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Comprehensive analysis of the wave function of a hadronic resonance and its compositeness

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We develop a theoretical framework to investigate the two-body composite structure of a resonance as well as a bound state from its wave function. For this purpose, we introduce both one-body bare states and two-body scattering states, and define the compositeness as a fraction of the contribution of the two-body wave function to the normalization of the total wave function. Writing down explicitly the wave function for a resonance state obtained with a general separable interaction, we formulate the compositeness in terms of the position of the resonance pole, the residue of the scattering amplitude at the pole, and the derivative of the Green function of the free two-body scattering system. At the same time, our formulation provides the elementariness expressed with the resonance properties and the two-body effective interaction, and confirms the sum rule showing that the summation of the compositeness and elementariness gives unity. In this formulation, Weinberg's relation for the scattering length and effective range can be derived in the weak binding limit. The extension to the resonance states is performed with the Gamow vector, and a relativistic formulation is also established. As its applications, we study the compositeness of the $\Lambda(1405)$ resonance and the light scalar and vector mesons described with refined amplitudes in coupled-channel models with interactions up to the next-to-leading order in chiral perturbation theory. We find that $\Lambda(1405)$ and $f_0(980)$ are dominated by the $\bar{K}N$ and $K\bar{K}$ composite states, respectively, while the vector mesons $\rho(770)$ and $K^*(892)$ are elementary. We also briefly discuss the compositeness of $N(1535)$ and $\Lambda(1670)$ obtained in a leading-order chiral unitary approach.

Subject Index D32

1. Introduction

In hadron physics, the internal structure of an individual hadron is one of the most important subjects. Traditionally, the excellent successes of constituent quark models lead us to the interpretation that baryons consist of three quarks (qqq) and mesons of a quark–antiquark pair ($q\bar{q}$) [1]. At the same time, however, there are experimental indications that some hadrons do not fit into the classification suggested by constituent quark models. One of the classical examples is the hyperon resonance $\Lambda(1405)$, which has an anomalously light mass among the negative parity baryons. In addition, the lightest scalar mesons [$f_0(500) = \sigma$, $K_0^*(800) = \kappa$, $f_0(980)$, and $a_0(980)$] exhibit a spectrum inverted from the naïve expectation with the $q\bar{q}$ configuration. These observations motivate us to consider more exotic structures for hadrons, such as hadronic molecules and multiquarks [2–7].

It is encouraging that there have been experimental reports on candidates for manifestly exotic hadrons, such as charged quarkonium-like states by the Belle collaboration [8]. Moreover, the LEPS

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collaboration observed the "Θ^+ signal" [9,10], but its interpretation is still controversial [11,12]. The accumulation of observations of unconventional states in the heavy quark sector reinforces the existence of hadrons with exotic structure [13,14]. In fact, recent detailed analyses of $\Lambda(1405)$ in various reactions [15–18] and of the $a_0^0(980)$–$f_0(980)$ mixing in J/ψ decay [19] are providing some clues for unusual structure of these hadrons. The exotic structure is also investigated by analyzing the theoretical models; the meson–baryon components of $\Lambda(1405)$ by using the natural renormalization scheme [20], the N_c scaling behaviors of scalar and vector mesons [21,22] and of $\Lambda(1405)$ [23,24], the spatial size of $\Lambda(1405)$ [25–27], σ meson [28], and $f_0(980)$ [27], the nature of the σ meson from the partial restoration of chiral symmetry [29], and the structure of σ and $\rho(770)$ mesons studied by their Regge trajectories [30]. Possibilities for extracting the hadron structure from the production yield in relativistic heavy ion collisions [31,32] and from high-energy exclusive productions [33,34] are also suggested.

Among various exotic structures, hadronic molecular configurations are of special interest. These states are composed of two (or more) constituent hadrons by strong interaction between them without losing the character of the constituent hadrons, in a similar way to atomic nuclei as bound states of nucleons. The $\bar{K}N$ quasi-bound picture for $\Lambda(1405)$ is one example. In contrast to the quark degrees of freedom, the masses and interactions of hadrons are defined independently of the renormalization scheme of QCD, because hadrons are color singlet states. This fact implies that the structure of hadrons may be adequately defined in terms of the hadronic degrees of freedom. This viewpoint originates in investigations of the elementary or composite nature of particles in terms of the field renormalization constant [35–37]. Indeed, it is shown in this approach that the deuteron is dominated by the loosely bound proton–neutron component [38]. The study of the structure of hadrons from the field renormalization constant has been further developed in Refs. [39–51].

Motivated by these observations, in this study we develop a framework to investigate hadronic two-body components inside a hadron by comprehensively analyzing the wave function of a resonance state. For this purpose, we explicitly introduce one-body bare states in addition to the two-body components so as to form a complete set within them and to measure the elementary and composite contributions. The one-body component has not been taken into account in the preceding studies on wave functions (see Refs. [43,52,53]). For the resonance state we employ the Gamow vector [54], which ensures a finite normalization of the resonance wave function. The wave function from a relativistically covariant wave equation is also discussed. Making good use of a general separable interaction, we analytically solve the wave equations.

In the present formulation, the compositeness and elementariness are respectively defined as the fractions of the contributions from the two-body scattering states and one-body bare states to the normalization of the total wave function. They are further expressed with the quantities in the scattering equation with a general separable interaction. As a consequence, the compositeness can be written in terms of the residue of the scattering amplitude at the pole position, i.e., the coupling constant, and the derivative of the Green function of the free two-body scattering system at the pole. This means that the compositeness can be obtained solely with the pole position of the resonance and the residue at the pole but without knowing the details of the two-body effective interaction. On the other hand, the elementariness is obtained with the residue of the scattering amplitude, the Green function, and the derivative of the two-body effective interaction at the pole. It is an interesting finding that with this expression we are allowed to interpret the elementariness as the contributions coming from one-body bare states and implicit two-body channels which do not appear as explicit degrees of freedom but are effectively taken into account for the two-body interaction in the practical model space. Through the discussion on the multiple bare states, we show that our formulation of the compositeness and elementariness can be applied to any separable interactions with arbitrary energy dependence. Based on this foundation, as applications we evaluate the compositeness of hadronic resonances, such as $\Lambda(1405)$, the light scalar mesons and vector mesons described in the chiral coupled-channel approach with the next-to-leading-order interactions so as to discuss their internal structure from the viewpoint of hadronic two-body components.

This paper is organized as follows. In Sect. 2, we formulate the compositeness and elementariness of a physical particle state in terms of its wave function, and show their connection to the physical quantities in the scattering equation. We first consider a two-body bound state in the nonrelativistic framework, and later extend the formulation to a resonance state in a relativistic covariant form with the Gamow vector. In Sect. 3, numerical results for the applications to physical resonances are presented. Section 4 is devoted to drawing the conclusions of this study.

2. Compositeness and elementariness from wave functions

In this section, we define the compositeness (and simultaneously elementariness) of a particle state, i.e., a stable bound state or an unstable resonance, using its wave function, and link the compositeness to the physical quantities in the scattering equation. For this purpose, we consider two-body scattering states[1] coupled with each other and one-body bare states. The one-body bare states have not been introduced in studies of wave functions before, and the introduction of the one-body bare states makes it possible to establish the meaning of the elementariness in the formulation. To solve the scattering equation analytically, we make use of the separable type of interaction. We will concentrate on an s-wave scattering system, and thus the two-body wave function and the form factors are assumed to be spherical.

In Sect. 2.1 we consider a bound state[2] in two-body scattering. We first introduce a one-body bare state and a single scattering channel, and give the expressions of the compositeness and the elementariness in terms of the wave function of the bound state. In Sect. 2.2 we extend the discussion to a system with multiple bare states and coupled scattering channels, in order to clarify further the meaning of the compositeness and elementariness obtained in Sect. 2.1. Here we also discuss a way to introduce a general energy-dependent interaction into the formulation. In Sect. 2.3 we consider the weak binding limit to derive Weinberg's relation for the scattering length and the effective range [38]. Generalization to resonance states is discussed in Sect. 2.4. Finally, we give a relativistic covariant formulation in Sect. 2.5.

2.1. Bound state in nonrelativistic scattering

We consider a two-body scattering system in which there exists a discrete energy level below the scattering threshold energy. We call this energy level a bound state since it is located below the two-body scattering threshold. We do not assume the origin and structure of the bound state at all. We take the rest frame of the center-of-mass motion, namely two scattering particles have equal and opposite momentum and the bound state is at rest with zero momentum. The system in this frame is described

[1] We note that the two-body wave functions are given by the asymptotic states of the system. In the application to QCD, the basis should be spanned by the hadronic degrees of freedom. The compositeness in terms of quarks cannot be defined in this approach.

[2] In general, there can be several bound states in the system. In such a case, we just focus on one of these bound states. Nothing changes in the following discussion.

by Hamiltonian $\hat{H}$ which consists of the free part $\hat{H}_0$ and the interaction term $\hat{V}$,

$$\hat{H} = \hat{H}_0 + \hat{V}. \tag{1}$$

We assume that the free Hamiltonian has continuum eigenstates $|\boldsymbol{q}\rangle$ for the scattering state and one discrete state $|\psi_0\rangle$ for the one-body bare state. The eigenvalues of the Hamiltonian are set to be

$$\hat{H}_0|\boldsymbol{q}\rangle = \left(M^{\text{th}} + \frac{q^2}{2\mu} \right) |\boldsymbol{q}\rangle, \quad \langle \boldsymbol{q}|\hat{H}_0 = \left(M^{\text{th}} + \frac{q^2}{2\mu} \right) \langle \boldsymbol{q}|, \tag{2}$$

$$\hat{H}_0|\psi_0\rangle = M_0|\psi_0\rangle, \quad \langle\psi_0|\hat{H}_0 = M_0\langle\psi_0|, \tag{3}$$

where μ is the reduced mass of the two-body system, M_0 is the mass of the bare state, and $q \equiv |\boldsymbol{q}|$. We include the sum of the scattering particle masses, M^{th}, which is just the scattering threshold energy, into the definition of the eigenenergy for later convenience. These eigenstates are normalized as

$$\langle \boldsymbol{q}' | \boldsymbol{q} \rangle = (2\pi)^3 \delta^3(\boldsymbol{q}' - \boldsymbol{q}), \quad \langle \psi_0 | \psi_0 \rangle = 1, \quad \langle \psi_0 | \boldsymbol{q} \rangle = \langle \boldsymbol{q} | \psi_0 \rangle = 0. \tag{4}$$

These states form the complete set of the free Hamiltonian, and thus we can decompose unity in the following way:

$$1 = |\psi_0\rangle\langle\psi_0| + \int \frac{d^3q}{(2\pi)^3} |\boldsymbol{q}\rangle\langle\boldsymbol{q}|. \tag{5}$$

The bound state is realized as an eigenstate of the full Hamiltonian:

$$\hat{H}|\psi\rangle = M_{\text{B}}|\psi\rangle, \quad \langle\psi|\hat{H} = M_{\text{B}}\langle\psi|, \tag{6}$$

where M_{B} is the mass of the bound state. The bound state wave function is normalized as

$$\langle\psi|\psi\rangle = 1. \tag{7}$$

We take the matrix element of Eq. (5) in terms of the bound state $|\psi\rangle$:

$$1 = \langle\psi|\psi_0\rangle\langle\psi_0|\psi\rangle + \int \frac{d^3q}{(2\pi)^3} \langle\psi|\boldsymbol{q}\rangle\langle\boldsymbol{q}|\psi\rangle. \tag{8}$$

The first term of the right-hand side is the probability of finding the bare state in the bound state and also corresponds to the field renormalization constant in the field theory. Thus, we call this quantity elementariness Z:

$$Z \equiv \langle\psi|\psi_0\rangle\langle\psi_0|\psi\rangle. \tag{9}$$

Because $\langle\psi|\psi_0\rangle = \langle\psi_0|\psi\rangle^*$, Z is always real and nonnegative. The second term, on the other hand, represents the contribution from the two-body state and we call it compositeness X:

$$X \equiv \int \frac{d^3q}{(2\pi)^3} \langle\psi|\boldsymbol{q}\rangle\langle\boldsymbol{q}|\psi\rangle. \tag{10}$$

The elementariness and compositeness satisfy the sum rule

$$1 = \langle\psi|\psi\rangle = Z + X. \tag{11}$$

Introducing the momentum space wave function for the two-body state, $\tilde{\psi}(q)$,

$$\tilde{\psi}(q) = \langle\boldsymbol{q}|\psi\rangle, \quad \tilde{\psi}^*(q) = \langle\psi|\boldsymbol{q}\rangle, \tag{12}$$

the compositeness X can be expressed as

$$X = \int \frac{d^3q}{(2\pi)^3} \left| \tilde{\psi}(q) \right|^2. \tag{13}$$

Again, X is real and nonnegative.

For the explicit calculation, we assume the separable form of the matrix elements of $\hat{V}$ in the momentum space. The matrix elements are given by

$$\langle \boldsymbol{q}' | \hat{V} | \boldsymbol{q} \rangle = v f^*(q'^2) f(q^2), \quad \langle \boldsymbol{q} | \hat{V} | \psi_0 \rangle = g_0 f^*(q^2), \quad \langle \psi_0 | \hat{V} | \psi_0 \rangle = 0, \tag{14}$$

where v is the interaction strength between the scattering particles, and g_0 is the coupling constant of the bare state to the scattering state. As we will see later, the one-body state is the source of the energy dependence of the effective interaction between the scattering particles. The matrix element $\langle \psi_0 | \hat{V} | \psi_0 \rangle$ is taken to be zero since it can be absorbed into $\hat{H}_0$ without loss of generality, and throughout this study the mass of the bare state, M_0, is not restricted to be smaller than M^{th} but is allowed to take any value with this condition. The form factor $f(q^2)$ is responsible for the off-shell momentum dependence of the interaction and suppresses the high momentum contribution to tame the ultraviolet divergence. The normalization is chosen to be $f(0) = 1$. The hermiticity of the Hamiltonian ensures that v is real and

$$\langle \psi_0 | \hat{V} | \boldsymbol{q} \rangle = g_0^* f(q^2). \tag{15}$$

In this study we further assume the time-reversal invariance of the scattering process, which constraints the interaction, with an appropriate choice of phases of the states, as

$$\langle \boldsymbol{q}' | \hat{V} | \boldsymbol{q} \rangle = \langle \boldsymbol{q} | \hat{V} | \boldsymbol{q}' \rangle = v f(q'^2) f(q^2), \quad \langle \boldsymbol{q} | \hat{V} | \psi_0 \rangle = \langle \psi_0 | \hat{V} | \boldsymbol{q} \rangle = g_0 f(q^2), \quad \langle \psi_0 | \hat{V} | \psi_0 \rangle = 0. \tag{16}$$

Thus all of the quantities v, g_0, and $f(q^2)$ are now real. We emphasize that the assumptions made in the present framework are just the factorization of the momentum dependence and the time-reversal invariance of the interaction. With the interaction (16), we obtain the exact solution of this system without introducing any further assumptions.

