Stochastic Modeling of Hydrological Events for Better Water Management

2016

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Abstract

Alteration of the hydrological cycles caused by climate change is increasingly affecting water resources. Offering sustainable solutions to the water management problems require hydrological process studies. The goal of this thesis is to provide a mathematical and numerical modeling framework for assessing hydrological events, deducing optimal control strategies, and studying their real world applications for better water management. This thesis focuses in particular on stochastic modeling and optimal control of reservoirs as well as rainfed agriculture.

In the first part of the thesis a hydrologic model is proposed based on the Langevin equation, which is a stochastic differential equation governing zero-reverting Ornstein-Uhlenbeck (OU) processes. The optimal control problem is formulated in the context of dynamic programming to deduce the optimal strategies for better water management. Then, computational methods are presented to numerically approximate the relevant equations. Considering the level of drought severity as the zero-reverting OU process, optimality of rainfed agriculture is investigated in the context of stochastic control theory. Occurrence of drought terminating growth of crops is modelled with the concept of first exit time. Next, a novel type of hydraulic structure is proposed for rainwater harvesting in arid area. Design, construction, and operation of the actual rainwater harvesting system are presented with model parameters identified from observed data.

Finally, a mathematical model is proposed to deduce optimal operational rules for reservoirs harvesting stochastic surface water flows to meet the demand from command areas. The mathematical model defines stochastic control problems in terms of dynamic programming, involving (Hamilton-Jacobi-Bellman) HJB equation systems. Then, the real life application is discussed with numerical solutions to a HJB equation. The zero-reverting OU process is applied to representing the stochastic surface water flows and the demand from command areas. The numerical solutions of the HJB equation yield optimal irrigation strategies represented in terms of rule curves prescribing water withdrawal limits.
Acknowledgement

I would like to thank my supervisor Professor Masayuki Fujihara for his invaluable support throughout my Ph.D. studies at Kyoto University. I tremendously benefited from his broad-range knowledge and experience, and I am thankful to him for that. It has been a real pleasure working with him.

I would also like to thank Professor Akira Murakami, and Associate Professor Koichi Unami for serving in my thesis committee. I am grateful for their valuable advice and feedback on my work and all the fruitful discussions we had. I am thankful to Dr. Unami for his great support throughout the past years at Kyoto University. Dr. Unami’s high academic standards and great vision will always be a source of inspiration to me. He has been a great mentor and I wish for many more opportunities to learn from him.

I express my sincere thanks to Assistant Professor Junichiro Takeuchi for his valuable comments and technical assistance. I would also like to thank the staff in the Department of Environmental Science and Technology who have always been there to help and guide how to do different administrative tasks at Kyoto University. My life at Kyoto University has been a great experience thanks to friends and lab-mates. I would like to thank former and current members of the Water Resources Engineering group.

I wish to express my deep gratitude to my family who have encouraged me with their love and care for the long time. I am indebted to my parents, for their love and understanding through these years. I am grateful to them for their sacrifice, unconditional support and dedication toward my education. Finally, I am deeply grateful to my husband, for his love, patience and faith in me. I could have not completed this work without his continuous support.

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June 2016
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CHAPTER 1

1 Introduction

1.1 Research Background

Water availability in many countries declined with growing population over the past decades. Food production, protecting the natural ecosystem and human life require water resources management. Alteration of the hydrological cycles caused by climate change is also increasingly affecting water resources. Offering sustainable solutions to the water management problems require hydrologic process studies. Many approaches of hydrologic modeling have been studied with the aim of climate change impact assessment and adaptation planning (Leavesley et al. 1992; Xu 1999; Georgakakos et al. 2012; Flint et al. 2013). Many researches including Kijne et al. (2003), Biswas (2004), Viala (2008), Gordon et al. (2010) studied water problems and management in different sectors such as energy, agriculture and ecosystem. Agriculture is the largest single user of fresh water, accounting for about 75% of current human water use, and at present almost 7% of the world’s population live in areas where water is scarce, therefore there is a great need for improving water use efficiency in agriculture, particularly in arid and semi-arid areas where water is scarce (Wallace 2000).

1.2 Stochastic Modeling of Hydrological Events

Hydrology is the science of describing the occurrence, distribution, behavior and movement of water above, over, and through the earth and their relationship with the environment (Savenije 2009). Stochastic nature of rainfall leads to the uncertainty and variability of the hydrological inputs. Stochastic differential equations (SDEs), can describe the occurrence of hydrological events such as drought and flood. Bodo et al. (1987) studied SDEs, applicable to different problems in water resources engineering, as well as the fundamental concepts of stochastic processes including Markov processes and Brownian motions. A stochastic process is a Markov process if its future evolution depends only on the latest state (Schuss 2009). SDEs have been studied with their application in environmental and water resources

1.3 Better Water Management

In general, water resources management process combines physical components of the system (i.e. hydrology) with analytical tools (i.e. optimization techniques) (Nikolic et al. 2013). The framework for water resources problem analysis and decision making uses the set of tools to determine preferred designs, plans and operation strategies for complex water resources systems (Simonovic 2012). Water resources management problems involve environmental, hydrologic and economic issues, which make the system complicated. Mathematical modeling ranging from linear differential equations to complex nonlinear differential equations can be used to deal with these complications (Belaineh et al. 1999).

Due to the inherent uncertainty in most real world systems, decision makers look for optimal decisions among all possible ones. Such optimization problems are stochastic control problems (Yong & Zhou 1999). Stochastic control deals with dynamical systems, which can be described by SDEs, subject to decision making (Åström 2012). In decision making, when the decision (control) is chosen based on the state at current time, it is called Markov control policy (Fleming & Soner 2006). Providing decision for the management of water resources systems, which are optimal, involves development and adaptation of optimization techniques (Yeh 1985). In reservoir studies, operation of reservoirs collecting surface water such as flash floods typically involves stochastic optimal control problems. In irrigation water management, the optimal control problems in the face of limited water supply encounter with the question of how much to irrigate (Yakowitz 1982; Ormsbee & Lansey 1994).

Computational fluid dynamics (CFD) is a powerful numerical tool that uses computer to provide numerical simulation of fluid flows. It enables the engineer to model and simulate the physical effects during the design process, and to analyze the behavior of the structure (Hirsch 2007). CFD is becoming widely used to simulate many processes in the civil and environmental engineering, agriculture and water resources management (Norton et al. 2007; Wilcox 1998; Lane et al. 1999; Bates et al. 2005, Bashiri-Atrabi et al. 2014).
surface flow along open channels, which is one of the most widely used approaches in environmental problems, can be described with shallow water equations (SWEs) (Tan 1992). SWEs can be solved numerically to model the shallow water flows in free surfaces (Szymkiewicz 2010; LeVeque 2002).

1.3.1 Optimality of Rainfed Agriculture

Cropping systems can be classified according to production ecology (upland, hydromorphic, and lowland areas), water control (rainfed, and irrigated), that lead to different water management practices based on the system. Rainfed agriculture is exclusively being practiced in many parts of sub-Saharan Africa (SSA) under semi-arid savanna climates, where most of the annual precipitation occurs during the few months of the rainy season, hence irregularity of rainfall occurrence leads to the vulnerability of subsistence rainfed agriculture. In West Africa, upland fields have traditionally been the focus for crop production, however due to the increasing demand for land, hydromorphic valley bottoms and lowland fields are coming under more intensive use particularly for rice production (Windmeijer & Andriesse 1993). Rainfed crop production uses infiltrated rainfall in the form of soil moisture in the root zone (the so-called green water resource) (Rockström et al. 2010). However due to the important role of rainfall in rainfed agriculture, modeling rainfall variability and effective water resources management strategies are required for the sustainable food production. In order to optimize rainfed crop production, different studies focused on the rainfall variability and management techniques that improve agricultural water productivity (Stewart 1988; Rockström 2003; Raes et al. 2007; Biazin et al. 2012; Sharifi et al. 2016). Prior to discussing strategies for future improvement of rainfed agriculture, attention should be paid to whether the current practice of rainfed agriculture is optimal or not. It can be said that rainfed agriculture without any substantial water management is the optimal, if the cost of water surpasses the benefits of farming under the risk of drought.
1.3.2 Rainwater Harvesting Systems for Irrigation in Harsh Environment

The dominant climate class by land area is arid, which is 30.2% globally (Peel et al. 2007). Based on the modified Köppen climate classification system, in arid areas evaporation exceeds precipitation (Hess 2013). Water scarcity is a major problem in arid countries, which are characterized by low erratic rainfall resulting in high risk of droughts, intra-seasonal dry spells, and frequent food insecurity. Rainwater harvesting (RWH), which is a technology where surface runoff is effectively collected during yielding rain periods, is a likely viable option to increase water productivity in water-scarce countries (Helmreich & Horn 2009). Boers & Asher (1982) reviewed different rainwater harvesting systems with the emphasis on its potential application for crop production. Their definition of rainwater harvesting show that water harvesting encompasses methods to induce, collect, and store runoff from various sources and for various purposes. Different researches have been conducted on the hydraulics and hydrological aspects of rainwater harvesting to improve the efficiency of harvested water in both rainfed and irrigated agriculture (Boers 1994; Biazin et al. 2012; AlAyyash et al. 2012; Pandey et al. 2003).

