}_{s(i,j)} = \frac{\sigma_{i,j} b_{s(i,j)} l_{s(i,j)}}{a_{s(i,j)}}
\]
and the cell Damköhler number
\[
\text{Da}_{s(i,j)} = \frac{c_{s(i,j)} l_{s(i,j)}^2}{a_{s(i,j)}} \geq 0.
\]
The trial functions \( \phi_{i,j,0} \) and \( \phi_{i,j,3} \) satisfy the nodal boundary conditions
\[
\phi_{i,j,0}(0) = \phi_{i,j,3}(1) = 1
\]
and
\[
\phi_{i,j,0}(1) = \phi_{i,j,3}(0) = 0.
\]
The test function associated with the node \( P_i \), which is denoted by \( w_i \), is analytically expressed in the element \( \Omega_{s(i,j)} \) as
\[
w_i = \frac{e^{\lambda_{s(i,j)}^+ (1-2z_{i,j})} - e^{\lambda_{s(i,j)}^- (1-2z_{i,j})}}{e^{\lambda_{s(i,j)}^+} - e^{\lambda_{s(i,j)}^-}}.
\]
The test function \( w_i \) satisfies the nodal boundary conditions
\[
w_i(0) = 1 - w_i(1) = 1.
\]
The trial and test functions presented in Eqs.(178), (179), and (185) with the degeneration of the coefficients \( a_{s(i,j)} \), \( b_{s(i,j)} \), and \( c_{s(i,j)} \) are accordingly derived with the l’Hospital’s rule.

### 7.3.3 Temporal Discretization

Application of the present CPGFE scheme to Eq.(164) leads to the linear system of ODEs
\[
M \frac{du}{dt} - f = -Lu = -\sum_i u_i A_{i_k} (w_i, \phi_i)
\]
with the nodal solution vector \( \mathbf{u} \), the mass matrix \( M \), the coefficient matrix \( L \), the vector of the source \( \mathbf{f} \), and the discrete bilinear form
\[
A_h\left( w_i, \phi \right) = \int_{\Omega} \left( -b_{x(i,j)} w_i + a_{x(i,j)} \frac{\partial \phi}{\partial x} \right) \frac{\partial \phi}{\partial x} \, dx + \int_{\Omega} c_{x(i,j)} w_i \phi \, dx. \tag{188}
\]
The exact expressions for the non-zero \( (i,m) \) th entry of the mass matrix \( M \) is given by
\[
M_{i,j} = \sum_{j=1}^{v(i)} \frac{h_{x(i,j)}}{\Delta t} \frac{e^{2 \lambda^{+}_{x(i,j)} \Delta t} - e^{2 \lambda^{-}_{x(i,j)} \Delta t} - 2 \left( \lambda^{+}_{x(i,j)} - \lambda^{-}_{x(i,j)} \right) e^{\lambda^{+}_{x(i,j)} \Delta t} e^{\lambda^{-}_{x(i,j)} \Delta t}} \left( \lambda^{+}_{x(i,j)} - \lambda^{-}_{x(i,j)} \right)^2 \tag{189}
\]
with \( m = i \) and
\[
M_{i,m(i,j)} = \frac{h_{x(i,j)}}{\Delta t} \frac{\left( \lambda^{+}_{x(i,j)} - \lambda^{-}_{x(i,j)} \right) \left( e^{\lambda^{+}_{x(i,j)} \Delta t} + e^{\lambda^{-}_{x(i,j)} \Delta t} \right) - 2 \left( e^{\lambda^{+}_{x(i,j)} \Delta t} - e^{\lambda^{-}_{x(i,j)} \Delta t} \right)} \left( \lambda^{+}_{x(i,j)} - \lambda^{-}_{x(i,j)} \right)^2 \tag{190}
\]
with \( m = \mu(i,j) \). Similarly, the exact expressions for the non-zero \( (i,m) \) th entry of the coefficient matrix \( L \) is given by
\[
L_{i,j} = \sum_{j=1}^{v(i)} \frac{\varepsilon^{-}_{x(i,j)} h_{x(i,j)} \lambda^{+}_{x(i,j)} e^{\lambda^{+}_{x(i,j)} \Delta t} - \lambda^{-}_{x(i,j)} e^{\lambda^{-}_{x(i,j)} \Delta t}} {e^{\lambda^{+}_{x(i,j)} \Delta t} e^{\lambda^{-}_{x(i,j)} \Delta t}} \tag{191}
\]
with \( m = i \) and
\[
L_{i,m(i,j)} = \frac{\varepsilon^{-}_{x(i,j)} h_{x(i,j)} \lambda^{+}_{x(i,j)} - \lambda^{-}_{x(i,j)}} {h_{x(i,j)} e^{\lambda^{+}_{x(i,j)} \Delta t}} \tag{192}
\]
with \( m = \mu(i,j) \). The l’Hospital’s rule is applied to Eqs.(189) through (192) for the degenerated cases.

### 7.4 Numerical diffusivity and decay of the CPGFE Scheme

For steady problems with a 1-D interval, analytical expressions of the numerical diffusivity \( \varepsilon_{h} \) and the numerical decay coefficients \( c_{h} \) in the CPGFE scheme are derived through an application of a modified equation analysis in each element (Ramos, 1999). These numerical coefficients imply that the net diffusivity and the net decay coefficient in the spatial discretization are evaluated as \( \varepsilon + \varepsilon_{h} \) and \( c + c_{h} \) in each element, respectively. Throughout this subsection, the subscript specifying the element \( \kappa(i,j) \) is omitted from the variables for the sake of brevity.