For the separable interaction, the wave function $\tilde{\psi}(q)$ can be analytically obtained [55]. To this end, we multiply $\langle \boldsymbol{q}|$ and $\langle \psi_0|$ by Eq. (6):

$$\langle \boldsymbol{q} | \hat{H} | \psi \rangle = \left(M^{\text{th}} + \frac{q^2}{2\mu} \right) \tilde{\psi}(q) + v f(q^2) \int \frac{d^3 q'}{(2\pi)^3} f(q'^2) \tilde{\psi}(q') + g_0 f(q^2) \langle \psi_0 | \psi \rangle = M_{\text{B}} \tilde{\psi}(q), \tag{17}$$

$$\langle \psi_0 | \hat{H} | \psi \rangle = M_0 \langle \psi_0 | \psi \rangle + g_0 \int \frac{d^3 q}{(2\pi)^3} f(q^2) \tilde{\psi}(q) = M_{\text{B}} \langle \psi_0 | \psi \rangle, \tag{18}$$

where we have inserted Eq. (5) between $\hat{V}$ and $|\psi\rangle$. Eliminating $\langle \psi_0 | \psi \rangle$ from these equations, we obtain the Schrödinger equation for $\tilde{\psi}(q)$ in an integral form:

$$\left(M^{\text{th}} + \frac{q^2}{2\mu} \right) \tilde{\psi}(q) + v^{\text{eff}}(M_{\text{B}}) f(q^2) \int \frac{d^3 q'}{(2\pi)^3} f(q'^2) \tilde{\psi}(q') = M_{\text{B}} \tilde{\psi}(q), \tag{19}$$

where we have defined the energy-dependent interaction v^{eff} as

$$v^{\text{eff}}(E) \equiv v + \frac{(g_0)^2}{E - M_0}. \tag{20}$$

Equation (19) is the single-channel Schrödinger equation for the relative motion of the scattering particles under the presence of the bare state interacting with them by $\hat{V}$. The effect of the bare state is incorporated into the energy-dependent interaction $v^{\text{eff}}(E)$.

The solution of Eq. (19) can be obtained as

$$\tilde{\psi}(q) = \frac{-cf(q^2)}{B + q^2/(2\mu)}, \tag{21}$$

where we have defined the binding energy $B \equiv M^{\mathrm{th}} - M_{\mathrm{B}} > 0$ and the normalization constant

$$c \equiv v^{\mathrm{eff}}(M_{\mathrm{B}}) \int \frac{d^3 q'}{(2\pi)^3} f(q'^2)\tilde{\psi}(q'). \tag{22}$$

In general, Eq. (19) is an integral equation to determine the wave function $\tilde{\psi}(q)$. For the separable interaction, however, the integral in Eq. (22), and hence the constant c, is independent of q. In this way, the wave function $\tilde{\psi}(q)$ is analytically determined by the form factor $f(q^2)$ and the constant c, which will be determined through the comparison with the scattering amplitude. Substituting the wave function (21) into Eq. (22), we obtain

$$c = -v^{\mathrm{eff}}(M_{\mathrm{B}}) \int \frac{d^3 q}{(2\pi)^3} \frac{[f(q^2)]^2}{B + q^2/(2\mu)} c. \tag{23}$$

For the existence of the bound state at $E = M_{\mathrm{B}}$, Eq. (23) should be satisfied with nonzero c. The nontrivial solution can be obtained by

$$1 = v^{\mathrm{eff}}(M_{\mathrm{B}}) G(M_{\mathrm{B}}), \tag{24}$$

where we have introduced a function

$$G(E) = \int \frac{d^3 q}{(2\pi)^3} \frac{[f(q^2)]^2}{E - M^{\mathrm{th}} - q^2/(2\mu)}, \tag{25}$$

which plays an important role in the following discussion and is called the loop function. As we will see later, the loop function is equivalent to the Green function of the free two-body Hamiltonian. We note that here and in the following the energy in the denominator of the loop function is considered to have an infinitesimal positive imaginary part $i\epsilon$: $E \to E + i\epsilon$.

The normalization constant c is equal to the square root of the residue of the scattering amplitude at the pole position of the bound state. To prove this, we first represent the compositeness X and elementariness Z using c. With the explicit form of the wave function (21) and the loop function (25), the compositeness for the separable interaction can be expressed with the derivative of the loop function as

$$X = \int \frac{d^3 q}{(2\pi)^3} \left| \tilde{\psi}(q) \right|^2 = -|c|^2 \left[\frac{dG}{dE} \right]_{E = M_{\mathrm{B}}}. \tag{26}$$

We note that both the wave function $\tilde{\psi}(q)$ and the loop function have the same structure of $1/(E - \hat{H}_0)$ at $E = M_{\mathrm{B}}$. Substituting the wave function into Eq. (18), we obtain

$$\langle \psi_0 | \psi \rangle = \frac{c g_0}{M_{\mathrm{B}} - M_0} G(M_{\mathrm{B}}), \tag{27}$$

and hence

$$Z = \langle \psi | \psi_0 \rangle \langle \psi_0 | \psi \rangle = |c|^2 G(M_{\mathrm{B}}) \frac{(g_0)^2}{(M_{\mathrm{B}} - M_0)^2} G(M_{\mathrm{B}}) = -|c|^2 \left[G \frac{dv^{\mathrm{eff}}}{dE} G \right]_{E = M_{\mathrm{B}}}, \tag{28}$$

where we have used the derivative of Eq. (20). We note that Eqs. (26) and (28) provide a sum rule

$$1 = -|c|^2 \left[\frac{dG}{dE} + G \frac{dv^{\mathrm{eff}}}{dE} G \right]_{E = M_{\mathrm{B}}}. \tag{29}$$

Next, the scattering amplitude $t(E)$ is obtained by taking the matrix element of the T-operator for the scattering state $|\boldsymbol{q}\rangle$ with the on-shell condition as $\langle \boldsymbol{q}' | \hat{T} | \boldsymbol{q} \rangle = t(E) f(q'^2) f(q^2)$ for the separable interaction. The T-operator satisfies the Lippmann–Schwinger equation

$$\hat{T} = \hat{V} + \hat{V}\frac{1}{E - \hat{H}_0}\hat{T}. \tag{30}$$

Inserting the complete set (5) between the operators and eliminating the bare state component from the equation, we obtain the Lippmann-Schwinger equation for the scattering state as

$$\hat{T} = \hat{V}^{\text{eff}}(E) + \hat{V}^{\text{eff}}(E)\frac{1}{E - \hat{H}_0}\hat{T}, \tag{31}$$

where we have introduced the operator of the effective interaction for the scattering state as

$$\hat{V}^{\text{eff}}(E) \equiv \hat{V} + \hat{V}|\psi_0\rangle\frac{1}{E - M_0}\langle\psi_0|\hat{V}. \tag{32}$$

This operator acts only on the two-body state and its matrix element leads to $\left\langle \boldsymbol{q}' \middle| \hat{V}^{\text{eff}}(E) \middle| \boldsymbol{q} \right\rangle = v^{\text{eff}}(E) f\left(q'^2\right) f\left(q^2\right)$. Taking matrix elements of the two-body state in Eq. (31), we obtain the amplitude $t(E)$ algebraically as

$$t(E) = v^{\text{eff}}(E) + v^{\text{eff}}(E)G(E)t(E) = \frac{v^{\text{eff}}(E)}{1 - v^{\text{eff}}(E)G(E)}, \tag{33}$$

where $G(E)$ is the same form as Eq. (25), i.e., the Green function of the free two-body Hamiltonian. The bound state condition (24) ensures that the amplitude $t(E)$ has a pole at $E = M_{\text{B}}$. The residue of the amplitude $t(E)$ at the pole reflects the properties of the bound state. The residue turns out to be real and positive, so we represent the residue as $|g|^2$:

$$|g|^2 \equiv \lim_{E \to M_{\text{B}}} (E - M_{\text{B}})t(E) = -\frac{1}{\left[\frac{dG}{dE} + \frac{1}{(v^{\text{eff}})^2}\frac{dv^{\text{eff}}}{dE}\right]_{E = M_{\text{B}}}}. \tag{34}$$

We can interpret g as the coupling constant of the bound state to the two-body state. Using the bound state condition (24), we obtain the relation

$$1 = -|g|^2\left[\frac{dG}{dE} + G\frac{dv^{\text{eff}}}{dE}G\right]_{E = M_{\text{B}}}. \tag{35}$$

Comparing this with Eq. (29), we find $c = g$ with an appropriate choice of phase.

The equality $c = g$ is also confirmed by the following form of the T-operator:

$$\hat{T} = \hat{V}^{\text{eff}}(E) + \hat{V}^{\text{eff}}(E)\frac{1}{E - \hat{H}_0 - \hat{V}^{\text{eff}}(E)}\hat{V}^{\text{eff}}(E). \tag{36}$$

As we have seen before, the operator $\hat{H}_0 + \hat{V}^{\text{eff}}$ corresponds to the full Hamiltonian for the two-body system with the implicit bare state. Near the bound state pole, the amplitude is dominated by the pole term in the expansion by the eigenstates of the full Hamiltonian as

$$\lim_{E \to M_{\text{B}}} \hat{T}(E) \sim \hat{V}^{\text{eff}}\left(M_{\text{B}}\right)|\psi\rangle\frac{1}{E - M_{\text{B}}}\langle\psi|\hat{V}^{\text{eff}}\left(M_{\text{B}}\right), \tag{37}$$

and hence, taking the matrix element of the scattering states, we have

$$\lim_{E \to M_{\text{B}}} t(E) \sim \int\frac{d^3q}{(2\pi)^3}\int\frac{d^3p}{(2\pi)^3} v^{\text{eff}}\left(M_{\text{B}}\right) f\left(q^2\right)\frac{\langle\boldsymbol{q}|\psi\rangle\langle\psi|\boldsymbol{p}\rangle}{E - M_{\text{B}}} f\left(p^2\right) v^{\text{eff}}\left(M_{\text{B}}\right) \to \frac{|c|^2}{E - M_{\text{B}}}, \tag{38}$$

where we have used Eq. (22). From the definition of the residue (34), this verifies $c = g$.

Here, we emphasize that, as seen in Eq. (26), the compositeness is expressed with the residue of the scattering amplitude at the pole position and the energy derivative of the loop function dG/dE, and hence the compositeness does not explicitly depend on the effective interaction v^{eff}.[3] Therefore, the compositeness can be obtained solely with the bound state properties without knowing the details of the effective interaction, once we fix the loop function, which coincides with fixing the model space to measure the compositeness via the Green function of the free two-body Hamiltonian.

We also note that, as seen in Eq. (28), the elementariness Z is proportional to the energy derivative of the interaction dv^{eff}/dE at the bound state energy. This is instructive in interpreting the origin of the elementariness Z. In quantum mechanics, the two-body interaction should not depend on the energy to have an appropriate normalization. In the present case, the energy dependence of v^{eff} stems from the bare state channel $|\psi_0\rangle$. Strong energy dependence of the interaction v^{eff} at the bound state pole position emerges when the involved bare state lies close to the physical bound state, and provides $Z \approx 1$. This means that the effect from the bare state is responsible for the formation of the bound state. Weak energy dependence, which corresponds to $Z \approx 0$, can be understood that the bare state exists far away from the pole position of the physical bound state, and is insensitive to the bound state. In this case, the bound state is composed dominantly of the scattering channels considered. This shares viewpoints with Ref. [20], where it was discussed that the energy-dependent Weinberg–Tomozawa term can provide the effect of the Castillejo–Dalitz–Dyson (CDD) pole [56].

