<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
<tbody>
<tr>
<td>タイトル</td>
<td>強力な正則dyadicサブベースおよびそのドメイン理論的性質 (要約)</td>
</tr>
<tr>
<td>著者名</td>
<td>Tsukamoto, Yasuyuki</td>
</tr>
<tr>
<td>引用</td>
<td>Kyoto University (京都大学)</td>
</tr>
<tr>
<td>発行日</td>
<td>2016-03-23</td>
</tr>
<tr>
<td>URL</td>
<td><a href="https://doi.org/10.14989/doctor.k19795">https://doi.org/10.14989/doctor.k19795</a></td>
</tr>
</tbody>
</table>
| 条件 | 学位規則第9条第2項により要約公開。許諾条件により本文は2017-05-14に公開。全文ファイル追加。

学位規則第 [条第 [項により要約公開。許諾条件により本文は [公開日に公開。全文ファイル追加。] |

<table>
<thead>
<tr>
<th>項目</th>
<th>内容</th>
</tr>
</thead>
</table>
Strongly proper dyadic subbases and their domain theoretic properties (Summary)

Yasuyuki Tsukamoto*

A real number can be represented by an infinite sequence of finitely many digits in several ways such as binary expansion, signed digit representation and Gray expansion. In binary expansion, one number can have two different representations, e.g., \(1 = 1.000\ldots = 0.111\ldots\). This duplication causes difficulties in computation. For example, when we add \(0.1000\ldots\) and \(0.0111\ldots\), we cannot determine the first digit from the finite prefixes of the two arguments. The modified Gray expansion of the unit interval, in contrast, avoids this disadvantage of binary expansion and provides a unique representation for each number by allowing undefinedness in a sequence \([2, 5]\). In this expansion, a real number is represented by a sequence \(\mathbb{T} = \{0, 1, \perp\}\), where the bottom character \(\perp\) denotes undefinedness. Let \(G_n^a\) \((n < \omega, a \in \{0, 1\})\) denote the set of real numbers whose \(n\)-th digit in this coding is \(a\). The family \(\{G_n^a \mid n < \omega, a \in \{0, 1\}\}\) is a subbase of the unit interval, the Gray subbase.

A dyadic subbase can be considered as a generalization of the Gray subbase. Let \(X\) be a second-countable Hausdorff space. We can see immediately that \(X\) has a subbase that is the union of a countable collection of pairs \((S_n^0, S_n^1)\) \((n < \omega)\) of disjoint open sets. We call such a subbase with a fixed enumeration a dyadic subbase. For a sequence \(\sigma \in \mathbb{T}^\omega\), its domain is defined as \(\text{dom}(\sigma) := \{n < \omega \mid \sigma(n) \neq \perp\}\). The set of all sequences whose domains are finite is denoted by \(\mathbb{T}^*\).

We study dyadic subbases and related domain representations. When a dyadic subbase \(S\) is given, every point of \(X\) is represented by a sequence of \(\mathbb{T}\). That is, we obtain a map \(\varphi_S : X \to \mathbb{T}^\omega\) defined as

\[
\varphi_S(x)(n) = \begin{cases} 0 & (x \in S_n^0) \\ 1 & (x \in S_n^1) \\ \perp & \text{(otherwise)} \end{cases}
\]

Suppose that \(\varphi_S(x)\) can be obtained on an infinite tape as follows. The output on the tape starts from \(\perp^\omega\), and if we get “\(x\) is in \(S_n^a\)” as a result of computation, then the contents of the \(n\)-th cell of the tape is replaced by \(a\). In such a computation, the \(n\)-th cell could be filled with 0 or 1 after the \(m\)-th cell is filled for some \(m > n\). We want to obtain \(\varphi_S(x)\) as the least upper bound of

---

*Graduate School of Human and Environmental Studies, Kyoto University, Japan.
all finite time states of this output. However, the least upper bound of a strictly increasing sequence in the set of compact predecessors of $\varphi_S(x)$ could be less than $\varphi_S(x)$. Hence, we restrict the finite time states of this output to a subset $K_S$ of $\mathbb{T}^\omega$ as follows. We set

$$K_S := \{ \varphi_S(x) \mid x \in X, n < \omega \},$$

$$D_S := \{ \sigma \in \mathbb{T}^\omega \mid (\forall n < \omega)(\sigma|_n \in K_S) \}.$$  

The set $D_S$ is an algebraic pointed dcpo that is the ideal completion of $K_S$. The set of limit elements of $D_S$ is denoted by $L_S$, and the set of minimal elements of $L_S$ is called the minimal limit set $M_S$ of $D_S$. A proper dyadic subbase is a dyadic subbase which satisfies

$$\forall \sigma \in \mathbb{T}^\omega \cdot \bigcap_{n \in \text{dom}(\sigma)} X \setminus S_n^{1-\sigma(n)} = \text{cl} \left( \bigcap_{n \in \text{dom}(\sigma)} S_n^{\sigma(n)} \right).$$

Note that if a dyadic subbase is proper, then $X$ is embedded in $L_S$. We have the following.

