Yang–Baxter invariance of the Nappi–Witten model

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1. Introduction

The Yang–Baxter sigma-model description, which was originally proposed by Klimcik [1], is a systematic way to consider integrable deformations of 2D non-linear sigma models. According to this procedure, the deformations are specified by skew-symmetric classical $r$-matrices satisfying the modified classical Yang–Baxter equation (mCYBE). The original work [1] has been generalized to symmetric spaces [2] and the homogeneous CYBE [3].

Yang–Baxter deformations of the $\text{AdS}_5 \times S^5$ superstring can be studied with the mCYBE [4] and the CYBE [5]. For the former case, the metric and $B$-field are derived in [6] and the
full background has recently been studied in [7,8]. For the latter case, classical $r$-matrices are identified with solutions of type IIB supergravity including $\gamma$-deformations of $S^5$ [9,10] and gravity duals of non-commutative gauge theories [11,12], in a series of works [13–20] (for a short summary, see [21]).

Lately, Yang–Baxter deformations of 4D Minkowski spacetime have been studied [22,23]. In [22], classical $r$-matrices are identified with exactly-solvable string backgrounds such as Melvin backgrounds and pp-wave backgrounds. In [23], Yang–Baxter deformations of 4D Minkowski spacetime are discussed by using classical $r$-matrices associated with $\kappa$-deformations of the Poincaré algebra [24]. Then the resulting deformed geometries include T-duals of (A)dS$_4$ spaces\(^1\) and a time-dependent pp-wave background. Furthermore, the Lax pair is presented for the general $\kappa$-deformations [23,26].

As a spin off from this progress, it would be interesting to study Yang–Baxter deformations of the Nappi–Witten model [27]. The target space of this model is given by a centrally extended 2D Poincaré group. Hence the Yang–Baxter deformed Nappi–Witten models can be regarded as toy models of the previous works [22,23], because the structure of the target space is much simpler than that of 4D Minkowski spacetime. This simplification makes it possible to study the most general Yang–Baxter deformation. As a matter of course, it is exceedingly complicated in general, hence such an analysis has not been done yet.

In this article, we investigate Yang–Baxter deformations of the Nappi–Witten model by following a prescription invented by Delduc, Magro and Vicedo [28]. We show that the sigma-model metric is invariant under the deformations (while the coefficient of $B$-field is changed) by utilizing the most general classical $r$-matrix. Furthermore, the coefficient of $B$-field is determined to be the original value from the requirement that the one-loop $\beta$-function should vanish. After all, the Nappi–Witten model is the unique conformal theory within the class of the Yang–Baxter deformations preserving the conformal invariance (i.e., Yang–Baxter invariance).

2. Nappi–Witten model

In this section, we shall give a concise review of the Nappi–Witten model [27].

The Nappi–Witten model is a Wess–Zumino–Witten (WZW) model whose target space is given by a centrally extend 2D Poincaré group. The associated extended Poincaré algebra $\mathfrak{g}$ is composed of two translations $P_i$ ($i = 1, 2$), a rotation $J$ and the center $T$. The commutation relations of the generators are given by

\[
[J, P_i] = \epsilon_{ij} P_j, \quad [P_i, P_j] = \epsilon_{ij} T, \quad [T, J] = [T, P_i] = 0, \tag{2.1}
\]

where $\epsilon_{ij}$ is an anti-symmetric tensor normalized as $\epsilon_{12} = 1$. It is convenient to introduce a notation of the generators with the group index $I$ like

\[
T_I = \{ P_1, P_2, J, T \} \quad (I = 1, 2, 3, 4). \tag{2.2}
\]

Let us introduce a group element represented by

\[
g = \exp(a_1 P_1 + a_2 P_2) \exp(u J + v T). \tag{2.3}
\]

By using this group element $g$, the left-invariant deformed current $A$ can be evaluated as

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\(^1\) T-dual of dS$_4$ can be derived as a scaling limit of $\eta$-deformed AdS$_5$ as well [25].
\[ A_\alpha \equiv g^{-1} \partial_\alpha g = A^I T_I \]
\[ = (\cos u \partial_\alpha a_1 + \sin u \partial_\alpha a_2) P_1 + (\cos u \partial_\alpha a_2 - \sin u \partial_\alpha a_1) P_2 
+ \partial_\alpha u J + \left[ \partial_\alpha v + \frac{1}{2} a_2 \partial_\alpha a_1 - \frac{1}{2} a_1 \partial_\alpha a_2 \right] T. \] (2.4)