Based on the modified equation analysis, the numerical diffusivity \( \varepsilon_{h} \) attributed to an
element is expressed as
\[
\frac{\varepsilon_h}{\varepsilon} = \frac{\text{Pe} e^{\text{Pe}} + 1}{2} - 1. \tag{193}
\]

The numerical diffusivity in Eq.(193) is identical to those derived in the literatures (Niijima, 1985; Stynes, 2005). By Eq.(193), the asymptotic behavior of the ratio \(\frac{\varepsilon_h}{\varepsilon}\) for the limit \(\varepsilon \to +0\) is given by
\[
\frac{\varepsilon_h}{\varepsilon} \approx \frac{|\text{Pe}|}{2}, \tag{194}
\]
showing that the numerical diffusivity \(\varepsilon_h\) asymptotically approaches that of the conventional fully upwind scheme for the limit \(\varepsilon \to +0\). In fact, Eq.(194) leads to the estimate
\[
\varepsilon_h = O(h). \tag{195}
\]

The numerical decay coefficient \(c_h\) attributed to an element is expressed as
\[
c_h = 2 \frac{\varepsilon_h}{\varepsilon} \left(1 + \frac{\varepsilon_h}{\varepsilon}\right) \frac{e^{\varepsilon_h} - 1}{e^{\varepsilon_h} + 1} - 1, \tag{196}
\]
which is an increasing function of the variables \(\text{Pe}\) and \(\text{Da}\). By Eq.(196), the limit behaviour
\[
c_h \to +0 \tag{197}
\]
holds for \(c \to +0\). Assuming the absence of the drift \((a \to +0)\) in particular, Eq.(196) degenerates to
\[
c_h = 2 \frac{\varepsilon_h}{\varepsilon} \frac{1}{\text{Da}^{1/2}} \sum_{n=1}^{\infty} \frac{1}{(2n)!} - 1, \tag{198}
\]
which leads to the order estimate
\[
c_h = c \left(12 \frac{1}{\text{Da}} + O\left(\text{Da}^{-2}\right)\right) = O(h^2). \tag{199}
\]

For a generic drift \(a\), by Eqs.(193), (194), and (196), the asymptotic behavior of the ratio \(\frac{c_h}{c}\) for \(\varepsilon \to +0\) is estimated as
\[
\frac{c_h}{c} \approx \frac{2}{\text{Da}} \left(1 + \frac{|\text{Pe}|}{2} \right) \left(e^{\varepsilon_h} - 1\right) = O(h). \tag{200}
\]

Figures 7-1 and 7-2 provide the plots for the numerical coefficients \(\varepsilon_h\) and \(c_h\). Eqs.(193) and (196) show that these numerical coefficients are of the order of \(O(h)\) and \(O(h)\) for a generic drift \(a\), and \(O(h)\) and \(O(h^2)\) for the limit \(a \to +0\), respectively. The coefficients satisfy \(a >> c\) for typical transport phenomena in surface water bodies, indicating larger con-
tribution of the numerical diffusivity $\varepsilon_n$ on computational accuracy for the problems. However, influences of the numerical decay coefficient are considered not to be negligible for the reaction-dominant problems appearing in the other research fields (Burioni et al., 2012; Noja, 2014).

Figure 7-1 The numerical diffusivity $\varepsilon_n$ as a function of the non-dimensional variable $Pe$.

Figure 7-2 The numerical decay coefficient $c_h$ as a function of the non-dimensional variables $Pe$ and $Da$.

7.5 Stability Analysis on the CPGFE Scheme
7.5.1 Steady Case
The steady counterpart of Eq.(187) is given by
\[ Lu = f. \] (201)

It is shown that the CPGFE scheme is unconditionally stable in space because the matrix \( L \) in Eq.(201) is an \( M \)-matrix, which is a diagonally-dominant square matrix whose inverse is positive definite. Diagonal and non-diagonal entries of an \( M \)-matrix are positive and negative, respectively. A proof for the unconditional stability is presented below. Firstly, Eqs.(191) and (192) are rewritten as

\[
L_{i,j} = \sum_{j=1}^{v(i)} \sigma_{i,j} \left( b_{k(i,j)} - a_{k(i,j)} \frac{\partial w_j}{\partial x} \right) \tag{202}
\]

and

\[
L_{i,p(i,j)} = \sigma_{i,j} a_{k(i,j)} \frac{\partial w_j}{\partial x} \tag{203}
\]

respectively. By Eq.(203) and the elliptic discrete maximum principle to the test function \( w_j \), the entry \( L_{i,p(i,j)} \) is negative. By Eqs. (185) and (188), Eq.(202) is rewritten in the integral form as

\[
L_{i,j} = \sum_{j=1}^{v(i)} \sigma_{i,j} \left( b_{k(i,j)} - a_{k(i,j)} \frac{\partial w_j}{\partial x} \right) \bigg|_{x=x_j} \bigg|_{x=x_{j_0}} \\
= \left[ \left( -b_{k(i,j)} w_j + a_{k(i,j)} \frac{\partial w_j}{\partial x} \right) w_j \right]_{x=x_{j_0}} \\
+ \int_{x_{j_0}}^{x_{j_1}} \left( c_{k(i,j)} w_j - b_{k(i,j)} \frac{\partial w_j}{\partial x} + a_{k(i,j)} \frac{\partial^2 w_j}{\partial x^2} \right) w_j dx \\
= \int_{x_{j_0}}^{x_{j_1}} \left( -b_{k(i,j)} w_j + a_{k(i,j)} \frac{\partial w_j}{\partial x} \right) \frac{\partial w_j}{\partial x} dx + \int_{x_{j_0}}^{x_{j_1}} c_{k(i,j)} w_j w_j dx \\
= A_j \left( w_j, w_j \right) > 0,
\]

showing that the entry \( L_{i,j} \) is positive. The diagonal dominance of the matrix \( L \) follows from the inequality

\[
|L_{i,j}| - \sum_{j=1}^{v(i)} |L_{i,j}| = L_{i,j} + \sum_{j=1}^{v(i)} L_{i,p(i,j)} \\
= \sum_{j=1}^{v(i)} \int_{x_{j_0}}^{x_{j_1}} c_{k(i,j)} w_j dx \geq 0.
\]