2.2. *Coupled scattering channels with multiple bare states*

The framework in the last subsection is straightforwardly generalized to the coupled-channel scattering with multiple one-body bare states. The eigenstates of the free Hamiltonian $\hat{H}_0$ now include several bare states $|\psi_a\rangle$ labeled by a and two-body scattering states of several channels labeled by j. We assume that the bound state whose components we want to examine is located below the lowest threshold of the two-body channels to make the state stable. The normalization and the completeness relation are given by

$$\langle \boldsymbol{q}'_j | \boldsymbol{q}_k \rangle = (2\pi)^3 \delta_{jk} \delta^3(\boldsymbol{q}' - \boldsymbol{q}), \quad \langle \psi_a | \psi_b \rangle = \delta_{ab}, \quad \langle \psi_a | \boldsymbol{q}_j \rangle = \langle \boldsymbol{q}_j | \psi_a \rangle = 0, \tag{39}$$

$$1 = \sum_a |\psi_a\rangle \langle \psi_a| + \sum_j \int \frac{d^3 q}{(2\pi)^3} |\boldsymbol{q}_j\rangle \langle \boldsymbol{q}_j|. \tag{40}$$

The matrix elements of the interaction are

$$\langle \boldsymbol{q}'_j | \hat{V} | \boldsymbol{q}_k \rangle = v_{jk} f_j(q'^2) f_k(q^2), \quad \langle \boldsymbol{q}_j | \hat{V} | \psi_a \rangle = \langle \psi_a | \hat{V} | \boldsymbol{q}_j \rangle = g_0^{a,j} f_j(q^2), \quad \langle \psi_a | \hat{V} | \psi_b \rangle = 0, \tag{41}$$

where, due to the time-reversal invariance, v_{jk} is a real symmetric matrix and $g_0^{a,j}$ and $f_j(q^2)$ are real with an appropriate choice of phases of states. The total normalization of the bound state wave function now leads to

$$1 = \sum_a Z_a + \sum_j X_j, \tag{42}$$

with the elementariness

$$Z_a \equiv \langle \psi | \psi_a \rangle \langle \psi_a | \psi \rangle, \tag{43}$$

[3] Since the bound state properties are determined by the interaction, the compositeness depends implicitly on the effective interaction v^{eff}.

and the compositeness given by the wave function for each channel

$$X_j \equiv \int \frac{d^3q}{(2\pi)^3} \left|\tilde{\psi}_j(q)\right|^2, \tag{44}$$

where

$$\tilde{\psi}_j(q) = \langle \boldsymbol{q}_j|\psi\rangle, \quad \tilde{\psi}_j^*(q) = \langle\psi|\boldsymbol{q}_j\rangle. \tag{45}$$

We follow the same procedure as the single-channel case; incorporating the one-body bare states in the effective interaction for the two-body states, we obtain the coupled Schrödinger equation as

$$\left(M_j^{\rm th} + \frac{q^2}{2\mu_j}\right)\tilde{\psi}_j(q) + \sum_k v_{jk}^{\rm eff}(M_{\rm B}) f_j\left(q^2\right) \int \frac{d^3q'}{(2\pi)^3} f_k\left(q'^2\right)\tilde{\psi}_k(q') = M_{\rm B}\tilde{\psi}_j(q), \tag{46}$$

where $M_j^{\rm th}$ and μ_j are the threshold and the reduced mass in channel j, respectively, and we have defined the energy-dependent effective interaction as

$$v_{jk}^{\rm eff}(E) \equiv v_{jk} + \sum_a \frac{g_0^{a,j} g_0^{a,k}}{E - M_a}, \tag{47}$$

which is a real symmetric matrix for a real energy, and M_a is the mass of the bare state. The Schrödinger equation can be solved algebraically again for the separable interaction:

$$\tilde{\psi}_j(q) = \frac{-c_j f_j\left(q^2\right)}{B_j + q^2/(2\mu_j)}, \tag{48}$$

where $B_j \equiv M_j^{\rm th} - M_{\rm B}$ is the binding energy measured from the j-channel threshold. The normalization constant is given by

$$c_j \equiv \sum_k v_{jk}^{\rm eff}\left(M_{\rm B}\right) \int \frac{d^3q}{(2\pi)^3} f_k\left(q^2\right)\tilde{\psi}_k(q). \tag{49}$$

With substitution of Eq. (48) in Eq. (49), the bound state condition for nonzero c_j can be summarized as

$$\det\left[1 - v^{\rm eff}(M_{\rm B})G(M_{\rm B})\right] = 0, \tag{50}$$

with the loop function

$$G_j(E) = \int \frac{d^3q}{(2\pi)^3} \frac{\left[f_j\left(q^2\right)\right]^2}{E - M_j^{\rm th} - q^2/(2\mu_j)}, \tag{51}$$

which is diagonal with respect to the channel index.

The coupled-channel scattering equation is, in matrix form,

$$t(E) = \left[1 - v^{\rm eff}(E)G(E)\right]^{-1} v^{\rm eff}(E), \tag{52}$$

where the channel index runs through only the scattering channels, since the one-body bare states are incorporated into the effective interaction $v^{\rm eff}$. Equation (50) ensures the existence of the bound state pole at $E = M_{\rm B}$. The residue of the amplitude at the pole, which is real for the bound state, is interpreted as the product of the coupling constants,[4]

$$g_j g_k = \lim_{E\to M_{\rm B}} (E - M_{\rm B}) t_{jk}(E). \tag{53}$$

On the other hand, using the coupled-channel version of Eq. (36), the amplitude near the bound state pole is given by

$$\begin{aligned}\lim_{E\to M_{\rm B}} t_{jk}(E) &\sim \sum_{l,m} \int \frac{d^3 q}{(2\pi)^3} \int \frac{d^3 p}{(2\pi)^3} v^{\rm eff}_{jl}(M_{\rm B}) f_l\left(q^2\right) \frac{\langle \boldsymbol{q}_l|\psi\rangle\langle\psi|\boldsymbol{p}_m\rangle}{E - M_{\rm B}} f_m\left(p^2\right) v^{\rm eff}_{mk}(M_{\rm B}) \\ &\to \frac{c_j c_k^*}{E - M_{\rm B}},\end{aligned} \tag{54}$$

which shows that $c_j = g_j$ with an appropriate choice of phase.

Now the compositeness in channel j can be expressed as

$$X_j = \int \frac{d^3 q}{(2\pi)^3} \left|\tilde{\psi}_j(q)\right|^2 = -|c_j|^2 \left[\frac{dG_j}{dE}\right]_{E=M_{\rm B}} = -|g_j|^2 \left[\frac{dG_j}{dE}\right]_{E=M_{\rm B}}. \tag{55}$$

The overlap of the bound state wave function with the bare state ψ_a is given by

$$\langle \psi_a|\psi\rangle = \frac{1}{M_{\rm B} - M_a} \sum_j c_j g_0^{a,j} G_j(M_{\rm B}), \tag{56}$$

and thus we obtain

$$Z_a = \langle\psi|\psi_a\rangle\langle\psi_a|\psi\rangle = \sum_{j,k} c_k c_j^* G_j(M_{\rm B}) G_k(M_{\rm B}) \frac{g_0^{a,j} g_0^{a,k}}{(M_{\rm B} - M_a)^2}. \tag{57}$$

The total elementariness $Z \equiv \sum_a Z_a$, which contains all contributions from the implicit channels, is

$$Z \equiv \sum_a Z_a = \sum_{j,k} c_k c_j^* G_j(M_{\rm B}) G_k(M_{\rm B}) \sum_a \frac{g_0^{a,j} g_0^{a,k}}{(M_{\rm B} - M_a)^2} = -\sum_{j,k} g_k g_j \left[G_j \frac{dv^{\rm eff}_{jk}}{dE} G_k\right]_{E=M_{\rm B}}. \tag{58}$$

From the normalization (42), we obtain the sum rule

$$-\sum_{j,k} g_k g_j \left[\delta_{jk} \frac{dG_j}{dE} + G_j \frac{dv^{\rm eff}_{jk}}{dE} G_k\right]_{E=M_{\rm B}} = 1. \tag{59}$$

This corresponds to the nonrelativistic counterpart of the generalized Ward identity derived in Ref. [26]. We note that the sum rule (59) as the normalization of the wave function can be obtained by the explicit treatment of both the two-body states and the one-body bare states, which complement the discussion of the bound state wave function with an energy-independent separable interaction in Ref. [52].

So far, we have regarded the components coming from the one-body bare states as the elementariness. On the other hand, sometimes it happens that some of the two-body channel thresholds are high enough that these channels may play a minor role. In such a case, these channels can be also included into implicit channels of the effective interaction $v^{\rm eff}$ by, e.g., the Feshbach method [57,58].

[4] Since an interaction of a symmetric matrix $v^{\rm eff}_{jk}$ leads to a symmetric t-matrix, $t_{jk} = t_{kj}$, the residue of the t-matrix is also symmetric and can be factorized as $g_j g_k$.

These implicit channels also provide energy dependence of the effective interaction which acts on the reduced model space (see also Ref. [48]), and accordingly we are allowed to interpret the contributions coming from these channels as the elementariness. For instance, the implementation of a scattering channel N into the effective interaction can be done by replacing v^{eff} as:

$$w_{jk}(E) = v_{jk}^{\mathrm{eff}} + v_{jN}^{\mathrm{eff}} \frac{G_N(E)}{1 - v_{NN}^{\mathrm{eff}} G_N(E)} v_{Nk}^{\mathrm{eff}}, \quad j,\, k \neq N, \tag{60}$$

where the Nth channel has been absorbed in the effective interaction w_{jk} in the same manner as in [59]. In this case the elementariness Z^w may be able to be calculated by the derivative of the effective interaction w_{jk} as

$$Z^w = - \sum_{j,k \neq N} g_k g_j \left[G_j \frac{d w_{jk}}{dE} G_k \right]_{E = M_{\mathrm{B}}}. \tag{61}$$

Interestingly, the elementariness Z^w can be expressed as the sum of the elementariness with the full two-body channels, Z, and the Nth channel compositeness X_N, namely,

$$Z^w = Z + X_N, \tag{62}$$

with

$$Z = - \sum_{j,k} g_k g_j \left[G_j \frac{d v_{jk}^{\mathrm{eff}}}{dE} G_k \right]_{E = M_{\mathrm{B}}}, \quad X_N = - g_N^2 \left[\frac{d G_N}{dE} \right]_{E = M_{\mathrm{B}}}. \tag{63}$$

The proof is shown in Appendix A. In this way, the elementariness can be redefined by Eq. (61). With this expression the elementariness measures contributions coming from both one-body bare states and two-body channels which are implemented into the effective interaction and do not appear as explicit degrees of freedom.

At the end of this subsection, we mention that our formulation of the compositeness and elementariness can be applied to any separable interactions with arbitrary energy dependence by interpreting that the energy dependence on the effective interaction comes from the implicit channels. Actually, when the compositeness and elementariness are formulated with multiple one-body bare states, all of these bare states are included in the effective two-body interaction $v^{\mathrm{eff}}(E)$ and the total elementariness is calculated as the sum of each bare-state contribution, which is essentially the derivative of the effective two-body interaction as in Eq. (58). It is important that in this case we can produce any energy-dependent interactions with suitable bare states. In order to see this, for instance, we assume that the mass of a bare state is large enough, and by expanding the bare-state term in the effective interaction as

$$\frac{1}{E - M_0} = - \frac{1}{M_0} \left(1 + \frac{E}{M_0} + \cdots \right), \tag{64}$$

we have polynomial energy dependence in the effective interaction. This fact enables us to apply the formulae of the compositeness and elementariness to interactions with an arbitrary energy dependence. This is the foundation of the analysis of physical hadronic resonances in Sect. 3.

2.3. *Weak binding limit and threshold parameters*

In this subsection, we consider the weak binding limit to derive Weinberg's compositeness condition [38] on the scattering length a and the effective range r_e. This ensures that the expression for the compositeness in this paper correctly reproduces the model-independent result of Ref. [38] in the weak binding limit. For simplicity we consider a system with one scattering channel, as in Sect. 2.1.

In the single-channel problem, the elastic scattering amplitude $\mathcal{F}(E)$ is written with the t-matrix $t(E)$ given in Eq. (33) as

$$\mathcal{F}(E) = -\frac{1}{(2\pi)^3}(2\pi)^2 \mu t(E)\Big[f\big(k^2\big)\Big]^2, \tag{65}$$

with $k \equiv \sqrt{2\mu(E - M^{\mathrm{th}})}$. The scattering length a is defined as the value of the scattering amplitude at the threshold:

$$a \equiv -\mathcal{F}\big(M^{\mathrm{th}}\big) = \frac{\mu}{2\pi} t\big(M^{\mathrm{th}}\big) = \frac{\mu}{2\pi} \frac{1}{v^{-1}\big(M^{\mathrm{th}}\big) - G\big(M^{\mathrm{th}}\big)}, \tag{66}$$

where we have abbreviated v^{eff} as v for simplicity. Now we perform the expansion in terms of the energy E around M_{B} by considering $B = M^{\mathrm{th}} - M_{\mathrm{B}}$ to be small. To expand the denominator, we write

$$v^{-1}\big(M^{\mathrm{th}}\big) = v^{-1}(M_{\mathrm{B}}) + B\left[\frac{dv^{-1}}{dE}\right]_{E=M_{\mathrm{B}}} + \Delta v^{-1}, \tag{67}$$

$$G\big(M^{\mathrm{th}}\big) = G(M_{\mathrm{B}}) + B\left[\frac{dG}{dE}\right]_{E=M_{\mathrm{B}}} + \Delta G, \tag{68}$$

where we have defined

$$\Delta v^{-1} \equiv \sum_{n=2}^{\infty} \frac{B^n}{n!}\left[\frac{d^n v^{-1}}{dE^n}\right]_{E=M_{\mathrm{B}}}, \quad \Delta G \equiv \sum_{n=2}^{\infty} \frac{B^n}{n!}\left[\frac{d^n G}{dE^n}\right]_{E=M_{\mathrm{B}}}. \tag{69}$$

Here we allow arbitrary energy dependence for v, as stated in the end of the last subsection, but assume that the effective range expansion is valid up to the energy of the bound state, which is a precondition for the formula in Ref. [38]. In this case there should exist no singularity of $v^{-1}(E)$ between $E = M_{\mathrm{B}}$ and M^{th}, and expansion (67) is safely performed up to the threshold; hence $\Delta v^{-1} = \mathcal{O}\big(B^2\big)$. Otherwise, the singularity of $v^{-1}(E)$ around the threshold spoils the effective range expansion, as the divergence of v^{-1} leads to the existence of the CDD pole. As a result, with the bound state condition (24), the scattering length is now given by

$$a = \frac{\mu}{2\pi}\left(B\left[\frac{dv^{-1}}{dE} - \frac{dG}{dE}\right]_{E=M_{\mathrm{B}}} - \Delta G + \mathcal{O}\big(B^2\big)\right)^{-1}. \tag{70}$$