Here the index \( \alpha = \tau, \sigma \) is for the world-sheet coordinates. It is also helpful to introduce the light-cone expression of \( A \) on the world-sheet like
\[ A_{\pm} \equiv A_{\tau} \pm A_{\sigma}. \] (2.5)

By using \( A_{\pm} \), the classical action of the Nappi–Witten model is given by
\[ S[A] = \frac{1}{2} \int_\Sigma d^2 \sigma \, \Omega_{IJ} A^I_+ A^J_+ + \frac{1}{6} B_3 \int d^3 \sigma \, \varepsilon^\hat{\alpha} \hat{\beta} \hat{\gamma} \, \Omega_{KL} f_{IJ} f^L_{KL} A_+^I A_+^J A^K_+. \] (2.6)

This action is basically composed of the two parts, 1) the sigma model part and 2) the Wess–Zumino–Witten (WZW) term.

The sigma model part is defined as usual on the world sheet \( \Sigma \), where we assume that \( \Sigma \) is compact and the periodic boundary condition is imposed for the dynamical variables. A key ingredient contained in this part is the most general symmetric two-form\(^2\)
\[ \Omega_{IJ} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & b & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \] (2.7)

which satisfies the following condition:
\[ f_{IJ}^K \Omega_{KL} + f_{IL}^K \Omega_{JK} = 0. \] (2.8)

Here \( f_{IJ}^K \) are the structure constants which determine the commutation relations
\[ [T_I, T_J] = f_{IJ}^K T_K. \]

The WZW term in (2.6) also contains \( \Omega_{IJ} \), but, apart from this point, it is the same as the usual. The symbol \( B_3 \) denotes a 3D space which has \( \Sigma \) as a boundary. Hence the domain of \( A^I \) is implicitly generalized to \( B_3 \) with \( \hat{\alpha} = \tau, \sigma \) and \( \xi \) in the WZW term,\(^3\) where the extra direction is labeled by \( \xi \).

A remarkable point is that the action (2.6) can be rewritten into the following form \([27]^{4}\):
\[ S = -\frac{1}{2} \int_\Sigma d^2 \sigma \left[ \gamma^{\alpha\beta} g_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu - \epsilon^{\alpha\beta} B_{\mu\nu} \partial_\alpha X^\mu \partial_\beta X^\nu \right]. \] (2.9)

Here \( X^\mu = \{ u, v, a_1, a_2 \} \) are the dynamical variables. The metric and anti-symmetric two-form on \( \Sigma \) are described by \( \gamma^{\alpha\beta} = \text{diag}(-1, 1) \) and \( \epsilon^{\alpha\beta} \) normalized as \( \epsilon^\tau\sigma = 1 \). Then the space–time metric \( g_{\mu\nu} \) and two-form field \( B \) are given by

\(^2\) The overall factor of \( \Omega_{IJ} \) (i.e., the level of the WZW model) is set to be 1 because it is irrelevant to the deformations we consider later.

\(^3\) For the detail of the WZW model, for example, see [29].

\(^4\) In this derivation, we have used the identities (12) and (13) in [27].
\[ ds^2 = g_{\mu\nu} dX^\mu dX^\nu = 2duds + b d\eta^2 + da_1^2 + da_2^2 + a_1 da_2 du - a_2 da_1 du , \]
\[ B = B_{\mu\nu} dX^\mu \wedge dX^\nu = u da_1 \wedge da_2 . \] (2.10)

This is a simple 4D background.

It would be helpful to further rewrite the background (2.10). By performing the following coordinate transformation with a real constant \( m \) [30]
\[
\begin{align*}
a_1 &\rightarrow a_1 \cos (m u) + a_2 \sin (m u), \\
a_2 &\rightarrow a_1 \sin (m u) - a_2 \cos (m u), \\
u &\rightarrow 2m u, \\
v &\rightarrow \frac{1}{2m} v - bm u ,
\end{align*}
\] (2.11)
the background (2.10) can be rewritten as
\[
\begin{align*}
ds^2 &= 2duds - m^2 (a_1^2 + a_2^2) du^2 + da_1^2 + da_2^2 , \\
B &= -2m u da_1 \wedge da_2 + 2m^2 a_1 u da_1 \wedge du + 2m^2 a_2 u da_2 \wedge du .
\end{align*}
\] (2.12)
This is nothing but a pp-wave background. Note here that the last two terms of \( B \)-field in (2.12) contribute to the Lagrangian as the total derivatives, which can be ignored in the present setup. For this background (2.12), the world-sheet \( \beta \)-function vanishes at the one-loop level [27].