1.4 Research Objectives

The goal of this thesis is to provide a mathematical and numerical modeling framework for assessing hydrological events, deducing optimal control strategies, and studying their real world applications for better water management. This thesis focuses in particular on stochastic modeling and optimal control of reservoirs as well as rainfed agriculture. To this end, the aims of this thesis are:

1) Propose hydrological models based on the Langevin equation, which is a stochastic differential equation governing zero-reverting Ornstein-Uhlenbeck (OU) processes.
2) Formulate optimal control problems in the context of dynamic programming to deduce the optimal strategies for better water management.
3) Present computational methods to numerically approximate solutions to the relevant differential equations.
4) Design, construct, and operate an actual rainwater harvesting system using the methodologies above with model parameters identified from observed data.

5) Obtain optimal irrigation strategies as a solution of HJB equation and represent them in terms of rule curves prescribing water withdrawal limits.

1.5 Thesis Structure

This thesis consists of six chapters.

Chapter 1 explains the background and the objectives of this thesis.

Chapter 2 provides a fundamental concept to comprehend alternation of dry and wet spells in rainy seasons of West African savanna. It presents a stochastic process model consisting of a stochastic differential equation governing a zero-reverting Ornstein-Uhlenbeck (OU) process to model the phenomena. The model with the parameter values is applicable to a variety of problems.

Chapter 3 investigates the optimality of rainfed agriculture in the context of stochastic control theory considering the level of drought severity as the zero-reverting OU process. It introduces the HJB equation governing the optimal control to identify the set of cost functions optimizing rainfed agriculture in terms of an inverse problem. It attempts to comprehend the rationale of situation, in Sub-Saharan Africa, that subsistence rainfed agriculture is the predominant source of food where the production uncertainty is associated with the stochastic nature of rainfall. Data and information were collected in the coastal savanna agro-ecological zone of Ghana, to identify model parameters, formulate the stochastic control problem, solve the inverse problem, and then verify optimality of rainfed agriculture.

Chapter 4 presents a novel type of hydraulic structure for rainwater harvesting in an arid area of Jordan to develop an irrigation scheme. The scheme consists of an irrigated farm, a reservoir, and an intake structure to divert ephemeral flood flows into the reservoir. This chapter focuses on hydraulic design and actual construction processes of the structure. Details of structure dimensions are designed with numerical and hydraulic model experiments. Hydraulic model tests were conducted at one of experimental stations in Japan. Then a numerical scheme is used to numerically reproduce the whole flow field during a rainwater
harvesting event with the maximum design discharge. The numerical experiments are conducted to confirm the actual structure showing desired performance.

Chapter 5 proposes a mathematical model to deduce optimal operational rules for reservoirs harvesting stochastic surface water flows to meet the demand from command areas. The mathematical model defines stochastic control problems in terms of dynamic programming, involving HJB equation systems. Then, it discusses a real life application in the context of the Jordan Rift Valley, where the rainwater harvesting system is developed and presented in Chapter 4. The identification procedure for the values of model parameters is as described in Chapter 2 and then Chapter 3.

Finally, Chapter 6 concludes the findings and provides future directions for research.
CHAPTER 2

2 Stochastic Modeling of the Alternation of Dry and Wet spells

2.1 Introduction

Irregular occurrence of dry spells in cropping rainy seasons leads to the vulnerability of subsistence rainfed agriculture. Rainfed agriculture is exclusively being practiced in many parts of sub-Saharan Africa under semi-arid savanna climates, where most of the annual precipitation occurs during the few months of the rainy season, hence irregularity of rainfall occurrence leads to the vulnerability of subsistence rainfed agriculture. Modern irrigation technology is not well adapted to the traditional farming system because of the low cost-effectiveness. In such a situation, vulnerability of crop production is due to irregularity of rainfall in the rainy seasons, rather than the presence of the dry seasons whose lasting periods are known almost surely. There are traditional farmers’ practices responding to alternation of dry and wet spells in the rainy seasons. For example, rice is cultivated in valley bottoms where the soil moisture is rather stable in the second half of the rainy season, and the method of tillage is chosen according to the hysteresis of preceding precipitation.

Scientific insight into alternation of dry and wet spells in general can be found in the literature, researching its stochastic structures. Najem (1988) applied continuous Markovian models to daily point rainfall data including dry spells observed in Lebanon. Mishra et al. (2009) characterized drought events in India’s West Bengal using the alternative renewable process and run theory. Unami et al. (2010) modeled the stochastic nature of annual point rainfall patterns at several Ghanaian sites as the OU processes.

The model presented here uses the standardized one-dimensional Langevin equation, which is a stochastic differential equation with very simple structure, in order to represent a virtual process governing alternation of dry and wet spells in West African savanna. However, the soil moisture observed at an upland crop field is considered as the indicator of the humidity in the rainfed agriculture area during rainy seasons, while an onset of a rainy season is assumed to terminate a dry season.
2.2 Stochastic Process Revisited

A stochastic process \( X_t \) is a collection of random variables parameterized with the time \( t \).

There are different types of stochastic processes, some of which are revisited in this section.

The deterministic decay process, which is actually not stochastic, is governed by the ordinary differential equation

\[
dX_t = -aX_t dt
\]

where \( a \) is the positive decay coefficient. The solution to (2.1) with the initial value \( X_0 = x \) is uniquely given as

\[
X_t = x \exp(-at). \tag{2.2}
\]

The standard one-dimensional Brownian motion \( B_t \) is a primary stochastic process, being almost surely continuous, having independent increments, and having the property such that \( B_t - B_s \) obeys to the normal distribution with zero mean and \( t - s \) variance for \( 0 \leq s \leq t \). Expectation of \( B_t \), \( \mathbb{E}[B_t] \), which starts from the initial value is equal to \( x \).

The Black-Sholes equation governing the geometric Brownian motion \( X_t \)

\[
dX_t = -aX_t dt + \sigma X_t dB_t \tag{2.3}
\]

is originally developed in financial engineering to represent dynamics of asset prices. Where \( \sigma \) is the volatility, the geometric Brownian motion models population growth under environmental disturbances as well. For \( X_0 = x > 0 \), the solution to (2.3) is given by

\[
X_t = x \exp \left( - \left( a + \frac{1}{2} \sigma^2 \right) t + \sigma B_t \right) \tag{2.4}
\]

from which it can be understood that \( X_t > 0 \) as well as that \( a = -\frac{1}{2} \sigma^2 \) is a threshold determining asymptotic behavior of \( X_t \).

The one-dimensional Langevin equation governing a zero-reverting stochastic process \( X_t \)

\[
dX_t = -aX_t dt + \sigma dB_t \tag{2.5}
\]
is mostly studied in statistical mechanics. However, zero-reverting stochastic processes are potentially applicable to different water related problems under uncertain conditions, which are common in agricultural environment (Unami et al. 2013). The solution to (2.5) with \( X_0 = x \) is given by

\[
X_t = x \exp(-at) + \sigma \int_0^t \exp(-a(t-s))dB_s,
\]
with

\[
E[X_t] = x \exp(-at)
\]
and

\[
\text{Var}[X_t] = E[(X_t - E[X_t])^2] = \frac{\sigma^2}{2a}(1 - \exp(-2at)).
\]

In terms of the zero-reverting nature (2.7) and the bounded variance (2.8), \( X_t \) is considered suitable for representing virtual humidity of rainfed upland farmlands under semi-arid savanna climates, where evapotranspiration surplus implies reverting to a dry state of soil with limited water retention capacity.

### 2.3 A Stochastic Process Model

The one-dimensional Langevin equation (2.5) governing the zero-reverting stochastic process \( X_t \) can be normalized as

\[
dX_t = -X_t dt + \sqrt{2} dB_t,
\]
where the time \( t \) is scaled so that \( a = 1 \) and \( \sigma = \sqrt{2} \). The transition probability that \( X_t \) is in a subset \( G \) of \( \mathbb{R} \) at time \( t \) provided that \( X_s = \xi \) at the time \( s < t \) is denoted by \( P(s, \xi, t, G) \).