The stability presented ensures the elliptic discrete maximum principles (Idelsohn et al., 1996) of the two schemes. Namely, they can preserve non-negativity of numerical solutions for arbitrary values of the known functions, which cannot be achieved in some of the FE schemes.
with stabilization techniques (Hauke et al., 2007). The proposed CPGFE scheme is superior to the other FE schemes in this sense.

### 7.5.2 Unsteady Case

Stability of the CPGFE schemes for the unsteady case in general depends not only on the element size \( h \) but also on the time increment \( \tau \). The stability here is defined in the Lax-Richtmyer sense, which provides a stronger stability criterion than the Neumann sense (Smith, 1985). The Lax-Richtmyer stability is equivalent to the absolute stability, with which numerical solutions computed with the CPGFE scheme comply with the parabolic discrete maximum principle (Mincsovics, 2010). The stability analysis is performed assuming the \( \theta \)-method in time with \( 0 \leq \theta \leq 1 \). Eq.(187) without the source is then discretized in time as

\[
\Xi u^{n+1} = \Psi u^n
\]

with the square matrices

\[
\Xi = M + \theta \tau L \quad \text{and} \quad \Psi = M - (1-\theta) \tau L.
\]

The scheme is stable if the matrices \( \Xi \) and \( \Psi \) are positive definite matrix and an \( M \)-matrix, respectively. Because \( M \) is a positive definite matrix and \( \Psi \) is an \( M \)-matrix, these conditions are equivalent to

\[
M_{i,i} + \theta \tau L_{i,i} < 0 \quad \text{and} \quad M_{i,i} - (1-\theta) \tau L_{i,i} > 0,
\]

showing that the increment \( \tau \) has a lower bound when \( 0 < \theta \leq 1 \). and has an upper bound if \( 0 \leq \theta < 1 \). By Eq.(208), the present CPGFE scheme is unconditionally stable if the condition

\[
1-\theta^2 > -L_{i,i}^{-1} L_{i,i} M_{i,i}^{-1}
\]

when \( 0 < \theta < 1 \) is satisfied. The condition Eq.(209) would be restrictive in applications as shown below. For example, consider the case where the domain \( \Omega = (0,1) \), the known functions \( a = \pm 1 \), \( c = 0 \), and \( f = 0 \) are given and \( \Omega \) is uniformly discretized into \( m \) elements. The minimum value of the parameter \( \theta \) that Eq.(209) holds is plotted in Figure 7-3 for the CPGFE schemes with and without lamping the mass matrix \( M \) where Pe in the figure is the cell Peclet number defined by

\[
\text{Pe} = \frac{a}{|a| m \epsilon}.
\]

Figure 7-3 shows that the parameter \( \theta \) has to be almost equal to 1 in the present scheme,
meaning that the fully implicit method ($\theta = 1$) seems to be the most reasonable choice, which although may suffer from the lower bound of the time increment by Eq.(209). The minimum value of the parameter $\theta$ with the previous scheme is smaller than that of the present scheme. The difference between them is more clearly seen for larger absolute value of the cell Peclet number $Pe$. The lumping perturbs the mass matrix $M$ with the order of $O(h)$.

![Graph](image.png)

Figure 7-3 Plots of the minimum values of the parameter $\theta$ for guaranteeing temporal stability of the CPGFE scheme with and without lumping the mass matrix.

### 7.6 Error Analysis on the Conforming Petrov-Galerkin Finite Element Scheme

Theoretical error analysis on the CPGFE scheme is performed in this section based on the mathematical concept of the discrete Green’s function (DGF).

#### 7.6.1 Discrete Green’s function

The DGF is a discrete counterpart of the Green’s function defined for continuous problems (Miller et al., 2012). The DGF has been introduced in the error analysis of elliptic problems in 1-D intervals in the literatures (O’Riordan and Stynes, 1986; Axelsson and Karatson, 2013); however, no or at most only a few applications have been made for the problems on connected graphs. This chapter therefore provides basic properties of the DGF on connected graphs. The DGF associated with the $j$th node in the domain $\Omega$, which is denoted by $G_j$, solves the elliptic problem
for arbitrary $\phi \in H^1_0(\Omega)$. The DGF $G_j$ is represented as a linear combination of the basis functions chosen from in the test space. Both unique existence and non-negativity of the DGF $G_j$ are guaranteed owing to the positive-definiteness of the coefficient matrix $L$. In addition, the DGF $G_j$ has the classical $C^2$-regularity in each element because of the element-wise regularity of the basis functions in the test space. The subscript for the coefficients specifying element is omitted for the sake of brevity in the following. The abbreviations

$$G'_j = \frac{dG_j}{dx} \quad \text{and} \quad G''_j = \frac{d^2G_j}{dx^2}$$

are utilized throughout this sub-section. Hereafter $C$ represents generic positive constants independent of the diffusivity $D = \varepsilon$, which is assumed to be constant for the sake of simplicity of the analysis.