The first term in the parenthesis in Eq. (70) is calculated as

$$\begin{aligned} B\left[\frac{dv^{-1}}{dE} - \frac{dG}{dE}\right]_{E=M_{\mathrm{B}}} &= -B\left[G^2\frac{dv}{dE} + \frac{dG}{dE}\right]_{E=M_{\mathrm{B}}} \\ &= \frac{B}{|g|^2} \\ &= -\frac{B}{X}\left[\frac{dG}{dE}\right]_{E=M_{\mathrm{B}}} \\ &= \frac{B}{X}\int \frac{d^3q}{(2\pi)^3}\frac{[f(0)]^2 + \mathcal{O}\big(q^2\big)}{[B + q^2/(2\mu)]^2} \\ &= \frac{\mu}{4\pi X}\frac{1}{R} + \mathcal{O}(B), \end{aligned} \tag{71}$$

where we have used Eqs. (24), (35), (26), and the normalization $f(0) = 1$, and we have defined $R \equiv 1/\sqrt{2\mu B}$ in the last line. To evaluate ΔG, we first note that

$$\left[\frac{d^n G}{dE^n}\right]_{E=M_{\mathrm{B}}} = \int \frac{d^3 q}{(2\pi)^3} \frac{(-1)^n n! \left[f\left(q^2\right)\right]^2}{[M_{\mathrm{B}} - M^{\mathrm{th}} - q^2/(2\mu)]^{n+1}}$$
$$= -n! \int \frac{d^3 q}{(2\pi)^3} \frac{\left[f\left(q^2\right)\right]^2}{[B + q^2/(2\mu)]^{n+1}}$$
$$= -\frac{n!}{B^n} \frac{2\mu}{R} \int \frac{d^3 q'}{(2\pi)^3} \frac{\left[f\left(2\mu B q'^2\right)\right]^2}{(q'^2 + 1)^{n+1}}, \tag{72}$$

where $\boldsymbol{q}' \equiv R\boldsymbol{q}$. Thus, summing up all contributions we have

$$\Delta G = -\sum_{n=2}^{\infty} \frac{2\mu}{R} \int \frac{d^3 q'}{(2\pi)^3} \frac{[f(0)]^2}{(q'^2 + 1)^{n+1}} + \mathcal{O}(B)$$
$$= -\frac{\mu}{\pi^2 R} \int_0^{\infty} dx\, x^2 \sum_{n=2}^{\infty} \frac{1}{(x^2 + 1)^{n+1}} + \mathcal{O}(B)$$
$$= -\frac{\mu}{4\pi} \frac{1}{R} + \mathcal{O}(B), \tag{73}$$

where we have used the summation relation

$$\sum_{n=2}^{\infty} \frac{1}{(x^2 + 1)^{n+1}} = \frac{1}{x^2(x^2 + 1)^2} \quad (x \neq 0). \tag{74}$$

As a consequence, we obtain the expression of the scattering length in terms of the compositeness X from Eqs. (70), (71), and (73):

$$a = \frac{\mu}{2\pi} \left(\frac{\mu}{4\pi X} \frac{1}{R} + \frac{\mu}{4\pi} \frac{1}{R} + \mathcal{O}(B) \right)^{-1} = R \frac{2X}{1 + X} + \mathcal{O}\left(B^0\right), \tag{75}$$

which agrees with the result in Ref. [38] with $X = 1 - Z$. It is important that in the weak binding limit the details of the form factor $f\left(q^2\right)$ are irrelevant to the determination of the compositeness of the bound state from the scattering length of two constituents. In contrast, the correction terms of $\mathcal{O}\left(B^0\right)$ depend on the explicit form of the function $f\left(q^2\right)$.

Because we have assumed that the bound state pole lies within the valid region of the effective range approximation, the relation between the scattering length and the effective range is given by[5]

$$r_e = 2R \left(1 - \frac{R}{a} \right). \tag{76}$$

Comparing it with Eq. (75), we find

$$r_e = R \frac{X - 1}{X} + \mathcal{O}\left(B^0\right). \tag{77}$$

This again corresponds to the expression in Ref. [38].

In this way, the structure of the bound state can be determined from a and r_e unambiguously in the weak binding limit. This means that, in principle, tuning a and r_e could lead to arbitrary structure of

[5] The relation (76) can be obtained from the condition $\mathcal{F}^{-1}(k) = -1/a - ik + r_e k^2/2 = 0$ at $k = i/R$.

the bound state. It is, however, shown in Ref. [60] that the bound state with $Z \sim 0$ naturally appears when the state exists near the threshold, and a significant fine tuning is required to realize $Z \sim 1$ in this small binding region. This behavior can be understood by considering the value of Z in the exact $B \to 0$ limit. Actually, the value of Z is shown to vanish in the $B \to 0$ limit, as far as the bound state pole exists in the scattering amplitude [61]. It is therefore natural to expect that the bound state should be $Z \sim 0$ in the small binding region.

2.4. *Generalization to resonances*

Now we generalize our argument to a resonance state. We first introduce the Gamow state [54] denoted as $|\psi)$ to express the resonance state. The eigenvalue of the Hamiltonian is allowed to be complex for the Gamow state:

$$\hat{H}|\psi) = \left(M_{\rm R} - i \frac{\Gamma_{\rm R}}{2} \right) |\psi). \tag{78}$$

Here, $M_{\rm R}$ and $\Gamma_{\rm R}$ are the mass and width of the resonance state, respectively. The state with a complex eigenvalue cannot be normalized in the ordinary sense. To establish the normalization, we define the corresponding bra-state as the complex conjugate of the Dirac bra-state:

$$(\psi| \equiv \langle \psi^{*}|, \tag{79}$$

which was first introduced to describe unstable nuclei [62–64]. As a consequence, the eigenvalue of the Hamiltonian is the same, with the ket vector:[6]

$$(\psi|\hat{H} = \left(M_{\rm R} - i \frac{\Gamma_{\rm R}}{2} \right) (\psi|. \tag{80}$$

These eigenvectors can be normalized as

$$(\psi|\psi) = 1. \tag{81}$$

With the same eigenstates of the free Hamiltonian in Eqs. (39) and (40), we can decompose this normalization as

$$1 = \sum_{a} (\psi|\psi_{a}\rangle\langle\psi_{a}|\psi) + \sum_{j} \int \frac{d^{3}q}{(2\pi)^{3}} (\psi|\boldsymbol{q}_{j}\rangle\langle\boldsymbol{q}_{j}|\psi) = \sum_{a} Z_{a} + \sum_{j} X_{j}, \tag{82}$$

where we have defined the elementariness Z_{a} and compositeness X_{j} as

$$Z_{a} \equiv (\psi|\psi_{a}\rangle\langle\psi_{a}|\psi), \quad X_{j} \equiv \int \frac{d^{3}q}{(2\pi)^{3}} (\psi|\boldsymbol{q}_{j}\rangle\langle\boldsymbol{q}_{j}|\psi). \tag{83}$$

In addition, we define the momentum space wave function $\tilde{\psi}_{j}(q) \equiv \langle\boldsymbol{q}_{j}|\psi)$. It follows from Eqs. (79) and (80) that

$$(\psi|\boldsymbol{q}_{j}\rangle = \langle\boldsymbol{q}_{j}|\psi) = \tilde{\psi}_{j}(q). \tag{84}$$

The compositeness is then given by

$$X_{j} = \int \frac{d^{3}q}{(2\pi)^{3}} \left[\tilde{\psi}_{j}(q) \right]^{2}. \tag{85}$$

In contrast to Eq. (44), where X_{j} is given by the absolute value squared, the compositeness of the resonance is given by the complex number squared. This is also the case for Z_{a}, because

[6] The eigenvectors $|\psi^{*})$ and $(\psi^{*}| = \langle\psi|$ have the eigenvalue $M_{\rm R} + i\Gamma_{\rm R}/2$.

$(\psi|\psi_a\rangle = \langle\psi_a|\psi) \neq \langle\psi_a|\psi)^*$. In this way, Z_a and X_j are in general complex, and the probabilistic interpretation of Z_a and X_j is not guaranteed.

To determine the wave function, we solve the Schrödinger equation

$$\left(M_{\mathrm{R}} - i\frac{\Gamma_{\mathrm{R}}}{2}\right)\tilde{\psi}_j(q) = \left(M_j^{\mathrm{th}} + \frac{q^2}{2\mu_j}\right)\tilde{\psi}_j(q) + \sum_k v_{jk} f_j\left(q^2\right)\int \frac{d^3q'}{(2\pi)^3} f_k\left(q'^2\right)\tilde{\psi}_k(q') + \sum_a g_0^{a,j} f_j\left(q^2\right)\langle\psi_a|\psi), \tag{86}$$

and

$$\left(M_{\mathrm{R}} - i\frac{\Gamma_{\mathrm{R}}}{2}\right)\langle\psi_a|\psi) = M_a\langle\psi_a|\psi) + \sum_k g_0^{a,k}\int\frac{d^3q}{(2\pi)^3} f_k\left(q^2\right)\tilde{\psi}_k(q). \tag{87}$$

Eliminating $\langle\psi_a|\psi)$, we obtain

$$\tilde{\psi}_j(q) = \frac{-c_j f_j\left(q^2\right)}{M_j^{\mathrm{th}} - M_{\mathrm{R}} + i\Gamma_{\mathrm{R}}/2 + q^2/(2\mu_j)}, \tag{88}$$

with the normalization constant

$$c_j \equiv \sum_k v_{jk}^{\mathrm{eff}}(M_{\mathrm{R}} - i\Gamma_{\mathrm{R}}/2)\int\frac{d^3q}{(2\pi)^3} f_k\left(q^2\right)\tilde{\psi}_k(q), \tag{89}$$

where $v_{jk}^{\mathrm{eff}}(E)$ is defined in the same way with Eq. (47). The condition for nonzero c_j is

$$\det\left[1 - v^{\mathrm{eff}}(M_{\mathrm{R}} - i\Gamma_{\mathrm{R}}/2)G(M_{\mathrm{R}} - i\Gamma_{\mathrm{R}}/2)\right] = 0. \tag{90}$$

This is the condition for the resonance pole at $E = M_{\mathrm{R}} - i\Gamma_{\mathrm{R}}/2$. We note that the loop function in the complex energy plane is defined on the 2^n-sheeted Riemann surface for an n-channel problem. The resonance pole can exist in any sheet, except for the one which is reached by choosing the first sheet for all channels. The most relevant Riemann sheet for the scattering amplitude at a given energy is reached by choosing the first sheet for the closed channels and the second sheet for the open channels. In the following, we concentrate on the poles in this Riemann sheet, while the framework is in principle applicable to the complex poles in the other Riemann sheets.

Also, for the resonance pole the residue of the scattering amplitude is interpreted as the product of the coupling constants $g_j g_k$:

$$g_j g_k = \lim_{E\to M_{\mathrm{R}} - i\Gamma_{\mathrm{R}}/2}(E - M_{\mathrm{R}} + i\Gamma_{\mathrm{R}}/2)t_{jk}(E), \tag{91}$$

where the complex conjugate should not be taken for the coupling constant g_k since t_{jk} is symmetric: $t_{jk} = t_{kj}$. In contrast to the bound states, the coupling constant g_j is in general complex. The amplitude near the resonance pole is also given by

$$\lim_{E\to M_{\mathrm{R}} - i\Gamma_{\mathrm{R}}/2} t_{jk}(E) \sim \sum_{l,m}\int\frac{d^3q}{(2\pi)^3}\int\frac{d^3p}{(2\pi)^3} v_{jl}^{\mathrm{eff}}(E) f_l\left(q^2\right)\frac{\langle \boldsymbol{q}_l|\psi)(\psi|\boldsymbol{p}_m\rangle}{E - M_{\mathrm{R}} + i\Gamma_{\mathrm{R}}/2} f_m\left(p^2\right) v_{mk}^{\mathrm{eff}}(E) \to \frac{c_j c_k}{E - M_{\mathrm{R}} + i\Gamma_{\mathrm{R}}/2}, \tag{92}$$

thus we find $c_j = g_j$. The compositeness in channel j is then given by

$$X_j = \int\frac{d^3q}{(2\pi)^3}\left[\tilde{\psi}_j(q)\right]^2 = -g_j^2\left[\frac{dG_j}{dE}\right]_{E=M_{\mathrm{R}} - i\Gamma_{\mathrm{R}}/2}. \tag{93}$$

The loop function in the complex energy plane should be evaluated by choosing the Riemann sheets consistently with the choice to obtain the pole condition (90).

From Eq. (87) and its counterpart coming from Eq. (80), we obtain

$$\langle \psi_a | \psi \rangle = (\psi | \psi_a \rangle = \sum_j \frac{c_j g_0^{a,j}}{M_{\mathrm{R}} - i\Gamma_{\mathrm{R}}/2 - M_a} G_j(M_{\mathrm{R}} - i\Gamma_{\mathrm{R}}/2), \tag{94}$$

so the total elementariness $Z \equiv \sum_a Z_a$ is obtained as

$$Z \equiv \sum_a Z_a = -\sum_{j,k} g_k g_j \left[G_j \frac{dv_{jk}^{\mathrm{eff}}}{dE} G_k \right]_{E=M_{\mathrm{R}}-i\Gamma_{\mathrm{R}}/2}. \tag{95}$$

Using Eqs. (82), we obtain

$$-\sum_{j,k} g_k g_j \left[\delta_{jk} \frac{dG_j}{dE} + G_j \frac{dv_{jk}^{\mathrm{eff}}}{dE} G_k \right]_{E=M_{\mathrm{R}}-i\Gamma_{\mathrm{R}}/2} = 1. \tag{96}$$

This corresponds to the nonrelativistic counterpart of the generalized Ward identity for resonance states derived in Ref. [26]. The special case of $Z = 0$ of Eq. (96) is obtained in Ref. [53] by using an energy-independent separable interaction without the bare-state contribution. Here we mention that we should obtain the same results in appropriate ways to treat resonance states such as the complex scaling method [65].[7]

By definition, the compositeness for the resonance state becomes complex. Therefore, strictly speaking, it cannot be interpreted as a probability of finding the two-body component. Nevertheless, because it represents the contribution of the channel wave function to the total normalization, the compositeness X_j will have an important piece of information on the structure of the resonance. For instance, consider a resonance such that the real part of a single X_j is close to unity with small imaginary part, and all the other components have small absolute values. In this case, the resonance wave function is considered to be similar to that of the bound state dominated by the jth channel. It is therefore natural to interpret the resonance state in this case as dominated by the component of the channel j. In general, however, all X_j and Z can be arbitrary complex numbers constrained by Eq. (82). The interpretation of the structure of such a state from X_j and Z is not straightforward.

2.5. *Relativistic covariant formulation*

Finally, we consider the coupled-channel two-body scattering in a relativistic form. Here we do not consider the intermediate states with more than two particles but simply solve the two-body wave equation.[8] To describe the wave function of the resonances, we extract the relative motion of the two-body system from a relativistic scattering equation with a three-dimensional reduction [66,67].

According to Appendix B, we introduce the state $|\boldsymbol{q}_j^{\mathrm{co}}\rangle$ as the two-body scattering state of the particles with masses m_j and M_j and the relative momentum $\boldsymbol{q}$, and its normalization is fixed as

$$\langle \boldsymbol{q}_j^{\prime\,\mathrm{co}} | \boldsymbol{q}_k^{\mathrm{co}} \rangle = \frac{2\omega_j(\boldsymbol{q}) \Omega_j(\boldsymbol{q})}{\sqrt{s_{qj}}} (2\pi)^3 \delta_{jk} \delta^3 (\boldsymbol{q}' - \boldsymbol{q}), \tag{97}$$

[7] In Sect. 3.2 we will compare the structure of $\Lambda(1405)$ in the present framework with that in the complex scaling method.