3. Yang–Baxter deformed Nappi–Witten model

In this section, let us consider Yang–Baxter deformations of the Nappi–Witten model.

3.1. A Yang–Baxter deformed classical action

As explained in the previous section, the Nappi–Witten model contains the WZW term. Hence it is not straightforward to study Yang–Baxter deformations of this model. Our strategy here is to follow a prescription invented by Delduc, Magro and Vicedo [28]. This is basically a two-parameter deformation. It is an easy task to extend their prescription to the Nappi–Witten model.

A deformed action we propose is the following:\(^5\):
\[
S = \frac{1}{2} \int_{\Sigma} d^2 \sigma \ \Omega_{IJ} A_I^J J_\pm + \frac{1}{2k} \int_{\Sigma} d^3 \sigma \ \Omega_{KL} f_{IJ} L A_I^K A_J^L A_\pm .
\] (3.1)

Here the deformed current \( J \) is defined as
\[
J_\pm = (1 + \omega \eta^2) \frac{1}{1 - \eta^2 R^2} A_\pm .
\] (3.2)

First of all, the classical action (3.1) includes three constant parameters \( \eta, A \) and \( k \). The deformation is measured by \( \eta \) and \( A \). The last parameter \( k \) is regarded as the level. When \( \eta = A = 0 \) and \( k = 1 \), the action (3.1) is reduced to the original Nappi–Witten model.

A key ingredient contained in \( J \) is a linear operator \( R \): \( g \rightarrow g \). In the context of Yang–Baxter deformations, it is supposed that \( R \) should be skew-symmetric and satisfy the (modified) Yang–Baxter equation [31]

\(^5\) Here we would like to start from the WZNW model (non-conformal) rather than the WZW model (conformal). Hence, a real constant \( k \) has been put in front of the WZW term to measure the non-conformality.
\[ [R(X), R(Y)] - R([R(X), Y] + [X, R(Y)]) = \omega [X, Y] \quad (X, Y \in g). \] (3.3)

The constant parameter \( \omega \) can be normalized by rescaling \( R \), hence it is enough to consider the following three cases: \( \omega = \pm 1 \) and 0. In particular, the case with \( \omega = 0 \) is the homogeneous CYBE.

### 3.2. The general solution of the (m)CYBE

In this subsection, we derive the general solution of the (m)CYBE.

Let us start from the most general expression of a linear \( R \)-operator:

\[ R(X) = M^{IJ} \Omega_{JK} X^K T_I, \quad M^{IJ} \equiv \sum_i \left( a_i^l b_i^j - b_i^l a_i^j \right). \] (3.4)

Here \( M^{IJ} \) is an anti-symmetric \( 4 \times 4 \) matrix which is parametrized as

\[
M^{IJ} = \begin{pmatrix}
0 & m_1 & m_2 & m_3 \\
-m_1 & 0 & m_4 & m_5 \\
-m_2 & -m_4 & 0 & m_6 \\
-m_3 & -m_5 & -m_6 & 0
\end{pmatrix}, \quad m_i \in \mathbb{R}. \quad (3.5)
\]

By using the expression (3.4) and the defining relation of \( \Omega_{IJ} \) (2.8), the (m)CYBE (3.3) can be rewritten into the following form:

\[
f_{LM}^K M^{LI} M^{MJ} + f_{LM}^I M^{LJ} M^{MK} + f_{LM}^J M^{LK} M^{MI} - \omega f_{LM}^K \Omega_{LI} \Omega_{MJ} = 0. \quad (3.6)
\]