A representation formula for the DGF $G_j$, which is used in this paper to prove its bound under the $L^\infty$-norm, is derived as follows. According to Eq.(211), the DGF $G_j$ satisfies

$$\varepsilon G''_j - aG'_j - cG_j = 0$$

in each element in the classical sense subject to the IBC

$$\sum_{j=1}^{N_1} (\varepsilon G'_j - aG_j)_j = -\delta_{j,j}.$$ 

By the IBC(214), integrating Eq.(213) over the domain $\Omega$ yields the equality

$$0 = \int_\Omega (\varepsilon G''_j - aG'_j - cG_j) \, dx$$

$$= \int_\Omega (\varepsilon G''_j - aG'_j) \, dx - \int_\Omega cG_j \, dx$$

$$= -\sum_{k=1}^{N_1} (\varepsilon G'_j - aG_j)_U_k + \sum_{k=1}^{N_1} (\varepsilon G'_j - aG_j)_D_k - \sum_{j=1}^{N_1} (\varepsilon G'_j - aG_j)_j - \int_\Omega cG_j \, dx$$

$$= -\sum_{k=1}^{N_1} (\varepsilon G'_j - aG_j)_U_k + \sum_{k=1}^{N_1} (\varepsilon G'_j - aG_j)_D_k - \int_\Omega cG_j \, dx + 1$$

(215)

where the subscripts $U_k$ and $D_k$ represent the $k$th upstream and downstream vertices that belong to the sets $\Gamma_u$ and $\Gamma_d$, respectively. The formula Eq.(215) accordingly reduces to the conventional model for the problems in single reach (Miller et al., 2012).

In the following, the bound of the DGF $G_j$ in the $L^\infty$-norm is derived for both the Dirichlet-Dirichlet and the Neumann-Dirichlet cases. In the former case, the homogenous Dirichlet condition
\[ G_j = 0 \]  \hspace{1cm} (216)

is specified at both the upstream-ends and the downstream-ends. On the other hand, in the latter case, the homogeneous Neumann condition
\[ \varepsilon G_j' = 0 \]  \hspace{1cm} (217)

is specified at the upstream-ends and the homogenous Dirichlet condition Eq.(216) at the downstream-ends. Eqs.(216) and (217) are consequences of the mathematical fact that the DGF \( G_j \) is given by a linear combination of the test functions subject to the specified boundary conditions for the unknown \( u \). By Eqs.(215), (216), and (217), the representation formulae for the DGF \( G_j \) in the Dirichlet-Dirichlet case and the Neumann-Dirichlet case are derived as
\[
\sum_{k=1}^{N_u} (\varepsilon G_j')_{u_k} = \sum_{k=1}^{N_u} (\varepsilon G_j')_{u_k} + \int_{\Omega} \varepsilon G_j \, dx - 1
\]  \hspace{1cm} (218)

and
\[
\sum_{k=1}^{N_u} (\varepsilon G_j')_{u_k} = -\sum_{k=1}^{N_u} (\varepsilon G_j')_{u_k} + \int_{\Omega} \varepsilon G_j \, dx - 1,
\]  \hspace{1cm} (219)

respectively. In both of the cases, the norm \( \|G_j\|_\Omega \) of the DGF \( G_j \) in the domain \( \Omega \) is bounded with respect to every value of the diffusivity \( \varepsilon (>0) \). This statement is proven in the following for the Dirichlet-Dirichlet and Neumann-Dirichlet cases separately.

7.6.1.1 Dirichlet-Dirichlet Case

The bound for the norm \( \|G_j\|_\Omega \) is derived with the proof of contradiction. The conditions
\[ G_j'\big|_{u_k} \geq 0 \quad \text{and} \quad G_j'\big|_{D_x} \leq 0 \]  \hspace{1cm} (220)

are satisfied because of the non-negativity of the DGF \( G_j \) in the domain \( \Omega \) and the elliptic maximum principle. Assuming the condition
\[
\|G_j\|_\Omega > \frac{1}{c_{\min} |\Omega|}
\]  \hspace{1cm} (221)

with Eqs.(218) and (220) yields the inequality
\[
\sum_{k=1}^{N_t} (\varepsilon G_j)_{t_k} = \sum_{k=1}^{N_t} (\varepsilon G_j)_{t_k} + \int_{\Omega} c G_j \, dx - 1 \\
\geq \int_{\Omega} c G_j \, dx - 1 \\
> c_{\min} |\Omega| \frac{1}{c_{\min} |\Omega|} - 1 \\
= 0,
\]
which is a contradiction, leading to the bound
\[
\|G_j\|_{\mathcal{V}} \leq \frac{1}{c_{\min} |\Omega|}. \tag{223}
\]

7.6.1.2 Neumann-Dirichlet case

By Eq.(220), Eq.(219) reduces to
\[
1 + \sum_{k=1}^{N_t} (a G_j)_{t_k} \geq \int_{\Omega} c G_j \, dx . \tag{224}
\]
By the non-negativity of the term \(a G_j\), Eq.(224) is satisfied when
\[
1 \geq \int_{\Omega} c G_j \, dx \tag{225}
\]
holds. Assuming the condition
\[
\|G_j\|_{\mathcal{V}} \leq \frac{1}{c_{\max} |\Omega|} \tag{226}
\]
holds true leads to Eq.(225). Eq.(226) thus serves as a bound for the Neumann-Dirichlet case. Eqs.(223) and (226) show that the DGF \(G_j\) is bounded as
\[
\|G_j\|_{\mathcal{V}} \leq C . \tag{227}
\]