Firstly, for a positive real number \( \kappa \), we assume that it is in a wet spell if \( |X_t| < K \) and is in a dry spell otherwise during a rainy season. Our discussion focuses on whether this mathematical model well represents alternation of dry and wet spells observed at the study site. Now, the transition probability that it is in a wet spell \( (|X_t| < K) \) at time \( t \) provided that the onset of the rainy season \( X_s = K \) is at the time \( s < t \) is \( P(s, K, t, (-K, K)) \), which can be calculated from the observed data series. It is known that the transition probability density
function \( p = p(s, \xi, t, x) \) such that \( P(s, \xi, t, G) = \int_G p(s, \xi, t, x) \, dx \) is governed by the forward Kolmogorov equation

\[
\frac{\partial p}{\partial t} = \frac{\partial (xp)}{\partial x} + \frac{\partial^2 p}{\partial x^2}
\]  

(2.10)

in the infinite domain \( \mathbb{R} \), with the initial condition

\[
p(s, \xi, s, x) = \delta(x - \xi)
\]  

(2.11)

where \( \delta \) represents the Dirac’s delta (Karatzas and Shreve 1991). The solution of the initial value problem (2.10) with (2.11) converges to the Gaussian distribution as

\[
p(s, \xi, \infty, x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)
\]  

(2.12)

from which the value of \( K \) is identified so that \( \int_{-K}^{K} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \, dx \) is equal to the proportion of wet spells to the rainy seasons.

Next, we consider dry seasons as the complement case. We assume that \(|X_j| < K\) during a dry season and that \(|X_j| \geq K\) during rainfall events binding the dry season \((T_0, T_i)\). However, defining another constant parameter \( K_\tau (> K) \), a substantial rainstorm event terminating the dry season is assumed to take place only if \(|X_j| = K_\tau\) is achieved at \( t = T_2\) during a period of \(|X_j| \geq K\). The value of the model parameter \( K \) is identified from

\[
\int_{-K}^{K} p(\infty, x) \, dx = \int_{-K}^{K} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \, dx \approx \frac{T_1 - T_0}{T_2 - T_0}.
\]  

(2.13)

The parameter \( K_\tau \) is identified by virtue of the Dynkin’s formula

\[
E^K[\tau] = \varphi(K_\tau) - \varphi(K)
\]  

(2.14)

where \( \tau \) is the first exit time from the domain \((-K_\tau, K_\tau)\). \( E^K \) represents the expectation with respect to the probability law of the zero-reverting stochastic processes starting from \( K \), and the function \( \varphi(x) \) is defined as

\[
\varphi(x) = \int \exp\left(\frac{x^2}{2}\right) \int \exp\left(-\frac{x^2}{2}\right) \, dx
\]  

(2.15)

which is calculated by numerical integration. Assuming that \( E^K[\tau] \) is estimated as the expectation of \((T_1 + T_2)/2 - T_0\) in the non-dimensional time space, the value of \( K_\tau \) is searched to fulfill (2.15).
2.4 Parameter Identification Using Numerical Solution

The unit for time, which has been scaled in (2.9), is determined so that the numerical solution to the initial value problem of the Langevin equation (2.10) with the initial condition (2.11) is compatible with the observed \( P(s, K, t, (-K, K)) \). The numerical approach proposed here is based on the finite element method (FEM). The FEM with the standard Galerkin scheme is applied to discretization of (2.10). However, a special set of basis functions is employed to deal with the infinite domain. The notation \( H^1_0(\Omega) \) is used for representing the Sobolev space, which is the complete normed space of functions having certain regularity properties and vanishing on the boundary, on a generic domain \( \Omega \).

A finite number of basis functions are substituted for the weight in the weak form of the Langevin equation. The weak form is written as

\[
\int_{\mathbb{R}} w \frac{\partial p}{\partial t} \, dx + \int_{\mathbb{R}} \frac{\partial w}{\partial x} \left( xp + \frac{\partial p}{\partial x} \right) \, dx = 0
\]  

(2.16)

for any weight \( w \in H^1_0(\mathbb{R}) \), which is the space of functions having certain regularity properties and vanishing at the points of infinity. A discretization scheme in the \( x \)-direction is proposed here, dividing the domain \( \mathbb{R} \) into \( 2N + 2 \) subdomains so as to define \( 2N + 1 \)-dimensional approximating space for \( H^1_0(\mathbb{R}) \), where \( N \) is a natural number. The basis functions \( w_i \) are set as

\[
w_i = \begin{cases} 
\frac{x - x_{i-1}}{\Delta x} & (x_{i-1} \leq x \leq x_i) \\
\frac{x_{i+1} - x}{\Delta x} & (x_i \leq x \leq x_{i+1}) \\
0 & \text{(otherwise)}
\end{cases}
\]  

(2.17)

for \( i = -N + 1 \sim N - 1 \),

\[
w_i = \begin{cases} 
\exp \left( -\frac{x_N - x}{\Delta x} \right) & (x \leq x_N) \\
\frac{x_{i+1} - x}{\Delta x} & (x_{N} \leq x \leq x_{N+1}) \\
0 & \text{(otherwise)}
\end{cases}
\]  

(2.18)

for \( i = -N \), and
for $i = N$. The $2N+1$-dimensional vector whose $i$th entry is $p_i$ is denoted by $\mathbf{p}$, to approximate $p$ as

$$p \approx \langle \mathbf{w}, \mathbf{p} \rangle$$

(2.20)

taking where $\mathbf{w}$ is the $2N+1$-dimensional vector whose $i$th entry is $w_i$, and $\langle , \rangle$ denotes the inner product. Then, substituting any $w_i$ into $\mathbf{w}$ of (2.16) results in

$$\int_{\mathbb{R}} \left\{ w_i \mathbf{w}, \frac{\partial \mathbf{p}}{\partial t} \right\} dx + \int_{\mathbb{R}} \left\{ \frac{\partial w_i}{\partial x} \left( \mathbf{xw} + \frac{\partial \mathbf{w}}{\partial x} \right), \mathbf{p} \right\} dx = 0$$

(2.21)

for the purpose of temporal discretization, the partial derivative $\frac{\partial \mathbf{p}}{\partial t}$ is approximated as

$$\frac{\partial \mathbf{p}}{\partial t} \approx \frac{p^{i+1} - p^i}{\Delta t}$$

(2.22)

where $\Delta t$ is the increment in the $t$-direction, and the superscript $j$ denotes the $j$th temporal stage. Using the implicit scheme that $p = p^{j+1}$, (2.21) is finally transformed into a linear equation system

$$M p^{j+1} = G p^j$$

(2.23)

where $M$ and $G$ are $2N+1$-dimensional square matrices, which are known at the $j$th temporal stage.

### 2.5 Application

A study site was chosen at the coordinates 09 29 57 N 000 59 17 W in a fallow farmland owned by Gung community, Tolon/Kumbungu District, the Northern Region of Ghana in the Guinea savanna agro ecological zone with monomodal rainfall pattern (Agyare et al. 2002; Unami et al. 2010). Subsistence rainfed agriculture is the dominant human activity of the community, and dry season irrigation is the least adopted. A soil moisture sensor (ML2x theta probe) with a data logger was installed at a depth of 15 cm. A tipping bucket raingauge connected to a pulse logger was installed to record events of 0.2 mm rainfall.
2.5.1 Rainy Seasons

Occurrence of dry spells during the rainy season, rather than the total amount of rainfall, is the major concern for the community members in each year. The soil moisture at the study site was monitored from 12 February 2006 through 26 March 2011, including five rainy seasons as shown in Figure 2-1. Here, dry and wet spells are defined as the periods where the volumetric water content of soil is less than and greater than 20%, respectively. Figure 2-2 is a photo taken in the vicinity of the study site on July 26th, 2010. Because of the continuing wet spell in the rainy season, upland crops including cassava and yam were growing well.

![Figure 2-1: Volumetric water content of soil observed at the study site.](image-url)
Figure 2-2: Photo taken in the vicinity of the study site on July 26th, 2010.

From the time series data of the volumetric water content shown Figure 2-1, alternation hours of dry and wet spells after the onset of each rainy season are extracted as in Table 2-1. The proportion of wet spells to the whole rainy seasons is 0.820, and the value of the model parameter $K$ is identified as 1.342 so as to be consistent with (2.12). Then, the numerical scheme is applied to computing the transition probability density function $p(0, K, t, x)$, setting the onset of a rainy season as the initial time $s = 0$. The discretization is performed with $N = 200$, $\Delta x = K/50$, and $\Delta t = 10^{-5}$. The computed $p(0, K, t, x)$ for $0 \leq t \leq 2$ are plotted in Figure 2-3. The transition probability $P(0, K, t, (K, -K))$ that the current time $t$ is in a dry spell is calculated from the computed $p(0, K, t, x)$ is also estimated from the observed alternation hours. The best fitting between the computed and the observed transition probabilities is achieved when the unit for time is chosen as 354 hours.
Table 2-1: Observed alternation hours of dry and wet spells.