**Theorem 1** ([6]). *If the space $X$ is regular Hausdorff and $S$ is proper, then $X$ is embedded in $M_S$. Moreover, if $X$ is compact in addition, then $X$ is homeomorphic to $M_S$.***

**Sketch of the proof.** Suppose that $X$ is regular, $x \in X$ is a point with $n \in \text{dom}(\varphi_S(x))$, $\sigma \in L_S$ is a sequence such that $\sigma$ and $\varphi_S(x)$ are consistent. By using the fact that $S_n^0 := X \setminus (S_n^0 \cup S_n^1)$ is a closed subset which does not contain $x$, we can show $\sigma(n) = \varphi_S(x)(n)$, which implies $\varphi_S(x) \sqsubseteq \sigma$.

Suppose that $X$ is compact. First, we show if $\sigma \in M_S$, then $\hat{S}(\sigma) := \bigcap_{n \in \text{dom}(\sigma)} \text{cl} S_n^{\sigma(n)}$ is not empty. Taking a point $x \in S(\sigma)$, we get $\sigma = \varphi_S(x) \in \varphi_S(X)$. \hfill \Box

We consider the consistent completeness of $D_S$. The dcpo $D_S$ might not be consistently complete. Moreover, even if $D_S$ is consistently complete, changing the enumeration of some pairs in $S$ might cause $D_S$ not to be consistently complete. We define another poset $\hat{D}_S$ which is always consistently complete, and we study a condition which ensures $D_S = \hat{D}_S$. The set of compact elements of $\hat{D}_S$ is denoted by $\hat{K}_S$. Let $S_n^0 (n < \omega)$ denote the common boundary of $S_n^0$ and $S_n^1$. We say that a proper dyadic subbase $S$ is **strongly proper** if

$$\forall \sigma \in \{0,1,\partial, \perp\}^*. \bigcap_{n \in \text{dom}(\sigma)} \text{cl} S_n^{\sigma(n)} = \text{cl} \left( \bigcap_{n \in \text{dom}(\sigma)} S_n^{\sigma(n)} \right).$$

We show that if a dyadic subbase $S$ is strongly proper, then we have $D_S = \hat{D}_S$ and $D_S$ is consistently complete. Conversely, if $D_S$ is consistently complete regardless of the enumeration of $S$, then $S$ is strongly proper. In fact, $D_S$ can be consistently complete even if $S$ is not strongly proper.
Theorem 2 ([7]). Suppose that $S$ is a proper dyadic subbase of a Hausdorff space $X$. The following are equivalent.

1. $S$ is strongly proper.

2. $K_S$ is conditional upper semilattice with least element regardless of the enumeration of $S$.

3. $K_S = \hat{K}_S$.

Sketch of the proof. (1 $\Rightarrow$ 2) Since the first condition does not depend on the enumeration, we have only to show that $K_S$ is a conditional upper semilattice.

(2 $\Rightarrow$ 3) Let $\sigma \in K_S$ be a sequence. For all ordinal $n$ that is less than the length of $\sigma$ but not belonging to $\text{dom}(\sigma)$, changing enumeration of $S$ by a transposition $(0\,n)$, we show that $n$-th digit of $\sigma$ can be replaced by 0 or 1, remaining in $K_S$.

(3 $\Rightarrow$ 1) By decomposing $\sigma \in \{0,1,\partial,\perp\}^*$ into two parts $\sigma|_{\sigma^{-1}(0,1)}$ and $\sigma|_{\sigma^{-1}(\partial)}$, we will show the strong properness under the condition that $K_S = \hat{K}_S$.

We give a characterization of the regularity of spaces through strongly proper dyadic subbases. As we said, if $X$ is regular, then $X$ is embedded in $M_S$. Moreover, the image of each point of $X$ is less than or equal to every consistent element in $L_S$.

Proposition 3. Suppose that $S$ is a strongly proper dyadic subbase of a Hausdorff space $X$. $X$ is regular if and only if $\varphi_S(x) \subseteq \sigma$ for all $\sigma \in L_S$ and $x \in \cap_{n<\omega} \text{cl} S_n^{(n)}$.