Note that we define \( \Omega_{IJ} \) as the inverse matrix of \( \Omega_{IJ} \). Then, by putting the expression (3.5) into (3.6), the most general solution can be determined like

\[
M^{IJ} = \begin{pmatrix}
0 & \sqrt{\omega} & 0 & m_3 \\
-\sqrt{\omega} & 0 & 0 & m_5 \\
0 & 0 & 0 & m_6 \\
-m_3 & -m_5 & -m_6 & 0
\end{pmatrix}. \quad (3.7)
\]

Here the condition (3.6) has led to the following constraints:

\[ m_1 = \sqrt{\omega}, \quad m_2 = m_4 = 0. \]

Then we have also supposed that \( \omega \geq 0 \) in order to preserve the reality of the background.\(^6\) After all, \( m_3, m_5 \) and \( m_6 \) have survived as free parameters of the \( R \)-operator as well as \( \omega \).

### 3.3. The general deformed background

Let us consider a deformation of the Nappi–Witten model with the general solution (3.7). The resulting background is given by

\[
ds^2 = da_1^2 + da_2^2 + \frac{(1 + \omega \eta^2) b - (m_3^2 + m_2^2) \eta^2}{1 - m_6^2 \eta^2} du^2 + 2 \frac{1 + \omega \eta^2}{1 - m_6^2 \eta^2} dudv + \frac{(1 + \omega \eta^2) a_2 + 2 \eta^2 ((m_3 m_6 + m_5 \sqrt{\omega}) \cos u - (m_5 m_6 - m_3 \sqrt{\omega}) \sin u)}{1 - m_6^2 \eta^2} da_1 du
\]

\(^6\) When we consider \( \omega < 0 \), we need to multiply the linear \( R \)-operator by the imaginary unit \( i \).
\[ -\frac{(1 + \omega \eta^2) a_1 - 2\eta^2((m_5m_6 - m_3\sqrt{\omega}) \cos u + (m_3m_6 + m_5\sqrt{\omega}) \sin u)}{1 - m_6^2 \eta^2} \, \, da_2 du, \]

\[ B = k u da_1 \wedge da_2 + \frac{\tilde{A}m_6(1 + \omega \eta^2)}{2(1 - m_6^2 \eta^2)} (a_2 da_1 \wedge du - a_1 da_2 \wedge du). \] (3.8)

Here we have ignored the total derivative terms that appeared in the \( B \)-field part. This background (3.8) can be simplified by performing a coordinate transformation

\[ a_1 \rightarrow a_1 \cos (m u) + a_2 \sin (m u) + C_1 \cos (C_3 u) + C_2 \sin (C_3 u), \]

\[ a_2 \rightarrow -a_2 \cos (m u) + a_1 \sin (m u) - C_2 \cos (C_3 u) + C_1 \sin (C_3 u), \]

\[ u \rightarrow C_3 u, \]

\[ v \rightarrow \frac{1}{2m} v - \frac{1}{2} b C_3 u - \frac{1}{2} \left[ C_2 \cos \left( \frac{C_3 - C_4}{2} u \right) - C_1 \sin \left( \frac{C_3 - C_4}{2} u \right) \right] a_1 \]

\[ + \frac{1}{2} \left[ C_1 \cos \left( \frac{C_3 - C_4}{2} u \right) + C_2 \sin \left( \frac{C_3 - C_4}{2} u \right) \right] a_2, \] (3.9)

where we have introduced the following quantities:

\[ C_1 \equiv \frac{m_5m_6 - m_3\sqrt{\omega}}{m_6^2 + \omega}, \quad C_2 \equiv \frac{m_3m_6 + m_5\sqrt{\omega}}{m_6^2 + \omega}, \]

\[ C_3 \equiv 2m \frac{1 - m_6^2 \eta^2}{1 + \omega \eta^2}, \quad C_4 \equiv 2m \frac{m_6^2 + \omega}{1 + \omega \eta^2}. \] (3.10)

After performing the transformation (3.9), the resulting background is given by the following pp-wave background equipped with a \( B \)-field:

\[ ds^2 = 2du dv - m^2(a_1^2 + a_2^2)du^2 + da_1^2 + da_2^2, \]

\[ B = -k C_3 u da_1 \wedge da_2 - m\tilde{A}m_6a_2 da_1 \wedge du + m\tilde{A}m_6a_1 da_2 \wedge du. \] (3.11)

Here we have ignored the total derivative terms again.