<table>
<thead>
<tr>
<th></th>
<th>2006</th>
<th>2007</th>
<th>2008</th>
<th>2009</th>
<th>2010</th>
</tr>
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</tbody>
</table>
2.5.2 Dry Seasons

Dry season irrigation is still challenging in the context of semi-arid savanna. Thanawong et al. (2014) investigated paddy rice cropping systems in Northeastern Thailand under different water management practices, and concluded that dry season irrigated systems were the least performing in terms of techno-economics and environmental impacts. A rainy season starts with a rainstorm, which is referred to as an onset, occurring at the end of each dry season. From the logger data, periods of four dry seasons and the consecutive onsets of rainy seasons are identified as shown in Table 2-2. For identification of model parameter $K$, the dry spell fraction is averaged for the four data series. The resulting value is 0.9989, from which $K = 3.262$ is deduced. The unit for time yielding the best fitting between the computed and the estimated transition probabilities is 9.06 h. The first exit time $E^K[\tau]$ is estimated at 135.4 days, which is equivalent to non-dimensional time 358.7, and $K_e = 3.773$ is identified.
<table>
<thead>
<tr>
<th>Year</th>
<th>$T_0$ (mm/dd hh:mm:ss)</th>
<th>$T_1$ (mm/dd hh:mm:ss)</th>
<th>$T_2$ (mm/dd hh:mm:ss)</th>
</tr>
</thead>
<tbody>
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<td>03/22 16:44:04</td>
<td>03/22 17:09:31</td>
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<tr>
<td>2007-2008</td>
<td>10/30 21:32:55</td>
<td>03/12 00:14:27</td>
<td>03/12 11:55:02</td>
</tr>
</tbody>
</table>

2.6 Conclusions

In this chapter, a stochastic model is proposed for the alternation of dry and wet spell using the Langevin equation. This is a novel approach toward scientifically understanding the subsistent rainfed agriculture. The numerical scheme to compute the transition probability density function is based on the rigorous finite element formulation. The model with the parameter values identified from the observed data series is applicable to a variety of problems. For example, the probability that a dry spell will not occur before time $t$ provided that it is currently in a wet spell is 

$$P = P(s, \xi, t, (-K, K)) = \int_{-K}^{K} p(s, \xi, t, x) \, dx,$$

which solves the backward Kolmogorov equation

$$\frac{\partial P}{\partial s} = x \frac{\partial P}{\partial \xi} - \frac{\partial^2 P}{\partial \xi^2}$$

in the domain $(-K, K)$ with appropriate final and boundary conditions. This probability will indicate the vulnerability of subsistence rainfed agriculture.
3 Verifying Optimality of Rainfed Agriculture under Drought Risk

3.1 Introduction

It may be paradoxical but subsistence rainfed agriculture is the predominant source of food in SSA where the production uncertainty is associated with the stochastic nature of rainfall. SSA is a remarkable region where subsistence rainfed agriculture is the mainstay of economies, having the least percentage of irrigated land (Biazin et al. 2012). Rainfed agriculture, with traditional rainwater harvesting techniques to reduce soil moisture stress during prolonged dry spells, is the most common in production of staple food such as maize, millet, yam, sorghum, cassava, and rice. Supplemental irrigation has a certain potentiality in some South Asian watersheds (Reshmidevi et al. 2010), but it is less promising in the context of SSA. The green water, which is the soil moisture in the root zone readily available for the crop water consumption, mostly stems from infiltrated local rainfall. Thus, variability of rainfall in the growing seasons, rather than the total annual precipitation, can severely affect productivity (Rockström et al. 2010). To make matters worse, data observed over a 30 years period (1965-1994) indicated a declining trend in annual rainfall and increasing trend in annual mean air temperature at four proxy stations in southern Ghana, with most of interviewed farmers claiming substantial changes in the climate, especially in volume and reliability of the rainfall, and intense sunshine (Ofori-Sarpong & Asante 2004). For food security, there is an urgent need in SSA to establish strategies to minimize the risk of drought induced crop failure in rainfed agriculture, without relying on the introduction of irrigation agriculture which may overexploit the limited resources.

Prior to discussing strategies for future improvement of rainfed agriculture, attention should be paid to whether the current practice of rainfed agriculture is optimal or not. It can be said that rainfed agriculture without any substantial water management is the optimal, if the cost of water surpasses the benefits of farming under the risk of drought. However, due to uncertainty regarding occurrence of drought, verification of such optimality is not
straightforward. The conventional concept of water productivity is prone to be misleading when the farmer’s goal is rather to reduce risks for dry spells (Rockström & Barron 2007). A proper model representing the stochastic behavior of rainfall, knowledge of potentially feasible water management strategies, and that of economic values of farm products are all prerequisite for consideration.

In order to comprehend irregularity of rainfall events and occurrence of drought, stochastic sequences of wet and dry spells have been the focus of interest. Half a century ago Green (1964) developed a model for rainfall occurrence assuming that the sequence of dry and wet spells forms an alternating renewal process, with exponential density functions for the durations of dry and wet spells. Feyerherm & Bark (1965) estimated the probability of occurrence for a given consecutive sequence of wet and dry days, by describing the sequence as a first order Markov chain. Lall et al. (1996) developed a nonparametric model for resampling daily precipitation, estimating the probability distribution functions of alternating wet and dry spells using kernel density estimators. Ochola & Kerkides (2003) embedded first order Markov chains into a computer model to determine the critical wet and dry spells in the Kano Plains of Kenya. However, some recent researches have introduced more sophisticated concepts of wet and dry regimes using continuous-time stochastic processes. Rodriguez-Iturbe et al. (1999) considered rainfall as a marked Poisson process driving the stochastic soil moisture dynamics at a point. Such stochastic dynamics can be discussed in terms of transition and equilibrium probability densities (Rodriguez-Iturbe & Porporato 2004). Unami et al. (2010) modelled the irregularity of rainfall intensity as well as duration of dry spells with the mean-reverting OU process evolving over the domain of cumulative rainfall depth. Harandi et al. (2014) modelled the temporal fluctuation of water level in a river as a dichotomous Markov process and suggested applying it to long-term management of the river. Continuous-time stochastic processes may not directly represent alternation of wet and dry regimes. Suweis et al. (2010) developed a stochastic model, where the Langevin equation governed the zero-reverting OU process disturbing storage-discharge relations that induced the alternation of wet and dry regimes. Sirangelo et al. (2015) proposed a non-homogeneous Poisson process model, which was applied to the arrival of rainfall events.
observed at a test site in southern Italy, in order to assess the occurrence probabilities of dry spells for specified durations. Agnese et al. (2014) introduced a methodology based on a generalization of the commonly adopted Bernoulli process for modeling inter-arrival time-series, to achieve a statistical inference of the alternation of wet and dry periods in daily rainfall records. The above reviewed literature indicates diverse approaches to stochastic sequences of wet and dry spells operated under different contexts.

It is also important to know the strategies currently studied or practiced in SSA to cope with drought. Adiku & Stone (1995) explored the occurrence of drought in five sites of the major agro-ecological zones in Ghana using Southern Oscillation Index (SOI) for rainfall prediction and subsequently proposed a criterion for introduction of irrigation schemes. Mishra et al. (2013) developed a procedure to downscale daily global climate model (GCM) rainfall to precipitation variables including dry spell length at a Kenyan site, so that the precipitation variables are linked with a crop model for a more objective evaluation of the yield of maize. Efficiency of supplemental irrigation to mitigate dry spells and to secure yield response to fertilization was more evident in the Sahel where the growing season lasted only 120 days (Fox & Rockström 2003). Namara et al. (2010) classified different types of irrigation systems in Ghana into two categories: conventional and emerging. Conventional irrigation systems are largely supply driven, while emerging irrigation systems are either autonomous or with little support from the government and/or non-governmental organizations (NGOs). Due to different availability of water sources and costs of irrigation, the cropping patterns differ between the two categories.

Establishing a performance index based on the above-mentioned knowledge leads to formulation of an optimization problem. Several different optimization models have been applied for tackling water management problems due to the randomly occurring hydrological events. Alarcón et al. (2014) analyzed and compared the efficiency of five allocation rules of irrigation water, defining the most efficient rule as the one that minimizes the economic losses arising from a reduction in water availability. However, comparative case studies do not yield an optimal solution where optimality is guaranteed. An optimal control for a dynamic system can only be deduced from the principle of optimality, as demonstrated in the

Dynamic programming is an approach to deduce optimal strategies in terms of maximization of a performance index. Karamouz & Houck (1987) compared deterministic and stochastic dynamic programming models to use for generate reservoir operating. They constructed real-time reservoir operation simulation models for three hydrologically different sites. Chandramouli & Raman (2001) developed a dynamic programming based neural network model for optimal multi-reservoir operation. In the suggested model, multi-reservoir operating rules are derived using a feedforward neural network from the results of three state variables’ dynamic programming algorithm. Operation of reservoirs collecting surface water such as flash floods typically involves stochastic control problems. Unami et al (2013) proposed a stochastic differential equation to represent the readily available soil moisture in the farm land, which was the command area irrigated by the reservoir.