7.6.2 Analysis on Steady Case

7.6.2.1 Pointwise Error Estimate

An estimate for the pointwise error with the CPGFE scheme, which does not depend on the value of the diffusivity \(\varepsilon\), is derived. It should be noted that such error estimates cannot be obtained with the conventional linear interpolation for the solution \(u\) as pointed out in Appendix A. The discretized coefficients and sources are represented with the subscript \(h\). Firstly, by Eq.(211), the equality
\[(u-u_h)(x_j)=(G,u)-(G,u_h)\]
\[=A_h(G,u)-A_h(G,u_h)+(A(G,u)-(G,f))\]
\[=(G_h,f_h)-A_h(G,u_h)+(A(G,u)-(G,f))\]
\[=(A-A_h)(G,u)-(G,f-h)\]

holds. Eq.(228) leads to the inequality
\[\|u-u_h\|(x_j)\leq\|(A-A_h)(G,u)-(G,f-h)\|\]
\[\leq\|(A-A_h)(G,u)\|+\|(G,f-h)\|\]
\[\leq\|(a-a_h\|\|u\|_{\varepsilon}+\|\varepsilon-c_h\|\|u\|_{\varepsilon}+\|f-f_h\|_{\varepsilon})\|G\|_{\varepsilon}\]
\[\leq\|(a-a_h\|\|u\|_{\varepsilon}+\|\varepsilon-c_h\|\|u\|_{\varepsilon}+\|f-f_h\|_{\varepsilon})C,\] (229)

explicitly relating the regularity conditions of the known functions and their discretization errors with the accuracy of the present CPGFE scheme. Eq.(229) also shows that numerical solutions with the scheme are nodally-exact if the discretized known functions do not contain the errors. This condition is satisfied for the problems with piecewise constant source terms.

The 1-D model of Eq.(229) has been presented in Miller et al. (2012).

In the following, the error estimates on the coefficient and sources
\[\|a-a_h\|_{\varepsilon}, \|\varepsilon-c_h\|_{\varepsilon}, \|f-f_h\|_{\varepsilon} \leq Ch\] (230)
are assumed to be satisfied, which are not considered to be so restrictive in applications. The first term in the left hand side of Eq.(230) is more restrictive than the others since it is given as an \(L^\varepsilon\)-error norm, which is stronger than the \(L^1\)-norm. Assuming the error estimates in Eq.(230), Eq.(229) reduces to
\[\|u-u_h\|(x_j)\leq C\left(Ch\|u\|_{H^{1/2}}+Ch\|u\|_{L^\varepsilon}+Ch\right)\]
\[\leq \max\{\|u\|_{H^{1/2}}, \|u\|_{L^\varepsilon}, C\}Ch.\] (231)

Because the two norms \(\|u\|_{L^\varepsilon}\) and \(\|u\|_{H^{1/2}}\) have the bounds that do not depend on the diffusivity \(\varepsilon\) as shown in Appendix B, the pointwise estimate
\[\|u-u_h\|(x_j)\leq Ch\] (232)
is derived. Eq.(232) means that the numerical solutions with the CPGFE scheme satisfies
\[\lim_{h\rightarrow0} \lim_{\varepsilon\rightarrow0} \|u-u_h\|(x_j) = \lim_{\varepsilon\rightarrow0} \lim_{h\rightarrow0} \|u-u_h\|(x_j) = 0,\] (233)
showing that it is nodally uniformly-convergent.
7.6.2.2 Global Error Estimate
For steady problems, an $L^\infty$-error norm between the exact and numerical solutions with the present CPGFE scheme can be derived. A key to derive the error estimate is the use of a weak maximum principle that the solution $u$ complies with. Under the assumed regularity conditions on the coefficients and the source, it is shown that the solution $u$ has $C^2$-regularity in each reach. The maximum principle of Stynes and O’Riordan (1989) is then satisfied. For the steady case, applying the maximum principle to Eq.(187) yields
\[ \left\| (u-u_h)(x) \right\| \leq r \leq Ch \] (234)
in the domain $\Omega$ with a barrier function $r=r(x)$, a positive function with piecewise $C^2$-regularity in each element. An example of the possible barrier function is provided in Hashimoto et al. (2005). Since the position $x$ in Eq.(234) is arbitrary in the domain $\Omega$ provided it does not coincide with any vertices, Eq.(232) with Eq.(234) consequently leads to the $L^\infty$-error estimate present CPGFE scheme as
\[ \left\| u-u_h \right\|_{L^\infty} \leq Ch, \] (235)
showing that the CPGFE scheme is uniformly-convergent with the $L^\infty$-error norm.

7.7 Conclusions
The CPGFE scheme for approximating solutions to the statistics equations, which are non-conservative PDEs, was presented in this chapter. In the CPGFE scheme, the dependent variable was assumed to be spatially continuous over the domains. Spatial discretization of the CPGFE scheme was based on the fitting technique where both test and trial functions are constructed on the basis of the analytical solutions to local two-point boundary value problems. The IBCs specified at junctions were consistently and efficiently incorporated into the discretized equations owing to appropriately defining the supports of the test and trial functions. Mathematical analysis demonstrated that numerical solutions to the CPGFE scheme rigorously satisfy the elliptic and parabolic discrete maximum principles with sufficiently regular known functions, indicating that the scheme is unconditionally stable in space. It was also shown that unconditional stability of the scheme in time could be guaranteed if an appropriate temporal integration method is utilized. Theoretical error analysis on the scheme using the concept of
the DGF was performed, showing that the scheme is uniformly convergent with respect to the dispersivity. Future researches will focus on real application of the presented CPGFE scheme to KBEs and the related PDEs that govern linear and non-linear transport phenomena in surface water bodies.
CHAPTER 8  Summary and Conclusions

8.1 Summary
This thesis addressed mathematical and numerical modeling of transport phenomena in surface water bodies, focusing in particular on the phenomena occurring in open channel networks. The obtained results in this thesis can be summarized as follows.