[8] In general relativistic field theory, there are infinitely many diagrams which contribute to the scattering amplitude. The present formulation picks up the summation of the s-channel two-body loop diagrams, which is the most dominant contribution in the nonrelativistic limit.

where $s_{qj} \equiv [\omega_j(\boldsymbol{q}) + \Omega_j(\boldsymbol{q})]^2$ with the on-shell energies $\omega_j(\boldsymbol{q}) \equiv \sqrt{\boldsymbol{q}^2 + m_j^2}$ and $\Omega_j(\boldsymbol{q}) \equiv \sqrt{\boldsymbol{q}^2 + M_j^2}$. This normalization is chosen so that the expression of the relativistic wave equation (103) becomes a natural extension of the nonrelativistic Schrödinger equation (see Appendix B). Furthermore, we also introduce the bare state Ψ_a, which satisfies the following orthonormal conditions:

$$\langle \Psi_a | \Psi_b \rangle = \delta_{ab}, \quad \langle \boldsymbol{q}_j^{\rm co} | \Psi_a \rangle = \langle \Psi_a | \boldsymbol{q}_j^{\rm co} \rangle = 0. \tag{98}$$

We note that with the normalization (97) and (98), the complete set of the system is given by

$$1 = \sum_a |\Psi_a\rangle \langle \Psi_a| + \sum_j \int \frac{d^3 q}{(2\pi)^3} \frac{\sqrt{s_{qj}}}{2\omega_j(\boldsymbol{q})\Omega_j(\boldsymbol{q})} |\boldsymbol{q}_j^{\rm co}\rangle \langle \boldsymbol{q}_j^{\rm co}|. \tag{99}$$

The scattering state $|\boldsymbol{q}_j^{\rm co}\rangle$ and the bare state $|\Psi_a\rangle$ span the space of the eigenstates of the kinetic energy operator $\hat{\mathcal{K}}$ which extracts the total energy squared of the state. Namely, for the two-body scattering state $|\boldsymbol{q}_j^{\rm co}\rangle$ we have

$$\hat{\mathcal{K}} |\boldsymbol{q}_j^{\rm co}\rangle = s_{qj} |\boldsymbol{q}_j^{\rm co}\rangle, \quad \langle \boldsymbol{q}_j^{\rm co}| \hat{\mathcal{K}} = \langle \boldsymbol{q}_j^{\rm co}| s_{qj}. \tag{100}$$

For the bare state, the eigenvalue of $\hat{\mathcal{K}}$ is the mass squared of the bare state Ψ_a, M_a^2:

$$\hat{\mathcal{K}} |\Psi_a\rangle = M_a^2 |\Psi_a\rangle, \quad \langle \Psi_a | \hat{\mathcal{K}} = \langle \Psi_a | M_a^2. \tag{101}$$

The dynamics of the system are determined by the interaction operator $\hat{\mathcal{V}}$. We again adopt the separable form as

$$\langle \boldsymbol{q}_j^{\prime\,\rm co} | \hat{\mathcal{V}} | \boldsymbol{q}_k^{\rm co} \rangle = V_{jk} f_j(q^{\prime 2}) f_k(q^2), \quad \langle \boldsymbol{q}_j^{\rm co} | \hat{\mathcal{V}} | \Psi_a \rangle = \langle \Psi_a | \hat{\mathcal{V}} | \boldsymbol{q}_j^{\rm co} \rangle = g_0^{a,j} f_j(q^2), \quad \langle \Psi_a | \hat{\mathcal{V}} | \Psi_b \rangle = 0, \tag{102}$$

where V_{jk} is a real symmetric matrix and $g_0^{a,j}$ and $f_j(q^2)$ are real with an appropriate choice of phases of the states.[9] In order to make a three-dimensional reduction of the scattering equation, we assume that the form factor $f_j(q^2)$ depends only on the magnitude of the three-momentum. We consider that the wave equation with the operator $\hat{\mathcal{K}} + \hat{\mathcal{V}}$ contains a resonance $|\Psi)$ with mass $M_{\rm R}$ and width $\Gamma_{\rm R}$ as an eigenstate [66,67]:

$$\left[\hat{\mathcal{K}} + \hat{\mathcal{V}}\right] |\Psi) = s_{\rm R} |\Psi), \quad (\Psi| \left[\hat{\mathcal{K}} + \hat{\mathcal{V}}\right] = (\Psi| s_{\rm R}, \tag{103}$$

where $(\Psi| = \langle \Psi^* |$ and $s_{\rm R} = (M_{\rm R} - i\Gamma_{\rm R}/2)^2$. By using Eq. (99) we can decompose the normalization of the resonance vector $(\Psi|\Psi) = 1$ as

$$1 = (\Psi|\Psi) = \sum_a Z_a + \sum_j X_j, \tag{104}$$

where we have defined the elementariness Z_a and compositeness X_j as:

$$Z_a \equiv (\Psi|\Psi_a\rangle\langle\Psi_a|\Psi), \quad X_j \equiv \int \frac{d^3 q}{(2\pi)^3} \frac{\sqrt{s_{qj}}}{2\omega_j(\boldsymbol{q})\Omega_j(\boldsymbol{q})} \left[\tilde{\Psi}_j(q)\right]^2, \tag{105}$$

[9] In relativistic field theory, the coupling $g_0^{a,j}$ can have an energy dependence from the derivative coupling. We do not consider the energy dependence of the coupling, in order to ensure a smooth reduction to the results in the previous section in the nonrelativistic limit.

with the momentum space wave function

$$\langle \boldsymbol{q}_j^{\text{co}} | \Psi \rangle = \tilde{\Psi}_j(q) = \big(\Psi | \boldsymbol{q}_j^{\text{co}} \big\rangle. \tag{106}$$

As in Sect. 2.3, the wave function is determined as

$$\tilde{\Psi}_j(q) = \frac{C_j f_j(q^2)}{s_{\text{R}} - s_{qj}}, \tag{107}$$

$$C_j = \sum_k V_{jk}^{\text{eff}}(s_{\text{R}}) \int \frac{d^3 q'}{(2\pi)^3} \frac{\sqrt{s_{q'k}}}{2\omega_k(\boldsymbol{q}')\Omega_k(\boldsymbol{q}'^2)} f_k(q'^2) \tilde{\Psi}_k(q'), \tag{108}$$

$$V_{jk}^{\text{eff}}(s) = V_{jk} + \sum_a \frac{g_0^{a,j} g_0^{a,k}}{s - M_a^2}. \tag{109}$$

The consistency condition for nonzero C_j is given by

$$\det \left[1 - V^{\text{eff}}(s_{\text{R}}) G(s_{\text{R}}) \right] = 0, \tag{110}$$

where the loop function G is diagonal with respect to the channel index and is expressed as

$$G_j(s) = \int \frac{d^3 q}{(2\pi)^3} \frac{\sqrt{s_{qj}}}{2\omega_j(\boldsymbol{q})\Omega_j(\boldsymbol{q})} \frac{\left[f_j(q^2) \right]^2}{s - s_{qj}} = \int \frac{d^4 q}{(2\pi)^4} \frac{i \left[f_j(q^2) \right]^2}{\left[(P/2 + q)^2 - m_j^2 \right] \left[(P/2 - q)^2 - M_j^2 \right]}, \tag{111}$$

with the energy squared $P^2 = s$. The energy squared s in the denominator of the loop function is considered to have an infinitesimal positive imaginary part $i\epsilon$: $s \to s + i\epsilon$. We note that the dimensional regularization of the loop function is achieved by setting $f_j(q^2) = 1$ and modifying the integration variable as $d^4 k \to \mu_{\text{reg}}^{4-d} d^d k$ with the regularization scale μ_{reg}.

In Appendix B we confirm that the wave equation (103) indeed describes a two-body system governed by the relativistic scattering equation. Namely, with the energy-dependent two-body interaction $V_{jk}^{\text{eff}}(s)$ (109) and the loop function $G_j(s)$ (111), the scattering amplitude $T_{jk}(s)$ can be calculated as

$$T_{jk}(s) = V_{jk}^{\text{eff}}(s) + \sum_l V_{jl}^{\text{eff}}(s) G_l(s) T_{lk}(s). \tag{112}$$

Therefore, Eq. (110) ensures that the scattering amplitude $T_{jk}(s)$ has a pole at $s = s_{\text{R}}$.

By comparing the residue of the resonance pole as in Eq. (92), we find $C_j = g_j$, where

$$g_j g_k = \lim_{s \to s_{\text{R}}} (s - s_{\text{R}}) T_{jk}(s). \tag{113}$$

Then, as in Sect. 2.3, we obtain

$$\langle \Psi_a | \Psi \rangle = (\Psi | \Psi_a \rangle = \sum_j \frac{g_j g_0^{a,j}}{s_{\text{R}} - M_a^2} G_j(s_{\text{R}}). \tag{114}$$

Therefore, we obtain the compositeness and elementariness as

$$X_j = -g_j^2 \left[\frac{dG_j}{ds} \right]_{s=s_{\text{R}}}, \tag{115}$$

$$Z \equiv \sum_a Z_a = -\sum_{j,k} g_k g_j \left[G_j \frac{dV_{jk}^{\text{eff}}}{ds} G_k \right]_{s=s_{\text{R}}}, \tag{116}$$

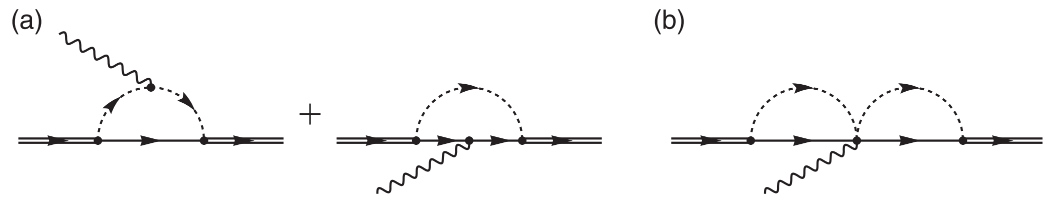

Fig. 1. Diagrammatic interpretation of the compositeness X (a) and elementariness Z (b). The double and wiggly lines represent the resonance state and the probe current, respectively, and the solid and dashed lines correspond to the constituent particles.

and the sum rule is derived from the normalization (104) as

$$-\sum_{j,k} g_k g_j \left[\delta_{jk} \frac{dG_j}{ds} + G_j \frac{dV_{jk}^{\text{eff}}}{ds} G_k \right]_{s=s_{\mathrm{R}}} = 1. \tag{117}$$

This is another derivation of the generalized Ward identity in Ref. [26], where Eq. (117) is obtained by attaching one probe current to the meson–baryon scattering amplitude. The derivative of the loop function corresponds to the diagrams in Fig. 1(a) in the soft limit of the probe current. It is therefore consistent to interpret the first term of Eq. (117) as compositeness, which reflects the contribution from the two-body molecule component. On the other hand, the derivative of the contact interaction corresponds to the attachment of the probe current to the interaction vertex [Fig. 1(b)], which represents something other than the compositeness and thus is understood as the elementariness.

We note that, although both the compositeness X_j and elementariness Z are complex for resonances, their sum should be unity, provided that the proper normalization of the wave function is adopted. As in the nonrelativistic case, the compositeness (elementariness) is expressed with the derivative of the loop function (interaction), and they can be determined by the local behavior of the interaction and loop function. Finally, we mention that the expression of the elementariness Z in Eq. (116) coincides with that derived by matching with the Yukawa theory in Ref. [42]. In this work, we derive Z and X_j without specifying the explicit form of the vertex and relate them with the wave function of the bound and resonance states.

3. Applications: structure of dynamically generated hadrons

3.1. Compositeness and elementariness in chiral dynamics

Having established the compositeness and elementariness in Eqs. (115) and (116), we now turn to the analysis of physical hadronic resonances by theoretical models with hadronic degrees of freedom. One of the most prominent models is the coupled-channel approach with the chiral perturbation theory. In this model the nonperturbative summation of the chiral interaction makes it possible to generate hadronic resonances dynamically, and hence these hadronic resonances are often called dynamically generated hadrons. This framework has been successfully applied to the description of low-energy hadron scatterings with resonance states. Among others, the $\Lambda(1405)$ resonance in the strangeness $S = -1$ meson–baryon scattering [59,68–75] and the lightest scalar and vector mesons in the meson–meson scattering [76–85] have been extensively studied in this approach.

The compositeness and elementariness have been evaluated in the chiral model with the simple leading order chiral interaction for $\Lambda(1405)$ and the scalar mesons in Ref. [27]. The compositeness of the $\rho(770)$ meson [43] and $K^*(892)$ [44] are also studied in phenomenological models. Here we aim at more quantitative discussion by using refined chiral models constrained by the recent experimental data. For this purpose, we employ the next-to-leading-order calculations for $\Lambda(1405)$ [73,74] and for scalar and vector mesons [85].

As we will show below, the scattering amplitudes in Refs. [73,74,85] can be reduced to the form of the coupled-channel algebraic equation

$$T_{jk}(s) = V_{jk}(s) + \sum_{l} V_{jl}(s) G_l(s) T_{lk}(s). \tag{118}$$

Here, the separable interaction kernel V_{jk} is a symmetric matrix with respect to the channel indices and depends on the Mandelstam variable s, and G_j is the two-body loop function. The explicit forms of V_{jk} and G_j will be given for each model. The resonances are identified by the poles of the scattering amplitude T_{jk}, and the scattering amplitude can be written in the vicinity of one of the resonance poles as:

$$T_{jk}(s) = \frac{g_j g_k}{s - s_{\mathrm{R}}} + T_{jk}^{\mathrm{BG}}(s), \tag{119}$$

where g_j and s_{R} are the coupling constant and the pole position for the resonance, respectively, and T_{jk}^{BG} is a background term which is regular at $s \to s_{\mathrm{R}}$.