The governing equation of the value function, which represents the estimated cost or benefit resulting from an optimal control strategy, is the HJB equation involving mathematical issues such as regularity problems of solutions. Finite element schemes are applicable to resolution of time-dependent HJB equations with continuous coefficients (Jensen & Smears 2013). Yoshioka et al. (2015) proposed a stochastic process model for analytical assessment of ascending behavior of individual fishes. They discussed about some numerical issues encountered in solving the HJB equation, governing the stochastic control problem for determining the optimal ascending strategy based on a minimization principle of physiological energy consumption of the fish during migration.

This chapter discusses the optimality of rainfed agriculture using the framework of
stochastic control. The zero-reverting OU process is applied for representing drought severity, which reverts to zero during growing seasons, to establish an abstract stochastic model. The concept of first exit time is used for representing a drought level which terminates growth of crops. The model parameters are identified from observed sequences of alternating wet and dry spells at a site located in the coastal savanna agro-ecological zone of Ghana. Benefit functions and water stress coefficients for different annual crops are considered for evaluating farm products when drought occurs. The set of cost functions for potential implementation of irrigation is determined in an inverse problem approach where the HJB equation assuming the optimality of rainfed agriculture is solved. Numerical solutions to the inverse problems finally verified that rainfed agriculture is optimal as common cost functions are significantly increasing with respect to irrigation effort. This result forms a striking contrast with the earlier work by Unami et al. (2013), where an optimal demand-side water management strategy is deduced for a micro-dam irrigation scheme under the condition of deterministic water depletion.

3.2 Methodology

3.2.1 Formulation of Stochastic Control Problem

The major risk of rainfed agriculture is the occurrence of drought that terminates growth of a crop. A farmer practicing rainfed agriculture would perceive it if such a terminating drought occurs during a growth period, by monitoring a variety of indicators on the farm. For example, the concept of permanent wilting point in soil science represents the status of soil moisture terminating growth of the crop. The time when growth of the crop is terminated is considered as a stopping time in terms of stochastic calculus. Irrigation, as the antithesis of rainfed agriculture, is also taken into account to formulate a stochastic control problem whose solution involves optimality of rainfed agriculture.

Let the time domain of a growth period be \((0,T)\) and the time of terminating drought be \(\hat{T}\). Assume that there is a real-valued one-dimensional stochastic process \(X_t\), whose value represents drought severity. The domain of wet state, or tolerable drought, is prescribed
as $\Omega^\ast = (-\infty, K)$ with a constant positive parameter $K$. The time $\hat{T}$ of terminating drought is considered as the first exit time of $X$, from $\Omega^\ast$ such that

$$\hat{T} = \min \left\{ \inf \{ t \mid t > 0, X \not\in \Omega^\ast \}, T \right\}.$$  \hfill (3.1)

The benefit which can be gained from the farm is a function of the time $t = \hat{T}$ when growth of the crop is terminated, and referred to as the benefit function $g(t)$. However, water stress negatively affecting the benefit should be allowed for at a rate of $\varphi(t, x)(dg/dt)$, where $\varphi(t, x)$ is a water stress coefficient satisfying $\varphi(t, -\infty) = 0$, $\varphi(t, K) = 1$, $0 \leq \varphi(t, x) \leq 1$, and $0 \leq \partial \varphi / \partial x$. While, before the time $t = \hat{T}$, the farmer is assumed to have choice of irrigation with an effort $u$ at a rate $f(u)$ of cost, in order to alleviate drought severity. The cost function $f(u)$ may depend on the time $t$ and the drought severity level $x$ as well. It is common in rainfed dominant rural areas that the cost of initiating irrigation is on high side, involving $f(u)$ having a discontinuity as

$$\begin{align*}
\left\{ f(0) = 0, \\
\lim_{u \to 0^+} f(u) > 0
\right\}
\end{align*}$$  \hfill (3.2)

prescribing these functions, the performance index $J^u(s, x)$ at the time $t = s$ with the drought severity $X_s = x$ is defined as

$$J^u(s, x) = E \left[ -\int_s^\hat{T} \left( f(u) + \varphi(t, x) \frac{dg}{dt} \right) dt + g(\hat{T}) \right]$$ \hfill (3.3)

where $E$ represents the expectation. The choice of $u$ is optimized to attain the maximum $J^u(s, x)$.

To model the dynamics of the drought severity $X_i$ under rainfed condition, the one-dimensional Langevin equation is assumed to govern it as

$$dX_i = -rX_idt + \sqrt{2D}dB_i$$ \hfill (3.4)

where $r$ is a constant reversion coefficient, $D$ is a constant diffusion coefficient, and $B_i$ is the standard Brownian motion (Øksendal 2007). Now, the drought severity $X_i$ is a zero-reverting OU process $X_i$, and the farm is assumed to be wet enough if $X_i \in \Omega^\ast$. This is the simplest stochastic model for drought severity reverting to the wet state, having only three parameters $K$, $r$, and $D$. The set of the parameters depends on the characteristics of local

36
climate, soil properties, and tillage practices. Alleviation of drought severity may be modelled as perturbation of the reversion coefficient \(r\) as

\[
dX_t = -b(u)rX_t dt + \sqrt{2D}dB_t
\]  

(3.5)

where \(b(u)\) is the perturbation coefficient satisfying

\[
b(u) \geq 1, \quad b(u) = 1 \quad \text{if and only if } u = 0. \quad (3.6)
\]

The function \(b(u)\) determines the effect of irrigation effort to magnification of reverting to the wet state. In most sophisticated irrigation schemes, it blows up at a finite irrigation effort \(u\) which completely controls the drought severity. While, it reaches the ceiling and even declines despite further irrigation effort if poorly organized. In formulation of an optimal control problem to maximize the performance index \(J^*(s,x)\), the irrigation effort \(u\) should be considered as the control variable constrained in the set of admissible control

\[
U = [0, \infty). \quad (3.7)
\]

It is also assumed that \(u\) is a Markov control whose choice at the time \(t\) depends on the current drought severity \(X_t\) only. In practice, the farmer is assumed to have up-to-date information on the drought severity and to make decision on irrigation accordingly. Then, the maximum \(\Phi = \Phi(s,x)\) of \(J^*(s,x)\) and the optimal control \(u^*\) attaining \(\Phi\) are governed by the HJB equation in the \(s-x\)-domain \(G = (0,T) \times \Omega^s\) as

\[
\begin{align*}
\frac{\partial \Phi}{\partial s} - b(u^*) rx \frac{\partial \Phi}{\partial x} + D \frac{\partial^2 \Phi}{\partial x^2} - f(u^*) - \varphi(s,x) \frac{dg}{ds} \\
= \sup_{u \in U} \left\{ \frac{\partial \Phi}{\partial s} - b(u) rx \frac{\partial \Phi}{\partial x} + D \frac{\partial^2 \Phi}{\partial x^2} - f(u) - \varphi(s,x) \frac{dg}{ds} \right\} = 0
\end{align*}
\]

(3.8)

with the terminal condition

\[
\Phi(T,x) = g(T) \quad (3.9)
\]

and the boundary condition

\[
\Phi(s,K) = g(s) \quad (3.10)
\]

For notational convenience, the uncontrolled advection term of (3.8) is denoted as

\[
\Psi = \Psi(s,x) = -rx \frac{\partial \Phi}{\partial x} \quad (3.11)
\]
3.2.2 Cost Functions Optimizing Rainfed Agriculture

The optimality condition of (3.8) asserts that

\[ u^* = 0, \quad \text{if } \Psi \leq 0 \]  

(3.12)

while, if \( \Psi > 0 \), \( u^* \) must be searched in \( U \) so as to minimize \(-b(u)\Psi + f(u)\). However, even though (3.12) is not the case, \( u^* = 0 \) still holds if

\[ \frac{f(u)}{b(u)-1} \geq \Psi, \quad \forall u > 0 \]  

(3.13)

when \( u^* = 0 \), the HJB equation (3.8) is identical with the parabolic partial differential equation

\[ \frac{\partial \Phi}{\partial s} - rx \frac{\partial \Phi}{\partial x} + D \frac{\partial^2 \Phi}{\partial x^2} - \varphi(s,x) \frac{dg}{ds} = 0 \]  

(3.14)

and the values of \( \Psi \) over the \( s \)-\( x \)-domain \( G \) can be calculated from the solution of \( \Phi \) to the terminal-boundary value problem consisting of (3.9), (3.10), and (3.14). Then, rainfed agriculture turns to be optimal for those given \( f(u) \) and \( b(u) \), if either (3.12) or (3.13) is verified over the \( s \)-\( x \)-domain \( G \). This presents an inverse problem to find the set \( F \) of cost functions \( f(u) \) such that rainfed agriculture is optimal, implying \( u^* = 0 \), for given \( g(t) \), \( \varphi(t,x) \), and \( b(u) \). The set \( F \) is finally given as

\[ F = \left\{ f \left| \begin{array}{ll} \text{any } f(t,x,u) & (\Psi \leq 0) \\ f(t,x,u) \geq \Psi \left( b(u)-1 \right) & (\Psi > 0) \end{array} \right\} . \]  

(3.15)

If positive \( \Psi \) approaches zero, then \( F \) approaches the set of all non-negative functions \( f(u) \).