1) The new shallow water model, which is referred to as the extended 1-D SWEs, was presented that can physically more consistently handle open channel network flows than the conventional mathematical models. They reduce to the conventional models for the flows in single straight open channels. A momentum flux evaluation scheme serving as an IBC for the momentum equation was also presented as an alternative to the conventional ones. Mathematical analysis based on the non-increase condition of the momentum, which was inspired from the momentum principle in classical physics, demonstrated that the present momentum flux evaluation scheme can unconditionally comply with this condition but the conventional ones do always not. (Chapter 3)

2) A new stochastic process model for analytically assessing solute transport phenomena in turbulent flows was presented. The stochastic process model was based on the SDE governing Lagrangian particle movements and its associated KFE and KBE. It was shown that the transport equation of solute concentration could be deduced from the linearity of the KFE without assuming the conventional Fick’s law. The derived solute transport equation was parabolic as with the conventional models, but has different diffusion terms. The governing equations of the spatially-distributed statistics, which were referred to as the statistics equations, were also presented. These equations were deduced from the KBE. This kind of equations cannot be in principle derived from the conventional, deterministic models. The shallow water counterparts of the model was also presented that are more practical to be used in scientific and engineering applications because of their mathematical
simplicity. The proposed stochastic process model was validated through the KS test with measured velocity time series data sampled in agricultural drainage canals. (Chapter 4)

3) The DFVF scheme for numerically solving the extended 1-D SWEs in locally one-dimensional open channel networks was presented and was examined with test cases involving the classical benchmark problems and the hydraulic experiments carried out by the authors. The most significant advantage of the scheme is its ability of consistently and efficiently incorporating the IBCs into the spatial discretization. The computational results of the test cases demonstrated its satisfactory computational accuracy, robustness, and versatility. Its application to numerical simulation of dam-break flash floods also supported its satisfactory computational performance in engineering applications. Both the flows with and without shock and depression waves were accurately handled and stable computation was performed even for transcritical flows in open channels with non-rectangular and non-prismatic cross-sections. Computational performance of the momentum flux evaluation schemes were also assessed as a part of the test problems, demonstrating that the solutions to the presented scheme outperforms the others. (Chapter 5)

4) The DFVS scheme for approximating solutions to the solute transport equations, which are conservative parabolic PDEs, was presented. In the DFVS scheme, the dependent variable was chosen as the solute concentration that is assumed to be spatially continuous over computational domains. Spatial discretization of the scheme was based on the fitting technique where each term in the finite volume formulation is evaluated with the analytical solutions to local two-point boundary value problems. The IBCs specified at junctions were consistently and efficiently incorporated into the discretized equations owing to the concurrent use of regular and dual meshes. Stability condition of the DFVS scheme was presented and it was then proved that the scheme gives positivity-preserving numerical solutions. Computational performance of the scheme was finally examined through its applications to test and hypothetical cases. (Chapter 6)
5) The CPGFE scheme for approximating solutions to the statistics equations, which are nonconservative PDEs, was presented. In the scheme, the dependent variable was assumed to be spatially continuous over computational domain. Spatial discretization of the CPGFE scheme was based on the fitting technique where both test and trial functions are constructed from the analytical solutions to local two-point boundary value problems. The IBCs specified at junctions were consistently and efficiently incorporated into the discretized equations owing to appropriately defining the supports of the test and trial functions. Mathematical analysis demonstrated that numerical solutions to the CPGFE scheme rigorously satisfy the elliptic and parabolic discrete maximum principles with sufficiently regular known functions, indicating that the scheme is unconditionally stable in space. It was also shown that unconditional stability of the scheme in time could be guaranteed if an appropriate temporal integration method is utilized. Theoretical error analysis on the scheme using the concept of the DGF was performed, showing that the scheme is uniformly convergent with respect to the dispersivity. (Chapter 7)

8.2 Future perspectives

This thesis developed mathematical and numerical models for assessing transport phenomena occurring in surface water bodies. However, there exist a number of unresolved issues, which have to be addressed in future researches. At least, the following issues remain to be resolved.

- Assessment of applicability and limitation of the presented models in real applications, the models for the stochastic process model should be performed in particular.

- Extendibility of the present mathematical models to the more complicated transport phenomena that are of importance in engineering applications should be addressed. Such examples include modeling water flows involving mud and debris materials, which have conventionally been analyzed with the 1-D SWEs in which the pressure and friction terms are modified. Another example is transport phenomena of nonlinearly reacting solutes, for which the present stochastic process model would have to be appropriately modified so that
the nonlinearity is considered in the governing equations. In some real applications, source and sink terms of the solute transport equations would be stochastic, which have to be dealt with in the framework of stochastic PDEs (Ancey et al., 2015; Bohorquez and Ancey, 2015). It is interesting if there is a link between the proposed stochastic process model and the stochastic PDE models.