In this study, we utilize the set of the one- and two-body states introduced in Sect. 2 as the basis for interpreting the structure of the hadronic resonances in the coupled-channel chiral model. On the assumption that the energy dependence of the interaction originates from channels which do not appear as explicit degrees of freedom, it has been shown that the final expression of the compositeness is given by Eq. (115) only with the quantities at the pole position. Namely, the j-channel compositeness is expressed with the pole position and the residue of the amplitude and the derivative of the loop function G_j, which is obtained with the two-body eigenstates of the free Hamiltonian $\hat{H}_0$:

$$X_j = -g_j^2 \left[\frac{dG_j}{ds} \right]_{s=s_{\mathrm{R}}}. \tag{120}$$

On the other hand, the elementariness Z is given by the rest of the component out of unity:

$$Z = 1 - \sum_{j} X_j. \tag{121}$$

By using the interaction V in the coupled-channel equation (118), the elementariness Z is also given as

$$Z = -\sum_{j,k} g_k g_j \left[G_j \frac{dV_{jk}}{ds} G_k \right]_{s=s_{\mathrm{R}}}, \tag{122}$$

which measures the contributions from one-body bare states and implicit two-body states on the basis in Sect. 2. Since the contribution of the bare state with a large mass gives an interaction with polynomial energy dependence, we are allowed to apply Eq. (122) for V with general energy dependence, which can be reproduced with suitable bare states.

Let us summarize the interpretation of the compositeness and elementariness for resonances. As shown in Sect. 2.3, X_j and Z for resonances are in general complex. This fact spoils the probabilistic interpretation in a strict sense. It is, however, possible to interpret the structure of the resonance when one of the real parts of X_j or Z is close to unity and all the other numbers have small absolute values. In this case, we interpret that the resonance is dominated by the jth channel component or something other than the two-body channels involved, respectively, on the basis of the similarity of the wave function of the stable bound state.

Table 1. Compositeness X_j and elementariness Z of $\Lambda(1405)$ in the isospin basis.

	$\Lambda(1405)$, higher pole	$\Lambda(1405)$, lower pole
$\sqrt{s_{\mathrm{R}}}$ [MeV]	$1424 - 26i$	$1381 - 81i$
$X_{\bar{K}N}$	$1.14 + 0.01i$	$-0.39 - 0.07i$
$X_{\pi\Sigma}$	$-0.19 - 0.22i$	$0.66 + 0.52i$
$X_{\eta\Lambda}$	$0.13 + 0.02i$	$-0.04 + 0.01i$
$X_{K\Xi}$	$0.00 + 0.00i$	$-0.00 + 0.00i$
Z	$-0.08 + 0.19i$	$0.77 - 0.46i$

3.2. *Structure of* $\Lambda(1405)$

In Refs. [73,74] the low-energy meson–baryon scattering in the strangeness $S = -1$ sector was studied in the chiral model. The meson–baryon interaction kernel was constructed in chiral perturbation theory up to the next-to-leading order, which consists of the Weinberg–Tomozawa contact term, the s- and u-channel Born terms, and the next-to-leading-order contact terms. After the s-wave projection, the interaction kernel V_{jk} depends only on the Mandelstam variable s as a real symmetric separable interaction. The explicit form of V_{jk} can be found in Refs. [74,75]. The loop function is regularized by the dimensional regularization:

$$\begin{aligned} G_j(s) &= i\mu_{\mathrm{reg}}^{4-d} \int \frac{d^d q}{(2\pi)^d} \frac{1}{\left[(P/2+q)^2 - m_j^2\right]\left[(P/2-q)^2 - M_j^2\right]} \\ &= a_j(\mu_{\mathrm{reg}}) + \frac{1}{16\pi^2}\left[-1 + \ln\left(\frac{m_j^2}{\mu_{\mathrm{reg}}^2}\right) + \frac{s + M_j^2 - m_j^2}{2s}\ln\left(\frac{M_j^2}{m_j^2}\right)\right. \\ &\qquad \left. - \frac{\lambda^{1/2}(s,\, m_j^2,\, M_j^2)}{s}\operatorname{artanh}\left(\frac{\lambda^{1/2}(s,\, m_j^2,\, M_j^2)}{m_j^2 + M_j^2 - s}\right)\right], \end{aligned} \tag{123}$$

with the Källen function $\lambda(x,\ y,\ z) = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx$. The finite part is specified by the subtraction constant $a_j(\mu_{\mathrm{reg}})$ at the regularization scale μ_{reg}. Because the meson–baryon one loop is counted as next-to-next-to-leading order in the baryon chiral perturbation theory, the amplitude is not renormalizable and hence it depends on the subtraction constants in this framework. The low-energy constants in the next-to-leading-order contact interaction terms and the subtraction constants of the loop function have been determined by fitting to the low-energy total cross sections of $K^- p$ scattering to elastic and inelastic channels, the threshold branching ratios, and the recent measurement of the $1s$ shift and width of kaonic hydrogen [86,87]. In this approach, the $\Lambda(1405)$ resonance is associated with two poles of the scattering amplitude in the complex energy plane [72]. For convenience we refer to the pole which has higher (lower) mass $M_{\mathrm{R}} = \mathrm{Re}\left(\sqrt{s_{\mathrm{R}}}\right)$ as the higher (lower) pole. It is expected from the structure of the Weinberg–Tomozawa interaction that the higher pole originates in a bound state caused by the $\bar{K}N$ attraction [59].

With the formulae in Sect. 3.1 we calculate the pole positions, compositeness X_j, and the elementariness Z of the $\Lambda(1405)$ resonance in this model; the results are summarized in Table 1. In Refs. [73,74], the isospin symmetry is slightly broken by the physical hadron masses. Therefore, we evaluate the compositeness in the charge basis and define the compositeness in the isospin basis by summing up all the channels in the charge basis, i.e., $X_{\bar{K}N} = X_{K^-p} + X_{\bar{K}^0 n}$, and so on. Although there are nonzero contributions from the $I = 1$ channels, $X_{\pi^0\Lambda}$ and $X_{\eta\Sigma^0}$, to the total normalization, these are negligible and hence not listed in Table 1. It is remarkable that the real part of the $X_{\bar{K}N}$

component of the higher $\Lambda(1405)$ pole is close to unity and its imaginary part is very small. In addition, the magnitude of the real and imaginary parts of all the other components is also small ($\lesssim$0.2). This indicates that the wave function of the higher $\Lambda(1405)$ pole is similar to that of the pure $\bar{K}N$ bound state which has $X_{\bar{K}N} = 1$, $X_i = 0$ ($i \neq \bar{K}N$), and $Z = 0$. It is therefore natural to interpret that the higher $\Lambda(1405)$ pole is dominated by the $\bar{K}N$ composite component. This is consistent with the non-qqq nature of this pole from the N_c scaling analysis [23,24].

On the other hand, for the lower pole, there is a certain amount of cancellation ($\sim$0.4) in the real part of the sum rule (117), and the absolute values of the imaginary parts are as large as $\sim$0.5. Although one may observe relatively large contributions in $X_{\pi\Sigma}$ and Z, the dominance of these components is comparable with the magnitude of the imaginary part. Therefore, it is not possible to clearly conclude the structure of the lower pole from the present analysis.

The compositeness and elementariness of $\Lambda(1405)$ were calculated in Ref. [27] using the simple chiral model with the leading-order Weinberg–Tomozawa interaction. The qualitative features of X_j and Z are not changed very much, so we confirm the earlier results in the present refined model. At the quantitative level, the results of the lower pole show relatively larger model dependence. This model dependence also implies the difficulty of clear interpretation of the structure of the lower pole.

Before closing this subsection, we mention that the structure of $\Lambda(1405)$ was investigated in the complex scaling method in Refs. [88,89]. In a $\bar{K}N$–$\pi\Sigma$ two-channel model, the norm of each component is evaluated from the wave function. It is found that the norm of the $\bar{K}N$ component of the higher $\Lambda(1405)$ pole is close to unity with a small imaginary part. Thus, the result for the higher pole is qualitatively consistent with ours. On the other hand, the result for the lower pole in Ref. [89] shows the dominance of the $\pi\Sigma$ component. This is because the complete set to decompose the resonance wave function in Ref. [89] does not contain the elementary component. Namely, the application of our formula to their amplitude would indicate a certain amount of the elementary component Z, as we have found here, since the interaction in Ref. [89] has an energy dependence. In fact, this is in accordance with the observation in Ref. [89] that the lower pole disappears when the energy dependence of the interaction is switched off.

3.3. Structure of the lightest scalar and vector mesons

The lowest-lying scalar and vector mesons in the meson–meson scattering have been studied in Ref. [85] using the inverse amplitude method (IAM) with the chiral interaction up to the next-to-leading order. The scattering amplitude in the coupled-channel IAM is given by

$$T(s) = T_2(s)\,[T_2(s) - T_4(s)]^{-1}\,T_2(s), \tag{124}$$

where T_2 and T_4 are respectively the leading- and next-to-leading-order amplitudes in matrix form with channel indices from chiral perturbation theory and have been projected to the orbital angular momentum $L = 0$ (scalar) and $L = 1$ (vector). In contrast to the model in the previous subsection, the meson–meson one loop is in the next-to-leading order, and hence the amplitude does not depend on the renormalization scale. Therefore, the parameters in this model are the renormalized low-energy constants in the next-to-leading-order chiral Lagrangians. These constants are determined by fitting the experimental meson–meson scattering data such as the $\pi\pi$ scattering up to $\sqrt{s} = 1.2$ GeV [85]. The lightest scalar mesons σ, $f_0(980)$, and κ are found as poles of the s-wave amplitude, while the $a_0(980)$ resonance appears as a cusp at the $K\bar{K}$ threshold, but the corresponding resonance pole is not found. The vector mesons $\rho(770)$ and $K^*(892)$ are also dynamically generated as poles of the p-wave amplitude.

To evaluate the compositeness and elementariness, we rewrite the amplitude (124) in the form of Eq. (118). To this end, we first notice that T_4 can be decomposed as the s-channel loop part and the rest:

$$T_4 = T_2 G T_2 + T_{4,\text{non-}G}, \tag{125}$$

where $T_{4,\text{non-}G}(s)$ consists of the next-to-leading-order tree-level amplitudes, tadpoles, and t- and u-channel loop contributions. The loop function $G_j(s)$ in Eq. (125) is given by

$$G_j(s) = \frac{1}{16\pi^2}\left[-1 + \left(\frac{M_j^2 - m_j^2}{2s} + \frac{m_j^2 + M_j^2}{2(m_j^2 - M_j^2)} \right) \ln\left(\frac{M_j^2}{m_j^2} \right) \right.$$
$$\left. - \frac{\lambda^{1/2}(s,\, m_j^2,\, M_j^2)}{s} \operatorname{artanh}\left(\frac{\lambda^{1/2}(s,\, m_j^2,\, M_j^2)}{m_j^2 + M_j^2 - s} \right) \right]. \tag{126}$$

Note that there are no degrees of freedom of the subtraction constant; the finite part is determined by the low-energy constants included in $T_{4,\text{non-}G}$. We then define

$$V \equiv T_2 (T_2 - T_{4,\text{non-}G})^{-1} T_2. \tag{127}$$

It is easily checked that the amplitude in IAM (124) is formally equivalent to Eq. (118) with the interaction (127) and the loop function (126). We thus interpret Eq. (127) as the effective interaction kernel used in IAM. Physically, this interaction kernel contains not only the chiral interaction up to the next-to-leading order but also the nonperturbative summation of contributions from the t- and u-channel loops. We also note that the interaction kernel (127) can have a nonzero imaginary part due to contributions from the t- and u-channel loops, which will disappear in the nonrelativistic limit.

Before evaluating the compositeness, let us focus on the structure of the interaction kernel (127). Because of the $(T_2 - T_{4,\text{non-}G})^{-1}$ factor, the interaction kernel can have a pole when $\det[T_2(s) - T_{4,\text{non-}G}(s)] = 0$ is satisfied. Thus, even though the IAM is constructed from chiral perturbation theory without bare fields of the scalar and vector mesons, there can be a pole contribution in the effective interaction V. In fact, we find poles in the vector channel V near the physical resonances as

$$\rho \text{ channel} : 746 - 11i \text{ MeV}, \qquad K^* \text{ channel} : 890 - 0i \text{ MeV}. \tag{128}$$

In contrast, in the scalar channel there is no pole contribution in the relevant energy region. The pole structure of the interaction V can be related to the origin of resonances in the full amplitude T.

Now let us evaluate the compositeness and elementariness of the lightest scalar and vector mesons described by the coupled-channel IAM developed in Ref. [85] on the basis of the one- and two-body states introduced in Sect. 2. The values obtained for the compositeness and elementariness are listed in Tables 2 (scalar channels) and 3 (vector channels). In the scalar channels, the $f_0(980)$ resonance

Table 2. Compositeness X_j and elementariness Z of scalar mesons in the isospin basis.

	$f_0(500) = \sigma$	$f_0(980)$	$K_0^*(800) = \kappa$
$\sqrt{s_{\rm R}}$ [MeV]	$443 - 217i$	$988 - 4i$	$750 - 227i$
$X_{\pi\pi}$	$-0.09 + 0.37i$	$0.00 - 0.00i$	—
$X_{K\bar{K}}$	$-0.01 - 0.00i$	$0.87 - 0.04i$	—
$X_{\eta\eta}$	$-0.00 + 0.00i$	$0.06 + 0.01i$	—
$X_{\pi K}$	—	—	$0.32 + 0.36i$
$X_{\eta K}$	—	—	$-0.01 - 0.00i$
Z	$1.09 - 0.37i$	$0.07 + 0.02i$	$0.70 - 0.36i$

Table 3. Compositeness X_j and elementariness Z of vector mesons in the isospin basis.

	$\rho(770)$	$K^*(892)$
$\sqrt{s_R}$ [MeV]	$760 - 84i$	$885 - 22i$
$X_{\pi\pi}$	$-0.08 + 0.03i$	—
$X_{K\bar{K}}$	$-0.02 + 0.00i$	—
$X_{\pi K}$	—	$-0.03 + 0.04i$
$X_{\eta K}$	—	$-0.03 + 0.00i$
Z	$1.10 - 0.04i$	$1.06 - 0.04i$

shows a clear property: the real part of the $K\bar{K}$ component is close to unity, while other components are smaller than 0.07. This indicates that the $f_0(980)$ resonance is dominated by the $K\bar{K}$ component. On the other hand, the results for the σ and κ resonances are subtle; the largest component seems to be Z, but its imaginary part is not small, ~ 0.37. We thus refrain from interpreting the structure of the σ and κ resonance from X and Z. In Refs. [21,22] the non-$q\bar{q}$ nature of the scalar mesons is implied from the N_c scaling behavior. Our conclusion of the $K\bar{K}$ dominance of $f_0(980)$ is consistent with the N_c scaling analysis.