3.3 Identification of Model Parameters

The model parameters can be estimated from observed sequences of alternating wet and dry spells.

The transition probability \( P(s,\xi,t,G) \) of \( X_t \) from time \( s \) to time \( t \) (\( t > s \)) is the probability such that \( X_s \) is in any subset \( G \) of \( \mathbb{R} \) at time \( t \) provided that \( X_s = \xi \). It is known
that the transition probability density function \( p = p(s, \xi, t, x) \) such that \( P(s, \xi, t, G) = \int_G p(s, \xi, t, x) \, dx \) is governed by the Kolmogorov forward equation (KFE)

\[
\frac{\partial p}{\partial t} = \frac{\partial \left( r x p \right)}{\partial x} + \frac{\partial^2 p}{\partial x^2}
\]

with the initial condition

\[
p(s, \xi, s, x) = \delta(x - \xi)
\]

where \( \delta \) is the Dirac’s delta. With the spatio-temporal transformation \( \tilde{t} = rt \) and \( \tilde{x} = \sqrt{r/Dx} \), (3.16) and (3.17) are normalized as

\[
\frac{\partial p}{\partial \tilde{t}} = \frac{\partial \left( \tilde{x} p \right)}{\partial \tilde{x}} + \frac{\partial^2 p}{\partial \tilde{x}^2}
\]

and

\[
p(\tilde{x}, \tilde{\xi}, \tilde{t}, \tilde{x}) = \delta(\tilde{x} - \tilde{\xi})
\]

where the tilde represents the variables after the spatio-temporal transformation. Even though the KFE is defined on the infinite domain \( \mathbb{R} \), the finite element scheme proposed by Sharifi et al. (2014) successfully computed a numerical solution to the initial value problem with satisfactory accuracy. Alternating wet and dry spells are supposed to be observed as a time series data of alteration time \( T_i \). For generic integer \( i \), the period starting from \( T_{2i} \) and ending at \( T_{2i+1} \) is a wet spell, while the period starting from \( T_{2i+1} \) and ending at \( T_{2i+2} \) is a dry spell. In the context of the model, the stochastic process \( X_t \) gets to the boundary of the domain \( \Omega^+ \) at each \( T_{2i} \) and at each \( T_{2i+1} \) from outside and inside, respectively. Under the assumption of stationarity, the transition probability \( P(0, \tilde{K}, \tilde{t}, \tilde{\Omega}^+) = \int_{-\infty}^{\tilde{K}} p(0, \tilde{K}, \tilde{t}, x) \, dx \) can be estimated from the observed time series data if each \( T_i \) is taken as the temporal origin. In particular, the ratio of the wet spells to the whole observation period approximates the stationary transition probability

\[
P(0, \tilde{K}, \infty, \tilde{\Omega}^+) = \int_{-\infty}^{\tilde{K}} p(0, \tilde{K}, \infty, x) \, dx = \int_{-\infty}^{\tilde{K}} \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{x^2}{2} \right) \, dx
\]

from which the value of the model parameter \( \tilde{K} = \sqrt{r/DK} \) is identified. The estimated transition probabilities \( P(0, \tilde{K}, \tilde{t}, \tilde{\Omega}^+) \) for different values of \( \tilde{t} \) are used for identifying the reversion coefficient \( r \) as the scaling factor for time, in comparison with the computed transition probabilities. The value of \( r \) is chosen so that the correlation between the transition
probabilities obtained by the two methods is the highest. Using these two independent methodologies, only two of the three parameters can be eventually identified. Therefore, without loss of generality, the parameter $K$ is set as unity. Then, the diffusion coefficient $D$ is computed as $r/K^2$.

### 3.4 Composition of Benefit Function and Water Stress Coefficient

To explore the suitability of different irrigation strategies including rainfed, benefits of farming under the risk of drought and water stress should be quantified in terms of crop yields and/or their economic values (Vico & Porporato 2011a). Vico & Porporato (2011b) discussed crop productivity functions which link soil moisture dynamics to crop yield, posteriorly evaluating water productivity. In the present framework of Markov control, however, the relationship of the benefit with the drought severity must not include any future information. Here, the benefit function $g(t)$ represents the virtual benefit when growth of the crop is terminated without experiencing any water stress. Knowledge of crop yields under irrigated conditions may be referred for determining $g(t)$. Without loss of generality, $g(t)$ is scaled so that $g(T) = 1$. Continuity of $g(t)$ is assumed as well. After establishing $g(t)$ for a particular crop, the water stress coefficient $\varphi(t,x)$ is searched so that $\Phi$ on the boundary of the $s\times x$-domain $G$ is consistent with data of the actual benefit under rainfed conditions. It may be assumed that $\varphi(t,x)$ is given as a power function

$$\varphi(t,x) = \varphi(x) = \exp\left(q\left(x-K\right)\right)$$

with an appropriate growth rate $q$, satisfying $\varphi(t,K) = 1$, and $\varphi(t,-\infty) = 0$.

### 3.5 Computational Methods for Solving the HJB Equation System

In order to approach the inverse problem, it is sufficient to evaluate $\Phi$ solving the HJB equation system (3.14) with (3.9), (3.10), and $u^* = 0$, asserting that rainfed agriculture is indeed optimal. Another transformation $y = \exp(x)$ results in

$$\frac{\partial \Phi}{\partial s} + \left(D - r \log y\right)y \frac{\partial \Phi}{\partial y} + Dy^2 \frac{\partial^2 \Phi}{\partial y^2} - \varphi(s, \log y) \frac{dg}{ds} = 0$$

(3.22)

with the terminal condition
\[ \Phi(T, y) = g(T) \]  
(3.23)

and the boundary conditions
\[ \Phi(s, e^K) = g(s), \quad \Phi(s, 0) = g(T) \]  
(3.24)

where the \( x \)-domain \((-\infty, K)\) is transformed into the \( y \)-domain \((0, e^K)\). The upwind finite element scheme developed by Unami et al. (2015) is versatile for advection dominant elliptic operators, and it is applied to discretizing the HJB equation (3.22) in the \( y \)-direction with the boundary conditions (3.24). The fully implicit scheme is employed for the temporal discretization. Starting from the terminal condition (3.23), values of \( \Phi \) are numerically obtained over the whole \( s- y \)-domain mapped from \( G \). Then, the central finite difference scheme is applied for approximating \( \partial \Phi / \partial y \) to calculate \( \Psi = -ry \log y \partial \Phi / \partial y \).

### 3.6 Application

This section presents application results of the stochastic process model and identification results of its parameters using observed data of the study site.

#### 3.6.1 Description of Study Site

For demonstration of the methodology presented above, the Soil and Irrigation Research Centre at Kpong (SIREC-Kpong), University of Ghana, is set as the study site. The 1,024 ha SIREC-Kpong is located at coordinates 06 07 N 00 04 E as shown in Figure 3-1, which delineates the lower Volta basin under the coastal savanna agro-ecological zone of Ghana (Campbell 2005). The annual amount of rainfall in the lower Volta basin is 800–1100 mm, and the rainfall pattern is bimodal with the major rainy season from March to July and the minor rainy season from September to November (Nyalemegbe & Osakpa 2012). Unreliability of rainfall during the minor rainy season in particular seriously limits sustainable rainfed agriculture in the lower Volta basin. The air temperature ranges between 20°C and 35°C year-round and is not a serious constraint for growth of crops. In contrast to the surrounding rainfed dominant rural area, the SIREC-Kpong is equipped with a self-reliant irrigation scheme including a micro-dam for research purposes (Kawachi et al. 2005; Unami
et al. 2005). The soils are mostly calcic Vertisols of high clay content (30-55 %), requiring proper management for high productivity (Yangyuoru et al. 2003).