- Development of more computationally accurate numerical schemes will be necessary for carrying out more reliable numerical simulation, for the problems where solutions have sudden transitions in particular. A possible way for tackling this issue is using some adaptive re-meshing techniques that automatically apply fine computational meshes where the solution profiles present sharp variations. Some elementary investigation on the adaptive re-meshing techniques based on the moving mesh partial differential equations (Huang and Russel, 2011) have already been made by the authors (Yaegashi et al., 2016; Yoshioka et al., 2016). More rigorous mathematical analysis on the presented schemes and the schemes to be developed will be necessary to be carried out for comprehending their performances.

- Development of a coupled 1D-2D mathematical model and its numerical counterparts, which would have potentially wider area of applications than the 1-D models dealt with in this thesis. A key in this research topic can be treatment of IBCs that connect dimensionally heterogeneous computational domains as pointed out in Chen et al. (2012). The integral-based mathematical formulations presented in this thesis can become a foundation in development of the model that has an ability to consistently handle the IBCs.

These issues will be addressed in the authors’ future researches.
APPENDIX A

This appendix presents two examples that the CPGFE scheme fails if the interpolation function for the solution \( u \) is linear. The domain \( \Omega \) is set as the interval \((0,1)\). Assume that the domain \( \Omega \) is uniformly discretized into \( m \) elements with the length of \( h = m^{-1} \) where the \( x \)-abscissa of the \( i \)th node is given by \( x_i = ih \) \( (0 \leq i \leq m) \).

The first example is the Dirichlet problem (Yoshioka et al., 2015a)

\[
\frac{1}{p} \frac{d^2u}{dx^2} - \frac{du}{dx} = 0
\]

with the coefficient \( p = \varepsilon^{-1} \) subject to the boundary conditions \( u(0) = 0 \) and \( u(1) = 1 \). The exact solution to Eq.\(\text{(236)}\) is expressed as

\[
u = \frac{e^{\varepsilon x} - 1}{e^{\varepsilon} - 1}.
\]

Assuming that the previous CPGFE scheme is applied to this problem, the \( L^\infty \)-error norm between the exact solution \( u \) and the numerical solution \( u_h \) is

\[
\|u - u_h\|_{L^\infty} = u_{m-1} + \frac{u_{m} - u_{m-1}}{h} (\alpha - x_{m-1}) - u(\alpha)
\]

with the variable \( \alpha \) given by

\[
\alpha = 1 + \frac{1}{p} \ln \left( \frac{1 - e^{-\varepsilon h}}{p h} \right).
\]

Substituting Eq.\(\text{(239)}\) to Eq.\(\text{(238)}\) yields

\[
\|u - u_h\|_{L^\infty} = \frac{e^{\varepsilon}}{e^{\varepsilon} - 1} \left[ 1 + \frac{1 - e^{-\varepsilon h}}{p h} \left( \ln \left( \frac{1 - e^{-\varepsilon h}}{p h} \right) - 1 \right) \right],
\]

which leads to the estimate

\[
\lim_{h \to 0} \|u - u_h\|_{L^\infty} = 0
\]

for fixed \( p \) but leads to

\[
\lim_{\varepsilon \to 0} \|u - u_h\|_{L^\infty} = \lim_{\varepsilon \to 0} \|u - u_h\|_{L^\infty} = 1
\]

for fixed \( h \).

The second example is the Dirichlet problem

\[
\frac{1}{p^2} \frac{d^2u}{dx^2} - u = 0
\]

with the coefficient \( p = \varepsilon^{-1} \) subject to the boundary conditions \( u(0) = 0 \) and \( u(1) = 1 \). The
The exact solution to Eq.(243) is given by
\[ u = \frac{e^{\mu} - e^{-\mu}}{e^\beta - e^{-\beta}}. \] (244)

The \( L^\infty \) -error norm between the exact solution \( u \) and the numerical solution \( u_h \) is
\[ \| u - u_h \|_{L^\infty} = u_{m-1} + \frac{u_m - u_{m-1}}{h} (\beta - x_{m-1}) - u(\beta) \] (245)

where
\[ \beta = \frac{1}{p} \ln \left[ \frac{1}{2} \left( \frac{\gamma + \sqrt{\gamma^2 - 4}}{\gamma} \right) \right] \] (246)

with
\[ \gamma = \frac{1}{ph} \left[ e^{\rho(1-k)}(e^{\rho h} - 1) + e^{-\rho(1-k)}(1 - e^{-\rho h}) \right] \geq 2. \] (247)

Substituting Eq.(246) into Eq.(245) yields
\[ \| u - u_h \|_{L^\infty} = \frac{(\beta - x_{m-1}) p \gamma - (e^{\rho h} - e^{-\rho h}) + e^{\rho x_m} - e^{-\rho x_{m-1}}}{e^\beta - e^{-\beta}}, \] (248)

again leading to Eqs.(241) and (242).

Miller et al. (2012) reported that the conventional fully upwind scheme does not comply with Eq.(235) as well. The cause of these failures is the use of the linear trial functions with which numerical schemes do not resolve the sharp transitions of the solutions. The above examples show that the finite element scheme with the polynomial test functions (Westerink and Shea, 1989) also fails the uniform-convergence because it uses the linear splines for the trial functions. The results indicate that the addition of an excessive numerical diffusivity does not always lead to the uniform-convergence. The situation is reported to be worse in multi-dimensional cases (Shih et al., 2011). Linear and non-linear multi-dimensional problems that the fully upwind scheme fails are presented in Brandt and Yavneh (1991). The disadvantages of the numerical schemes presented in this appendix can be improved if the fitting mesh technique (Garcia-Archilla, 2013) is employed. However, this technique necessitates \textit{a priori} knowledge on the locations of sharp transitions in the solutions, which cannot be available in real applications. Adaptive re-meshing techniques appropriately considering regularity conditions on the solutions will be a possible way to overcome this problem.