In the vector channels,[10] we find that, for both the $\rho(770)$ and $K^*(892)$ mesons, the real part of the elementariness Z is close to unity and the magnitude of the imaginary part is less than 0.1. This indicates that the structure originates in the elementary component. This is consistent with the finding of the pole contribution in the interaction kernel V for the vector channels. In fact, the physical pole position in Table 3 is very close to that in the effective interaction (128). We thus conclude that these vector mesons are not dominated by the two-meson composite structure. This is consistent with the N_c scaling analysis in Refs. [21,22], which indicates the $q\bar{q}$ structure of vector mesons.

The compositeness of scalar mesons [σ, $f_0(980)$, and $a_0(980)$] has been studied in Ref. [27] using the leading-order chiral interaction. The qualitative tendency of the results for σ and $f_0(980)$ is similar to the present calculation, while the dominance of the $K\bar{K}$ component of $f_0(980)$ is much clearer in the present results. Also, for the vector mesons, the present calculation in IAM with the next-to-leading-order chiral interaction is consistent with the previous phenomenological ones in Refs. [43,44], which suggest that $\rho(770)$ and $K^*(892)$ are elementary.

3.4. *Structure of other hadrons*

In the preceding subsections we have evaluated the compositeness and elementariness of $\Lambda(1405)$, light scalar mesons, and light vector mesons using the scattering amplitudes calculated in chiral dynamics with systematic improvements by higher-order contributions. In this subsection we also discuss the compositeness and elementariness of $N(1535)$ and $\Lambda(1670)$ in a simplified model with

[10] In the framework of IAM, the loop function in p wave is identical to that in s wave. On the other hand, with a nonrelativistic separable interaction, the loop function in the lth partial wave should contain the q^{2l} factor in the integrand [43,44]. This is to ensure the correct low-energy behavior of the amplitude $\mathcal{F}_l(q) \sim q^{2l}$. The difference of the loop function may be regarded as the difference of the definition of Z and X_j (basis to form the complete set). We note, however, that the present definition leads to $Z = 0$ in the $B \to 0$ limit even in p wave, while the definition in Refs. [43,44] does not constrain the value of Z at threshold for nonzero l. The general threshold behavior is consistent with the latter [61], so the present definition would lead to special behavior near the threshold. In practice, the $\rho(770)$ and $K^*(892)$ mesons locate away from the threshold energies of meson–meson channels, so the special nature of the definition would not cause a problem in the present analysis.

Table 4. Compositeness X_j and elementariness Z of $N(1535)$ and $\Lambda(1670)$ in the isospin basis.

$N(1535)$		$\Lambda(1670)$	
$\sqrt{s_{\mathrm{R}}}$ [MeV]	$1529 - 37i$	$\sqrt{s_{\mathrm{R}}}$ [MeV]	$1678 - 21i$
$X_{\pi N}$	$-0.02 - 0.01i$	$X_{\bar{K}N}$	$0.03 + 0.00i$
$X_{\eta N}$	$-0.03 + 0.23i$	$X_{\pi\Sigma}$	$0.00 + 0.00i$
$X_{K\Lambda}$	$0.09 - 0.04i$	$X_{\eta\Lambda}$	$-0.09 + 0.16i$
$X_{K\Sigma}$	$0.26 - 0.09i$	$X_{K\Xi}$	$0.53 - 0.10i$
Z	$0.70 - 0.09i$	Z	$0.53 - 0.06i$

the lowest-order Weinberg–Tomozawa interaction. Although a systematic analysis is not performed for these resonances, the model with appropriate subtraction constants [90,91] describes $N(1535)$ and $\Lambda(1670)$ reasonably well.

Using $V(s)$ and $G(s)$ with the subtraction parameters given in Ref. [90] for $N(1535)$ and Ref. [91] for $\Lambda(1670)$, we calculate the compositeness and elementariness of $N(1535)$ and $\Lambda(1670)$. The results are listed in Table 4. First of all, interestingly, for both resonances the imaginary parts of the values of the compositeness X_j and elementariness Z are relatively small. This may allow us to interpret X_j and Z as the components of the resonance state. For $N(1535)$, Z is a dominant piece with a relatively small imaginary part. This suggests that $N(1535)$ in the present model has a large component originating from contributions other than the pseudoscalar meson–baryon dynamics considered, in accordance with Ref. [20]. In contrast, for $\Lambda(1670)$ the $K\Xi$ compositeness $X_{K\Xi}$ and the elementariness Z share unity half-and-half. This implies that in the present model the $K\Xi$ composite state plays a substantial role for the $\Lambda(1670)$ pole together with a bare state coming from components other than meson–baryon systems. This conclusion for $\Lambda(1670)$ is consistent with the discussion with the natural renormalization scheme in Ref. [92].

Here we emphasize that both $N(1535)$ and $\Lambda(1670)$ discussed in this subsection are described by scattering amplitudes which do not fully reproduce the experimental data at relevant energies [93,94]. For a more realistic discussion, it is desirable to improve the theoretical models so as to reproduce the experimental data well, for instance by taking into account the interplay between $N(1535)$ and $N(1650)$ [95], by including the vector meson–baryon channels [96], and by implementing higher-order terms.

4. Conclusion

In this study we have developed a framework to investigate the internal structure of bound and resonance states with their compositeness and elementariness by using their wave functions. For this purpose we have explicitly taken into account both one-body bare states and two-body scattering states as the basis to interpret the structure of bound and resonance states. Compositeness and elementariness are respectively defined as the contributions from the two-body scattering states and the one-body bare states to the normalization of the total wave function. After reviewing the formulation for the bound state, we have discussed the extension to the resonance state.

Because the wave function is analytically obtained for a separable interaction, we have explicitly written down the wave function for a bound state in a general separable interaction and obtained the expressions for the compositeness and elementariness. We have demonstrated that the compositeness is determined by the residue of the scattering amplitude and the energy dependence of the loop function at the pole position. Therefore, once one has the loop function, which is the Green function of the free two-body Hamiltonian, one can obtain the compositeness only from the bound state properties. On the other hand, we have found that the elementariness is obtained with the energy dependence of the effective two-body interaction. It is an interesting finding that the energy dependence of the two-body effective interaction arises from implicit channels which do not appear as explicit degrees of freedom but are effectively taken into account for the two-body interaction in the practical model space. These implicit channels contain the two-body scattering states as well as the one-body bare states. We have also shown the sum rule of the compositeness and elementariness. We have proved that, with multiple bare states, the formulae of the compositeness and elementariness can be applied to interactions with an arbitrary energy dependence. Of particular value is the derivation of Weinberg's relation for the scattering length and effective range in the weak binding limit. In the present formulation, thanks to the separable interaction, the scattering amplitude is analytically obtained. With this fact we have explicitly performed the expansion of the amplitude around the threshold to derive Weinberg's relation. In this derivation, the higher-order corrections come from the explicit expression of the form factor as well as higher-order derivatives in the expansion. The limitation of the formula due to the existence of the CDD pole is clearly linked to the breakdown of the effective range expansion.

Our discussion on the wave function has been extended to resonance states with Gamow vectors. The use of the Gamow vector enables us to have finite normalization of the resonance wave function. For a resonance state, by definition both the compositeness and elementariness become complex, which are difficult to interpret. Nevertheless, utilizing the fact that the compositeness and elementariness are defined by the wave functions, we have proposed the interpretation of the structure of a certain class of resonance states, on the basis of the similarity of the wave function of the bound state. Namely, if the compositeness in a channel (elementariness) is close to unity with small imaginary part and all the other components have small absolute values, this resonance state can be considered to be a composite state in the channel (an elementary state). Finally, we have given the expressions for the compositeness and elementariness with a general separable interaction in a relativistic covariant form by considering a relativistic scattering with a three-dimensional reduction.

As applications, the expression for the compositeness in a relativistic form has been used to investigate the internal structure of hadronic resonances, on the assumption that the energy dependence of the interaction originates from the implicit channels. By employing chiral coupled-channel scattering models with interactions up to the next-to-leading order, we have observed that the higher pole of $\Lambda(1405)$ and $f_0(980)$ are dominated by the $\bar{K}N$ and $K\bar{K}$ composite states, respectively, while the vector mesons $\rho(770)$ and $K^*(892)$ are elementary.

Finally, we emphasize that the fact that constituent hadrons are observable as asymptotic states in QCD is essential to constructing the two-body wave functions and to determining the compositeness for hadronic resonances.

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Appendix A. Proof of Eq. (62)

In this appendix we prove the relation in Eq. (62). In order to specify the problem, we consider a nonrelativistic stable bound system described with N two-body channels, in which the jth channel compositeness X_j and the elementariness Z can be expressed as

$$X_j = -g_j^2 \left[\frac{dG_j}{dE} \right]_{E=M_{\rm B}} \quad (j = 1, \ldots, N), \tag{A1}$$

$$Z = - \sum_{j,k=1}^{N} g_k g_j \left[G_j \frac{dv_{jk}^{\rm eff}}{dE} G_k \right]_{E=M_{\rm B}}, \tag{A2}$$

with the coupling constant g_j, the loop function G_j, and the two-body effective interaction $v_{jk}^{\rm eff}$. Then we make an implementation of a scattering channel N into the effective interaction, in the same manner as in [59]:

$$w_{jk}(E) = v_{jk}^{\rm eff} + v_{jN}^{\rm eff} \frac{G_N(E)}{1 - v_{NN}^{\rm eff} G_N(E)} v_{Nk}^{\rm eff} \quad (j,\ k = 1, \ldots, N-1). \tag{A3}$$

When we adopt the effective interaction w_{jk} for the $N-1$ two-body channels, the elementariness Z^w may be able to be calculated by the derivative of the effective interaction w_{jk} as

$$Z^w = - \sum_{j,k=1}^{N-1} g_k g_j \left[G_j \frac{dw_{jk}}{dE} G_k \right]_{E=M_{\rm B}}. \tag{A4}$$

Now we would like to prove that Z^w can be expressed as

$$Z^w = Z + X_N. \tag{A5}$$

For this purpose we first note that the coupling constant g_j satisfies the following bound state condition:

$$\sum_k \left[\delta_{jk} - v_{jk}^{\rm eff} G_k \right]_{E=M_{\rm B}} g_k = 0. \tag{A6}$$

In the following equations we omit the argument of the functions $v_{jk}^{\rm eff}$, G_j, and so on, since we always take $E = M_{\rm B}$ in this appendix. From the condition (A6) we can express g_N in terms of other coupling constants g_j $(j \neq N)$ as

$$g_N = \frac{1}{1 - v_{NN}^{\rm eff} G_N} \sum_{j=1}^{N-1} g_j G_j v_{jN}^{\rm eff} = \frac{1}{1 - v_{NN}^{\rm eff} G_N} \sum_{k=1}^{N-1} g_k G_k v_{Nk}^{\rm eff}. \tag{A7}$$

We prove the relation (A5) by first calculating the derivative of the effective interaction w. From Eq. (A3), its derivative can be evaluated as

$$\begin{aligned} \frac{dw_{jk}}{dE} = {} & \frac{dv_{jk}^{\rm eff}}{dE} + \frac{dv_{jN}^{\rm eff}}{dE} \frac{G_N}{1 - v_{NN}^{\rm eff} G_N} v_{Nk}^{\rm eff} + v_{jN}^{\rm eff} \frac{dG_N}{dE} \frac{1}{1 - v_{NN}^{\rm eff} G_N} v_{Nk}^{\rm eff} \\ & + v_{jN}^{\rm eff} \frac{G_N}{\left(1 - v_{NN}^{\rm eff} G_N\right)^2} \frac{d\left(v_{NN}^{\rm eff} G_N\right)}{dE} v_{Nk}^{\rm eff} + v_{jN}^{\rm eff} \frac{G_N}{1 - v_{NN}^{\rm eff} G_N} \frac{dv_{Nk}^{\rm eff}}{dE}. \end{aligned} \tag{A8}$$

Therefore, the elementariness Z^w becomes

$$Z^w = -\sum_{j,k=1}^{N-1} g_k g_j G_j G_k \left[\frac{dv_{jk}^{\text{eff}}}{dE} + \frac{dv_{jN}^{\text{eff}}}{dE} \frac{G_N}{1 - v_{NN}^{\text{eff}} G_N} v_{Nk}^{\text{eff}} + v_{jN}^{\text{eff}} \frac{dG_N}{dE} \frac{1}{1 - v_{NN}^{\text{eff}} G_N} v_{Nk}^{\text{eff}} \right.$$
$$\left. + v_{jN}^{\text{eff}} \frac{G_N}{\left(1 - v_{NN}^{\text{eff}} G_N\right)^2} \frac{d\left(v_{NN}^{\text{eff}} G_N\right)}{dE} v_{Nk}^{\text{eff}} + v_{jN}^{\text{eff}} \frac{G_N}{1 - v_{NN}^{\text{eff}} G_N} \frac{dv_{Nk}^{\text{eff}}}{dE} \right]. \tag{A9}$$

Then using Eq. (A7) we can rewrite Z^w as

$$Z^w = -\sum_{j,k=1}^{N-1} g_k g_j G_j G_k \frac{dv_{jk}^{\text{eff}}}{dE} - \sum_{j=1}^{N-1} g_j G_j \frac{dv_{jN}^{\text{eff}}}{dE} G_N g_N - \sum_{j=1}^{N-1} g_j G_j v_{jN}^{\text{eff}} \frac{dG_N}{dE} g_N$$
$$- g_N G_N \frac{d\left(v_{NN}^{\text{eff}} G_N\right)}{dE} g_N - \sum_{k=1}^{N-1} g_k G_k G_N g_N \frac{dv_{Nk}^{\text{eff}}}{dE}. \tag{A10}$$

The third term of the right-hand side can be further translated by multiplying $1 = (1 - v_{NN}^{\text{eff}} G_N)/(1 - v_{NN}^{\text{eff}} G_N)$ as

$$-\sum_{j=1}^{N-1} g_j G_j v_{jN}^{\text{eff}} \frac{dG_N}{dE} g_N = -g_N \left(1 - v_{NN}^{\text{eff}} G_N\right) \frac{dG_N}{dE} g_N, \tag{A11}$$

which is combined with the fourth term to give

$$-g_N \left(1 - v_{NN}^{\text{eff}} G_N\right) \frac{dG_N}{dE} g_N - g_N G_N \frac{d\left(v_{NN}^{\text{eff}} G_N\right)}{dE} g_N = -g_N^2 \frac{dG_N}{dE} - g_N^2 G_N^2 \frac{dv_{NN}^{\text{eff}}}{dE}. \tag{A12}$$

As a consequence, the elementariness Z^w becomes

$$Z^w = -\sum_{j,k=1}^{N-1} g_k g_j G_j \frac{dv_{jk}^{\text{eff}}}{dE} G_k - \sum_{j=1}^{N-1} g_j G_j \frac{dv_{jN}^{\text{eff}}}{dE} G_N g_N - g_N^2 \frac{dG_N}{dE} - g_N^2 G_N^2 \frac{dv_{NN}^{\text{eff}}}{dE}$$
$$- \sum_{k=1}^{N-1} g_k g_N G_N \frac{dv_{Nk}^{\text{eff}}}{dE} G_k$$
$$= -\sum_{j,k=1}^{N} g_k g_j G_j \frac{dv_{jk}^{\text{eff}}}{dE} G_k - g_N^2 \frac{dG_N}{dE}, \tag{A13}$$

which completes the proof of Eq. (A5). By repeating the above procedure, one can make an implementation of two or more two-body channels. Moreover, in a similar way, one can prove that the contribution of the bare states can be expressed by the derivative of the Green function like Eq. (A1) when the bare states are not counted into the implicit channels.