Mango seedlings were planted on the 1st of August, 2003, at a planting interval of 10 m in the field immediately downstream of the micro-dam. A soil moisture sensor (ML2x theta probe) was set beside one of the mango trees (06 06 48.0 N 00 04 21.3 E) at a depth of 50 cm on the 5th of September, 2005, and it was continuously logging until the 30th of June, 2008, when it developed a technical problem. Rainfall intensity was monitored on the embankment of the micro-dam (06 06 46.6 N 00 04 27.1 E), using a tipping bucket raingauge with 0.2 mm resolution. Figure 3-2 shows the hyetograph and temporal variation of soil moisture in terms of volumetric water content. Irregularity of rainfall dominantly affected the variable soil moisture, which significantly decreased in each dry season. In the following sections, focus will be on annual crops for discussing optimality of rainfed agriculture, and wet spells are identified with the assumption that the permanent wilting point of soil in the root zones of annual crops corresponds to the monitored volumetric water content of 25 %.

The onset of the minor rainy season of the year 2005 was late at the beginning of October, but it lasted until early January of the following year without experiencing any substantial dry spell. Serious drought was observed in the minor rainy season of the year 2006. It was constantly wet enough in the year 2007, and the shade of the well grown mango tree facilitated high soil moisture.
Figure 3-1: Location of the study site (SIREC-Kpong) in the lower Volta basin covering the southeastern Ghana, West Africa.

Figure 3-2: Hyetograph (blue) and temporal variation of soil moisture (green), indicating the permanent wilting point with the red line.
3.6.2 Results of Model Parameter Identification

Focusing on annual crops cultivated under rainfed conditions, identification of the model parameters is based on time series data of soil moisture observed during two minor rainy seasons of 2005 (first year) and 2006 (second year). The wet spells are extracted as shown in Table 3-1.

According to the procedures described above, the ratio of the wet spells was calculated and then the model parameters were identified for every individual year as well as the case where both years were combined, as shown in Table 3-2. Since posteriorly perceived drought severity was low in the first year and high in the second year, the set of model parameters for both years is finally adopted.

Table 3-1: Observed wet spells at Kpong site during minor rainy seasons.

<table>
<thead>
<tr>
<th></th>
<th>$T_{2i}$ (DD/MM/YYYY hh:mm)</th>
<th>$T_{2i+1}$ (DD/MM/YYYY hh:mm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>First year</td>
<td>05/10/2005 22:00</td>
<td>04/01/2006 17:00</td>
</tr>
<tr>
<td></td>
<td>04/01/2006 22:00</td>
<td>07/01/2006 13:00</td>
</tr>
<tr>
<td></td>
<td>31/08/2006 17:00</td>
<td>02/09/2006 17:00</td>
</tr>
<tr>
<td>Second year</td>
<td>03/09/2006 17:00</td>
<td>08/11/2006 15:00</td>
</tr>
<tr>
<td></td>
<td>26/11/2006 18:00</td>
<td>30/11/2006 12:00</td>
</tr>
</tbody>
</table>

Table 3-2: Model parameters identified from observed data series for different periods.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>First year</th>
<th>Second year</th>
<th>Both years</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio of wet spells</td>
<td>0.9978</td>
<td>0.8995</td>
<td>0.9494</td>
</tr>
<tr>
<td>$\bar{K}$</td>
<td>2.844</td>
<td>1.279</td>
<td>1.639</td>
</tr>
<tr>
<td>$r$ (hour$^{-1}$)</td>
<td>263.2</td>
<td>149.2</td>
<td>179.3</td>
</tr>
<tr>
<td>$D$ (hour$^{-1}$)</td>
<td>32.54</td>
<td>91.21</td>
<td>66.75</td>
</tr>
</tbody>
</table>
3.6.3 Benefit Functions and Water Stress Coefficients for Different Crops

Maize (Zea mays) and okra (Abelmoschus esculentus) are two different major annual crops cultivated and locally consumed under the coastal savanna agro-ecological zone of Ghana. Growths of these two crops were tested in the experimental farm at the study site during the period of soil moisture measurement. The farm is 200-300 m away from the soil moisture sensor, however, the indicated drought severity and the identified model parameters are considered applicable to the farms as well. It is assumed that the benefit function \( g(t) \) for each crop is prescribed as a smooth monotone increasing function having the form

\[
g(t) = \exp\left(\frac{-\alpha}{\tan(\pi t/2T)}\right)
\]

(3.25)

where \( \alpha \) is a shape parameter. This benefit function \( g(t) \) satisfies \( g(0) = 0 \) and \( g(T) = 1 \).

An improved maize variety, Obatanpa (Badu-Apraku et al. 2006), was selected as the test crop for the first year. Rainfed and irrigated plots were prepared and respectively subdivided into four blocks for yield data collection. The fields were ridged into Broad Beds and Furrows (BBF) (Erkossa et al. 2006) in both plots, and furrow irrigation using water from the micro-dam was implemented in the irrigated plot. FAO (2013) described growth stages for maize in West African savannas: establishment for 20 days, vegetative stage for 35 days, flowering stage for 20 days, yield formation stage for 40 days, and ripening for 10 days. These growth stages were consistently observed in the test. Therefore, \( T = 125 \) days is set to determine the growth period. Considering that there will be no yield if permanent wilting occurs during the early 75 days, the value of \( \alpha \) is taken as 10 so that substantially stays at zero accordingly before increasing during the latter 50 days. The test results of grain yields, in relation to the potential yield 3,135 kg/ha estimated under supplementary irrigation conditions without special cultural practices (Yangyuoru et al. 2003), are summarized in Table 3-3. The value of \( q \) is chosen as 0.7270 so that the average of \( \Phi(0,x) \) for \( x < -K \) is equal to 0.1411, the average relative yield under the rainfed condition with a sufficiently wet initial stage.
The latter half of the minor rainy season was so dry in the second year that more critical conditions for rainfed agriculture were realized. Okra was expected to be highly tolerant to heat and drought and was selected as the test crop. Most farmers plant okra during the rainy seasons for the fruits because it is a local delicacy, though selling the fruits involves another risk of bumper harvest resulting in very poor market price. Therefore, extracting seeds for the next growing season without expecting any benefit from fruits is a more promising strategy which should be considered in the model. The growth period of okra purposing seeds extraction is set as $T = 60$ days, however, actual seeds harvesting is performed 10 days after the end of the growth period. This implies that the benefit function $g(t)$ remains zero except for the last few days of the growth period and therefore the value of $\alpha$ is set as 100. The same BBF as for maize of the first year were prepared for okra cultivation using irrigation water from the micro-dam. Seeds of okra were planted in four rows on the BBF on January 10th, 2007, when the drought severity was so high that $X > K$ in the model. The planting interval was 30 cm on each of the rows. There were two beds binding a single furrow, and each bed had two of the rows 60 cm apart. Rows No.2 and No.3 were closer to the irrigated furrow than Rows No.1 and No.4, respectively. The bed of Rows No.1 and No.2 were 3 cm
lower than that of Rows No.3 and No.4 and therefore soil moisture was higher in general. These differences were considered as different irrigation efforts. Irrigation was performed every day except Sundays and two consecutive rainy days (February 7th and 8th). The crop was monitored until maturation to extract seeds for sale as in Table 3-4. Normally, seeds of rainfed okra could not have germinated under such dry condition, for the generation of neither benefit nor loss. The value of \( q \) for okra should be larger than that for maize because water stress affects less the growth of okra, but it should not be small enough that the yield difference among rows would be significant as in Table 3-4. Therefore, different \( q \) -values 0.7270 (same as for maize), 2.000, and 5.000 are chosen for comparison.

Table 3-4: Benefit from dry seeds of irrigated okra.

<table>
<thead>
<tr>
<th></th>
<th>Seed (kg)</th>
<th>Sale (Ghana Cedis)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row No.1</td>
<td>0.37</td>
<td>3.24</td>
</tr>
<tr>
<td>Row No.2</td>
<td>0.48</td>
<td>4.19</td>
</tr>
<tr>
<td>Row No.3</td>
<td>0.33</td>
<td>2.90</td>
</tr>
<tr>
<td>Row No.4</td>
<td>0.16</td>
<td>1.37</td>
</tr>
<tr>
<td>Average</td>
<td>0.33</td>
<td>2.93</td>
</tr>
<tr>
<td>Standard deviation</td>
<td>0.13</td>
<td>1.17</td>
</tr>
<tr>
<td>Coefficient of variation</td>
<td>40 %</td>
<td></td>
</tr>
</tbody>
</table>

Figure 3-3 shows the benefit functions \( g(t) \) respectively set up for maize and okra as functions of time \( t \). The prescribed water stress coefficients \( \varphi(x) \) in the form of exponential functions over the space \( x \) are plotted in Figure 3-4.
Figure 3-3: Benefit functions for the cases of maize and okra.