Note that the \( B \)-field in (3.11) can be rewritten as (up to total derivative terms)

\[ B = -(k C_3 - 2m \tilde{A}m_6) u da_1 \wedge da_2. \] (3.12)

Comparing (3.12) with the \( B \)-field in (2.12), one can find that only the difference is the coefficient of \( B \)-field. From the viewpoint of the original Nappi–Witten model, the coefficient of the WZW term has been changed and the resulting theory should be regarded as a WZNW model. According to this observation, it is obvious that the deformed model is exactly solvable.\(^7\)

3.4. Yang–Baxter deformations and conformal invariance

Finally, let us show that the original Nappi–Witten model is the unique conformal theory within the class of the Yang–Baxter deformations preserving the conformal invariance.

Due to the requirement of the vanishing \( \beta \)-function at the one-loop level, the two \( B \)-fields should be identical as follows:

\(^7\) In the case of [28], it is necessary to impose a condition for \( \tilde{A} \) so as to preserve the integrability (cf., [32]). However, such an extra condition is not needed in the present case.
\[ 2m \quad \text{(the original)} = \quad k C_3 - 2m \overset{\sim}{C}_6 \quad \text{(the deformed)}. \] (3.13)

This condition indicates two interesting results. The first one is the Yang–Baxter invariance of the Nappi–Witten model. If we start from the case with \( k = 1 \), then the original system is invariant under the Yang–Baxter deformations preserving the conformal invariance, which are specified by the parameters satisfying the condition

\[ 1 = \frac{1 + \omega \eta^2}{1 - m_6 \overset{\sim}{\eta}^2} (1 + m_6 \overset{\sim}{A}). \]

In other words, the Yang–Baxter invariance follows from the conformal invariance.

The second is that the Yang–Baxter deformation may map a non-conformal theory to the conformal Nappi–Witten model. Suppose that we start from the case with \( k \neq 1 \). Then, by performing a Yang–Baxter deformation with parameters satisfying the condition

\[ k = \frac{1 + \omega \eta^2}{1 - m_6 \overset{\sim}{\eta}^2} (1 + m_6 \overset{\sim}{A}). \] (3.14)

the resulting system becomes the Nappi–Witten model. In other words, the coefficient of \( B \)-field can be set to the conformal fixed point by an appropriate Yang–Baxter deformation.

4. Conclusion and discussion

In this article, we have studied Yang–Baxter deformations of the Nappi–Witten model. By considering the most general classical \( r \)-matrix, we have shown the invariance of the sigma-model metric under arbitrary deformations, up to two-form \( B \)-fields. That is, the effect coming from the deformations is reflected only as the coefficient of \( B \)-field. Then, the coefficient of \( B \)-field has been determined to be the original value from the requirement that the one-loop \( \beta \)-function should vanish. After all, it has been shown that the Nappi–Witten model is the unique conformal theory within the class of the Yang–Baxter deformations preserving the conformal invariance (i.e., Yang–Baxter invariance).

There are many future directions. It would be interesting to consider a supersymmetric extension of our analysis by following [33]. The number of the remaining supersymmetries should depend on Yang–Baxter deformations because the coefficient of \( B \)-field is changed. It is also nice to investigate higher-dimensional cases (e.g., the maximally supersymmetric pp-wave background [34]). As another direction, one may consider non-relativistic backgrounds such as Schrödinger spacetimes [35] and Lifshitz spacetimes [36]. Although there is a problem of the degenerate Killing form similarly, it can be resolved by adopting the most general symmetric two-form [37] as in the Nappi–Witten model. It would be straightforward to apply the techniques presented in [37] to Yang–Baxter deformations by following our present analysis.

It should be remarked that the most interesting indication of this work is the universal aspect of the dual gauge-theory side. According to our work, pp-wave backgrounds would have a kind of rigidity against Yang–Baxter deformations. This result may indicate that the ground state and lower-lying excited states of the spin chain associated with the \( N = 4 \) super-Yang–Mills theory are invariant. It is quite significant to extract such a universal characteristic after classifying various examples. This is the standard strategy in theoretical physics and would be much more important than identifying the associated dual gauge theory for each of the deformations.

We hope that our result could shed light on a universal aspect of Yang–Baxter deformations from the viewpoint of the invariant pp-wave geometry.
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References