We study the case in which the minimal limit set $M_S$ is empty. Since every regular Hausdorff space is embedded in $M_S$, the space cannot be regular. Moreover, if $M_S$ is empty, then $X$ is covered by subsets in which no pair of two points can be separated by closed neighborhoods. We say that a dyadic subbase $S$ is strongly independent if $D_S$ is equal to $\mathbb{T}^\omega$. If $S$ is strongly independent, then $M_S$ is empty. We construct an example of a Hausdorff space with a strongly independent dyadic subbase. Let $\mathbb{N}$ be the set of positive integers, $(p_n)_{n<\omega} = (3, 5, 7, 11, \ldots)$ the sequence of odd prime numbers. We set

$$U_r^0 := \{ n \in \mathbb{N} \mid n \equiv r \pmod{p} \text{ for } 0 < r < \frac{p_n}{2} \}.$$ 

$$U_r^1 := \{ n \in \mathbb{N} \mid n \equiv r \pmod{p} \text{ for } \frac{p_n}{2} < r < p_n \}.$$ 

The topology on $\mathbb{N}$ generated by the family $\{U_r^a \mid n < \omega, a \in \{0,1\}\}$ is denoted by $\mathcal{P}$. We consider the topological space $(\mathbb{N}, \mathcal{P})$.

Theorem 4 ([7]). The space $(\mathbb{N}, \mathcal{P})$ is a Hausdorff space, and has a strongly independent dyadic subbase $S : \omega \times \{0,1\} \to \mathcal{P}$ defined as $S_n^a := U_r^a$ for all $n < \omega$ and $a \in \{0,1\}$.
We show that the Hausdorff property of this space can be deduced from a theorem of Sylvester and Schur.

**Theorem 5** (Sylvester, 1912; Schur, 1929; Erdős, 1934 [1]). Let \( m \) and \( n \) be two natural numbers. If \( n \geq m \), then there exists a number containing a prime divisor greater than \( m \) in the sequence \( n + 1, n + 2, \ldots, n + m \).

The case \( n = m \) corresponds to Bertrand’s postulate. Theorem 5 was first proved by Sylvester and Schur independently, and an elementary proof was given by Erdős [1].

**Sketch of the proof of Theorem 4.** The space has a dyadic subbase \( S \) defined as \( S^a_n := U^{a_n}_n \), and by Chinese Remainder Theorem, \( S \) is strongly independent.

We show that \((N, \mathfrak{H})\) is Hausdorff. Suppose that \( m, n \) are natural numbers with \( m < n \). There are two cases, \( 2m \leq n \) or \( m < n < 2m \). If \( 2m \leq n \), we have \( n \\geq 2 \). From Bertrand’s postulate (or Theorem 5), there exists a prime number \( p \) such that \( n < p < 2n \). Since \( p \geq 3 \), we can put \( p = p_k \) with \( k < \omega \). We have \( m < p_k / 2 < n < p_k \), and hence \( m \in S^0_k \) and \( n \in S^1_k \).

If \( m < n < 2m \), we have \( 0 < 2n - 2m - 1 < 2m \). By Theorem 5, there exists a number \( pq \) containing a prime divisor \( p \) greater than \( 2n - 2m - 1 \) in the sequence \( 2m + 1, 2m + 2, \ldots, 2m + (2n - 2m - 1) = 2n - 1 \). We can see that \( p \) is always odd, and we set \( p = p_k \). We obtain

\[
\frac{q - 1}{2} p_k < m < \frac{q}{2} p_k < n < \frac{q + 1}{2} p_k.
\]

Therefore, \( m \) and \( n \) are separated by \( S^0_k \) and \( S^1_k \).

We can see that the set \( \{ n \in N \mid n + 1 \in S^a_k \} \) is not open for all \( k > 0 \), \( a \in \{0, 1\} \). Therefore, the increment function is not continuous with respect to this topology.

Finally, we study the existence of strongly proper dyadic subbases.

**Theorem 6.** Every locally compact separable metric space has a strongly proper dyadic subbase.

It has been proved that every second-countable regular Hausdorff space has a proper dyadic subbase [3, 4]. By Urysohn’s metrization theorem, such a space is metrizable. First, we give another proof of this fact by using the metric. For a continuous function \( f : X \to \mathbb{R} \) and a real number \( c \), we can define two disjoint open subsets \( \{ x \in X \mid f(x) < c \} \) and \( \{ x \in X \mid f(x) > c \} \). We construct a dyadic subbase \( S \) of the form

\[
S^0_n := \{ x \in X \mid f_n(x) < c_n \}, \quad S^1_n := \{ x \in X \mid f_n(x) > c_n \},
\]

where \((f_n)_{n<\omega}\) is a sequence of continuous functions and \((c_n)_{n<\omega}\) is a sequence of real numbers. We do not fix \( c_n \) first, but we give a sequence \((f_n)_{n<\omega}\) of open intervals from which \( c_n \) will be taken. To obtain the properness, we have only to avoid countably many real numbers as \( c_n \) at each step.
In the construction of a strongly proper dyadic subbase, we assume that the space $X$ is locally compact, and show that the set of real numbers which we have to avoid as $c_n$ is meagre for all $n < \omega$. By Baire category theorem, every meagre subset of $\mathbb{R}$ is codense. We can inductively take $c_n$ such that $S$ is strongly proper.

References