143
APPENDIX B

This appendix provides a proof that the two norms $\|u\|_\infty$ and $\|u\|_{w^1}$ have the bounds that do not depend on the diffusivity $\varepsilon$ as mentioned at just below Eq.(231) in Chapter 8. The notation used in this appendix is same with that in Chapter 8. The procedure presented in this appendix is an extended version of Roos et al. (2008). Firstly, the estimate

$$\|u\|_\infty \leq C \quad (249)$$

follows from the weak maximum principle in each reach, owing to the boundedness and coercivity of the bilinear form $A_\varepsilon$. Secondly, denote a reach in the domain $\Omega$ by $R$. In this appendix, the length of the reach is denoted by $R$. The two vertices that bound the reach $R$ are denoted by $P_0$ and $P_1$. Without the loss of generality, the values of the $x$ abscissa at these nodes are specified as $0$ and $R$ so that the reach is identified with the 1-D interval $(0,R)$. The values of the solution $u$ at the vertices $x=0$ and $x=R$ are denoted by $u_0$ and $u_R$, respectively. Following the 1-D analogue (Roos et al., 2008), the solution $u$ in the reach $R$ satisfies

$$u=u_R - \int_0^R \psi(t)\,dt - \frac{du}{dx} \int_{x=R}^{x=0} \eta(R,t)\,dt \quad (250)$$

with

$$\psi(t) = \int_t^R \frac{1}{\varepsilon}(f-cu)\eta(t,x)\,dt, \quad (251)$$

$$\eta(t,x) = \exp\left(-\frac{1}{\varepsilon}(\hat{a}(t)-\hat{a}(x))\right) > 0, \quad (252)$$

and

$$\hat{a}(x) = \int_0^x a(t)\,dt. \quad (253)$$

By Eqs.(252) and (253), the inequality

$$\exp\left(-\frac{a_{\max}}{\varepsilon}(t-x)\right) \leq \eta(t,x) \leq \exp\left(-\frac{a_{\min}}{\varepsilon}(t-x)\right) \quad (254)$$

follows, which leads to the estimates

$$\|\psi(t)\|_\infty \leq \frac{1}{a_{\min}} \|f-cu\|_\infty \leq C \quad (255)$$

and

$$\left|\frac{du}{dx}\right|_{x=R} \leq C \frac{1}{\varepsilon} \quad (256)$$
By Eq.(256), the equality
\[
\frac{du}{dx} = \psi(x) + \frac{du}{dx}_{x=R-0} \eta(R,x)
\] (257)
holds, leading to the estimate
\[
\int_0^R \left| \frac{du}{dx} \right| dx \leq \int_0^R \psi(x) dx + \left| \frac{du}{dx} \right|_{x=R-0} \int_0^R \eta(R,x) dx
\]
\[
\leq C + \frac{C}{\varepsilon} \int_0^R \exp \left( -\frac{a_{\min}}{\varepsilon} (t-x) \right) dx
\]
(258)
\[
= C.
\]
Eq.(258) leads to the estimate
\[
\|u\|_{H^1} \leq C.
\] (259)
Eqs.(249) and (259) show that the norms \(\|u\|_{H^1} \) and \(\|u\|_{L^\infty} \) are bounded independent of the diffusivity \(\varepsilon\).
Acknowledgements

The author would like to offer my deepest thanks to express his sincerest appreciation and deepest gratitude to Dr. Masayuki FUJIHARA, Professor of Water Resources Engineering, Division of Environmental Science and Technology, Graduate School of Agriculture, Kyoto University, and Chairperson of the Dissertation Committee, for his dedicated supports, helpful suggestions and comments, and encouragement in the completion of my dissertation. The author would also like to extend his sincere appreciation to the members of his Dissertation Committee: Professor Dr. Akira MURAKAMI and Associate Professor Dr. Koichi Unami, for their valuable discussions and advices for reviewing and improving contents of this thesis. The author wishes to express his special thanks to Dr. Toshihiko KAWACHI of former professor of Graduate School of Agriculture, Kyoto University, Dr. Ken MORI of former Professor of Division of Bioproduction and Environment Information Sciences, Department of Bioproduction and Environmental Science, Factory of Agriculture, Kyushu University, and Associate Professor Dr. Kunihiko HAMAGAMI of Iwate University, for their long-term supports from my bachelor period at Faculty of Agriculture, Kyushu University. Heartfelt thanks are also due to Assistant Professor Dr. Junichiro TAKEUCHI of Water Resources Engineering, Division of Environmental Science and Technology, Graduate School of Agriculture, Kyoto University, for his fruitful comments and technical assistance. The author would also like to express his appreciation to the researchers and professors of the Faculty of Life and Environmental Science, Shimane University. The author also likes to thank all members of Water Resources Engineering Laboratory of Kyoto University and Environmental Dynamics Laboratory of Shimane University, secretaries of the Laboratory of Kyoto University, for their assistance, cooperation, and inspiration in all respects of my study period. Finally, the author expresses his deep gratitude to my family for their moral support and warm encouragement.

Hidekazu Yoshioka
References


simulations, American journal of physiology, Heart and Circulatory Physiology, ajpheart-00857.


[104] Kumar, P. and Narayanan, S. (2006) Solution of Fokker-Planck equation by finite ele-


[204] USGS (2011)


2 Society of Civil Engineers, Ser. B1, Vol. 71. (in press)