Appendix B. Conventions of relativistic two-body state and two-body equation

In this appendix we summarize our conventions of the two-body state in the relativistic kinematics and confirm that the wave equation (103) indeed describes a two-body system whose motion is governed by the Klein–Gordon equation. In the following we concentrate on single-channel kinematics of the two-body system, but generalization to multi-channel kinematics is straightforward.

B.1. Normalization of states

First, we consider an on-shell one-body state of a scalar field of mass m with definite three-dimensional momentum $\boldsymbol{p}$, $|\boldsymbol{p}\rangle$, whose normalization is defined as follows:

$$\langle \boldsymbol{p}' | \boldsymbol{p} \rangle = 2\sqrt{\boldsymbol{p}^2 + m^2}(2\pi)^3 \delta^3(\boldsymbol{p}' - \boldsymbol{p}). \tag{B1}$$

Since we do not explicitly treat spin components of scattering baryons in this paper, we also use the above normalization for baryons.

Next, we construct a two-body state, in which both particles are on the mass shell and the relative momentum is denoted as $\boldsymbol{q}$ in the center-of-mass frame used in Sect. 2.4. In this kinematical condition, the momenta of the two particles are given by $p_1^\mu = (\omega(\boldsymbol{q}),\ \boldsymbol{q})$ and $p_2^\mu = (\Omega(\boldsymbol{q}),\ -\boldsymbol{q})$, where $\omega(\boldsymbol{q}) \equiv \sqrt{\boldsymbol{q}^2 + m^2}$ and $\Omega(\boldsymbol{q}) \equiv \sqrt{\boldsymbol{q}^2 + M^2}$ are the on-shell energies of the two particles with m (M) being the mass of the first (second) particle, and the total momentum becomes $P^\mu \equiv p_1^\mu + p_2^\mu = (\sqrt{s_q},\ \boldsymbol{0})$ with $\sqrt{s_q} \equiv \omega(\boldsymbol{q}) + \Omega(\boldsymbol{q})$. Then the two-body state with relative momentum $\boldsymbol{q}$, $|\boldsymbol{q}^{\mathrm{co}}\rangle$, can be defined by using the product of two one-body states, $|\boldsymbol{p}_1\rangle \otimes |\boldsymbol{p}_2\rangle$. In this study we adopt the following normalization of $|\boldsymbol{q}^{\mathrm{co}}\rangle$:

$$|\boldsymbol{q}^{\mathrm{co}}\rangle \equiv \mathcal{N}_{sq} |\boldsymbol{q}_1\rangle \otimes |-\boldsymbol{q}_2\rangle, \quad \langle \boldsymbol{q}^{\mathrm{co}}| \equiv \mathcal{N}_{sq}^* \langle \boldsymbol{q}_1| \otimes \langle -\boldsymbol{q}_2|, \quad |\mathcal{N}_{sq}|^2 \equiv \frac{1}{2\mathcal{V}_3 \sqrt{s_q}}. \tag{B2}$$

In the normalization factor $\mathcal{N}_{sq}$, $\mathcal{V}_3$ is the total spatial volume and is related to the delta function for the momentum as $\mathcal{V}_3 = (2\pi)^3 \delta^3(\boldsymbol{0})$. The advantage to adopting this normalization factor is that the expression of the relativistic two-body wave equation becomes a natural extension of the nonrelativistic Schrödinger equation, as we will see in the next subsection.

With the definition of the two-body state $|\boldsymbol{q}^{\mathrm{co}}\rangle$ in Eq. (B2) and the normalization of the one-body state in Eq. (B1), we can calculate the normalization for $|\boldsymbol{q}^{\mathrm{co}}\rangle$ in a straightforward way as

$$\left\langle \boldsymbol{q}'^{\,\mathrm{co}} | \boldsymbol{q}^{\mathrm{co}} \right\rangle = \frac{2\omega(\boldsymbol{q})\Omega(\boldsymbol{q})}{\sqrt{s_q}} (2\pi)^3 \delta^3(\boldsymbol{q}' - \boldsymbol{q}). \tag{B3}$$

This normalization leads to the projection operator to the two-body state:

$$\hat{\mathcal{P}}_{\mathrm{two}} = \int \frac{d^3 q}{(2\pi)^3} \frac{\sqrt{s_q}}{2\omega(\boldsymbol{q})\Omega(\boldsymbol{q})} |\boldsymbol{q}^{\mathrm{co}}\rangle \langle \boldsymbol{q}^{\mathrm{co}}|, \tag{B4}$$

which corresponds to a part of the completeness condition.

B.2. Relativistic wave equation and scattering equation

Now we would like to confirm that the wave equation (103) indeed describes a two-body system whose motion is governed by the Klein–Gordon equation, by deriving the scattering equation from the operators in the wave equation. Here, in the same manner as in Sect. 2, we introduce a one-body bare state and a two-body scattering state, and assume that the bare state contribution is effectively contained in the two-body interaction V^{eff}. In this sense, the relation in Eq. (B4) coincides with the completeness condition; $\hat{\mathcal{P}}_{\mathrm{two}} = 1$.

In general, the wave equation can be composed of the free two-body Green's operator $\hat{\mathcal{G}}(s)$ and the two-body interaction operator $\hat{\mathcal{V}}^{\mathrm{eff}}(s)$. The two-body Green's operator $\hat{\mathcal{G}}(s)$ is defined as

$\hat{\mathcal{G}}(s) \equiv 1/(s - \hat{\mathcal{K}})$ with the kinetic energy operator $\hat{\mathcal{K}}$ so that[11]

$$\hat{\mathcal{G}}(s)|\boldsymbol{q}^{\rm co}\rangle = \frac{1}{s - s_q}|\boldsymbol{q}^{\rm co}\rangle, \quad \langle \boldsymbol{q}^{\rm co}|\hat{\mathcal{G}}(s) = \frac{1}{s - s_q}\langle \boldsymbol{q}^{\rm co}|. \tag{B5}$$

On the other hand, the two-body interaction operator $\hat{\mathcal{V}}^{\rm eff}(s)$ has a general separable interaction as in Eq. (102), thus we have

$$\left\langle \boldsymbol{q}'^{\,\rm co}|\hat{\mathcal{V}}^{\rm eff}(s)|\boldsymbol{q}^{\rm co}\right\rangle = V^{\rm eff}(s) f\left(q^2\right) f\left(q'^2\right), \tag{B6}$$

where $V^{\rm eff}(s)$ corresponds to the interaction in Eq. (109), which contains the implicit contribution from the bare state.[12] Here we also assume that the form factor $f(q^2)$ depends only on the three-momentum so as to make a three-dimensional reduction of the scattering equation. Then, by using $\hat{\mathcal{G}}$ and $\hat{\mathcal{V}}^{\rm eff}$, we can express the wave equation for a relativistic resonance state $|\Psi)$, whose mass and width are described by an eigenvalue $s_{\rm R}$, as

$$\hat{\mathcal{G}}^{-1}(s_{\rm R})|\Psi) = \hat{\mathcal{V}}^{\rm eff}\left(s_{\rm R}\right)|\Psi), \quad (\Psi|\hat{\mathcal{G}}^{-1}(s_{\rm R}) = (\Psi|\hat{\mathcal{V}}^{\rm eff}(s_{\rm R}), \tag{B7}$$

which is equivalent to the wave equation in Eq. (103) with the implicit bare-state degree of freedom.

Let us now derive the scattering equation with the above normalizations. To this end, we define the T-operator $\hat{\mathcal{T}}$ by the interaction $\hat{\mathcal{V}}^{\rm eff}$ and two-body Green's operator $\hat{\mathcal{G}}$ as:

$$\hat{\mathcal{T}} = \hat{\mathcal{V}}^{\rm eff} + \hat{\mathcal{V}}^{\rm eff}\hat{\mathcal{G}}\hat{\mathcal{T}}. \tag{B8}$$

This corresponds to the two-body scattering equation in an operator form. For the separable interaction (B6), the matrix element of the T-operator is given in the form $\langle \boldsymbol{q}'^{\,\rm co}|\hat{\mathcal{T}}|\boldsymbol{q}^{\rm co}\rangle = T(s) f\left(q^2\right) f\left(q'^2\right)$. The scattering equation is then obtained from Eq. (B8) as

$$T(s) = V^{\rm eff}(s) + V^{\rm eff}(s)G(s)T(s). \tag{B9}$$

where $G(s)$ corresponds to the loop function and is defined as

$$G(s) \equiv \int \frac{d^3q}{(2\pi)^3}\frac{\sqrt{s_q}}{2\omega(\boldsymbol{q})\Omega(\boldsymbol{q})}\frac{\left[f\left(q^2\right)\right]^2}{s - s_q} = i\int \frac{d^4q}{(2\pi)^4}\frac{\left[f\left(q^2\right)\right]^2}{\left[\left(P/2 + q\right)^2 - m^2\right]\left[\left(P/2 - q\right)^2 - M^2\right]}, \tag{B10}$$

with $P^\mu \equiv (\sqrt{s}, \boldsymbol{0})$. The second term of the right-hand side in Eq. (B9) can be obtained by inserting the operator $\hat{\mathcal{P}}_{\rm two} = 1$ (B4) between $\hat{\mathcal{V}}^{\rm eff}$ and $\hat{\mathcal{G}}$ as

$$\begin{aligned}
\langle \boldsymbol{q}'^{\,\rm co}|\hat{\mathcal{V}}^{\rm eff}\hat{\mathcal{G}}\hat{\mathcal{T}}|\boldsymbol{q}^{\rm co}\rangle &= \int \frac{d^3q''}{(2\pi)^3}\frac{\sqrt{s_{q''}}}{2\omega(\boldsymbol{q}'')\Omega\left(\boldsymbol{q}''\right)}\frac{\langle \boldsymbol{q}'^{\,\rm co}|\hat{\mathcal{V}}^{\rm eff}|\boldsymbol{q}''^{\,\rm co}\rangle\langle \boldsymbol{q}''^{\,\rm co}|\hat{\mathcal{T}}|\boldsymbol{q}^{\rm co}\rangle}{s - s_{q''}} \\
&= \int \frac{d^3q''}{(2\pi)^3}\frac{\sqrt{s_{q''}}}{2\omega(\boldsymbol{q}'')\Omega\left(\boldsymbol{q}''\right)}\frac{V^{\rm eff}(s) f\left(q''^2\right) f\left(q'^2\right) \times T(s) f\left(q^2\right) f\left(q''^2\right)}{s - s_{q''}} \\
&= V^{\rm eff}(s)G(s)T(s) f\left(q^2\right) f\left(q'^2\right).
\end{aligned} \tag{B11}$$

As seen in the last expression of the loop function $G(s)$ in Eq. (B10), Eq. (B9) is nothing but the scattering equation with the Klein–Gordon propagators, and hence the wave equation (103) indeed describes a two-body system whose motion is governed by the Klein–Gordon equation.

[11] In the nonrelativistic framework the two-body Green's operator is $\hat{\mathcal{G}}(E) = 1/(E - \hat{H}_0)$, with $\hat{H}_0$ being the free Hamiltonian, and Eq. (B7) is reduced to the Schrödinger equation.

[12] By using the notations in Sect. 2.4, the two-body interaction operator $\hat{\mathcal{V}}^{\rm eff}(s)$ can be defined as $\hat{\mathcal{V}}^{\rm eff}(s) \equiv \hat{\mathcal{V}} + \hat{\mathcal{V}}|\Psi_0\rangle\langle\Psi_0|\hat{\mathcal{V}}/(s - M_0^2)$ in a similar manner to the operator $\hat{V}^{\rm eff}(E)$ in Sect. 2.1.

At last we emphasize that the normalization (B3) is consistent with the two-body Green's operator $\hat{\mathcal{G}}(s) = 1/(s - \hat{\mathcal{K}})$, which is a natural extension of the nonrelativistic Green's operator $\hat{\mathcal{G}}(E) = 1/(E - \hat{H}_0)$. Otherwise, we should redefine $\hat{\mathcal{G}}(s)$ so as to absorb a kinematical factor coming from $\sqrt{s_q}/[2\omega(\boldsymbol{q})\Omega(\boldsymbol{q})]$ in the loop integral (B10). This allows us to determine the coefficient of the relativistic two-body wave function in Sect. 2.4 in a straightforward way as in the nonrelativistic case.

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