Figure 3-4: Water stress coefficients prescribed in the spatial domain.

3.6.4 Solution of the Inverse Problem

Tendency for rainfed agriculture as the only practice among the majority of farmers in the coastal savanna agro-ecological zone of Ghana is rationalized through examining the set $F$, which is obtained as a solution to the inverse problem for each of these two crops.
As already discussed above, $\Psi$ bounds the increase rate of cost in relation to irrigation effort. Therefore, the solution $\Phi$ to the terminal-boundary value problem consisting of (3.9), (3.10), and (3.14) with the identified model parameters, as well as $\Psi$, is computed using the methods as in Subsection 3.5. The resulting $\Phi$ and $\Psi$ for the case of maize are plotted in Figure 3-5, while those for the cases of okra with $q$-values 0.7270, 2.000, and 5.000 are plotted in Figure 3-6, Figure 3-7, and Figure 3-8, respectively.

Figure 3-5: Computed $\Phi$ and $\Psi$ over the spatio-temporal domain for the case of maize.

Figure 3-6: Computed $\Phi$ and $\Psi$ over the spatio-temporal domain for the case of okra with $q = 0.7270$. 
Figure 3-7: Computed $\Phi$ and $\Psi$ over the spatio-temporal domain for the case of okra with $q = 2.000$.

Figure 3-8: Computed $\Phi$ and $\Psi$ over the spatio-temporal domain for the case of okra with $q = 5.000$. 
As can be seen from Figures 3-5 through 3-8, the general behavior of $\Phi$ and $\Psi$ is the same in all the cases. Monotonicity in the temporal direction is inherited from the benefit function to $\Phi$, which indeed represents the maximum expected benefit of farming. It also decreases as the drought severity is high. The overall $\Psi$ is positive in the $s-x$-domain $G$, and the set $F$ of cost functions optimizing rainfed agriculture is composed as in the second case of (3.15). While, though not trivial, it eventually takes place that $\Psi$ attains the supremum as $x$ approaches $K$. The $\Psi$-values gradually increased during most of the growth period but intensely decreased to zero at the terminal time. The computed maximum values of $\Psi$ and the times where these are attained are summarized in Table 3-5. The set $F$ for a larger $\Psi$-value is a proper subset of that for a smaller $\Psi$-value. The $\Psi$-values are generally small in the case of maize, meaning that necessity of irrigation is less. It is striking that larger $\Psi$-values are attained for larger $q$-values in the cases of okra, implying that rainfed agriculture is not optimal when the crop is more tolerant to water stress. This can be understood if it is noticed that the cost of irrigation to maintain the benefit is relatively cheaper for drought tolerant crops. However, the actual situation in the common rural areas of SSA is the prevalence of $f(u)$ as in (3.2) and bounded $b(u)$, and thus rainfed agriculture is compelled to be optimal.

<table>
<thead>
<tr>
<th>Crop</th>
<th>$q$</th>
<th>$\Psi$</th>
<th>Attained time (day)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Maize</td>
<td>0.7270</td>
<td>1.165</td>
<td>109</td>
</tr>
<tr>
<td>Okra</td>
<td>0.7270</td>
<td>1.729</td>
<td>58</td>
</tr>
<tr>
<td>Okra</td>
<td>2.000</td>
<td>3.025</td>
<td>59</td>
</tr>
<tr>
<td>Okra</td>
<td>5.000</td>
<td>4.064</td>
<td>59</td>
</tr>
</tbody>
</table>

3.7 Conclusions

In this chapter a stochastic concept has been introduced for rigorously defining the optimality of rainfed agriculture in the context of SSA. A stochastic control problem consists of a
performance index to be maximized and a model representing the dynamics of states. The performance index here allowed for the virtual cost of irrigation, loss in value of agricultural products due to water stress, and benefit when growth of the crop is terminated at different stages. The zero-reverting OU process represented the drought severity, and the constant parameter $K$ defined the drought level which terminated growth of crops with the concept of first exit time. The irrigation effort is considered as the control variable, and its effect appeared as the magnification of the reversion coefficient, reducing the drought severity. The HJB equation governed the maximum of the performance index as well as the optimal control attaining it, and the vanishing optimal control implied optimality of rainfed agriculture. Therefore, the inverse problem was formulated to find the set of cost functions such that rainfed agriculture was optimal. Application of the methodology included identification of model parameters, composition of benefit functions and water stress coefficient, and computational solution of the HJB equation. These were demonstrated with data observed at the study site located in the coastal savanna agro-ecological zone of Ghana. The two annual crops, maize and okra, were different in terms of the growth period, the benefit function, and tolerance to water stress. The solutions of the inverse problem for these different cases verified that the necessity of irrigation was less in maize, in good accordance with farmers’ perception. The results also indicated that rainfed agriculture was not optimal when the crop is more tolerant to water stress. This therefore prompts stakeholders involved in agricultural development in rainfed dominant rural areas of SSA to reconsider adaptability of irrigation technologies under the risk of drought.
CHAPTER 4

4 Design and Construction of a Hydraulic Structure for Rainwater Harvesting in Arid Environment

4.1 Introduction

Water scarcity is a major problem in arid countries, which are characterized by low erratic rainfall resulting in high risk of droughts, intra-seasonal dry spells, and frequent food insecurity. Rainwater harvesting (RWH), which is a technology where surface runoff is effectively collected during yielding rain periods, is a likely viable option to increase water productivity in water-scarce countries (Helmreich & Horn 2009). Different RWH models have been used for enhancing water supply in different arid countries including Jordan. Abdulla and Al-Shareef (2009) emphasized the importance of roof-top RWH systems for domestic water supply in Jordan. They evaluated the potential use of harvested water for potable water savings and provided some suggestions to improve harvested rainwater quality and quantity. In the concept of rainfed agriculture, Abu-Zreig et al. (2000) conducted a study in northern Jordan to investigate the effect of sand ditches on improving water harvesting efficiency. Their results indicated that sand ditches can increase the percentage of harvested rainfall and the infiltration depth of water. Gammoh (2013) evaluated a mechanized water harvesting furrow opening technique, comparing it with the deep furrow technique for the East Mediterranean arid environment.

Jordan Rift valley is considered as the food basket for Jordan. The altitude of the Jordan Rift Valley ranges from 32 m below mean sea level to the Dead Sea level which is about 430 m below mean sea level in a significant declining trend. The Shuttle Radar Topography Mission (SRTM) digital elevation data (Farr et al. 2007) are used for depicting the contours in Figure 4-1 as well as in the succeeding Figure 4-2. The Lisan Peninsula (LP) separating the North and the South basins of the Dead Sea is now expanding due to the recent decrease of the Dead Sea’s water level, which was at 390 m below sea level (mbsl) before 1960 but at 415 mbsl as of 2000 (Closson et al. 2007). Temperature in summer reaches around 40°C and rarely falls below 10°C in winter, which is several degrees warmer than the rest of the country
because of its location in the lee side of the Judean mountains with a westerly descending dry and hot wind. This region, at the southern end of the Dead Sea in particular, has the least precipitation due to its deep location below mean sea level and the accentuated lee effect of the western mountains (Tarawneh & Kadioğlu 2003). Low rainfall and its uneven distribution, high losses due to evaporation and surface runoff, and increased demand of water due to population growth are the major problems of the country (Abu-Awwad & Shatanawi 1997). In addition, high salinity is the serious concern along the coast of the Dead Sea. Therefore, a RWH system is being developed in the LP of the Dead Sea, including a novel type of hydraulic structure (31 15 41 N 35 29 37 E) to divert ephemeral flood flows of the catchment area of 112 ha into a reservoir (Unami et al. 2015). The harvested water is expected to be fresh in comparison with the groundwater in the vulnerable freshwater lens.

Runoff processes in the catchment area could be researched only through basic hydrological knowledge in the region available in the literature (AlAyyash et al. 2012). The precipitation pattern in the east bank of the Dead Sea can be inferred from the monthly data observed for 30 years at Ghor Safi, 25 km south of the RWH system, as shown in Figure 4-3. The average annual precipitation depth is 75 mm. There is substantially no precipitation during the months from June to September, and the occurrence of precipitation events in the other months is highly irregular.

The structure consists of a gutter cutting across a 16 m wide valley bottom and a conveyance channel of 60 m long to guide the water to the reservoir. The conveyance channel is equipped with a spillway part. Numerical and model experimental approaches are complimentarily employed to design and validate details of structure dimensions. The two-dimensional (2D) shallow water equations (SWEs) have been widely used to describe the shallow free surface flows such as in the structure. There have been many computational techniques to solve the 2D SWEs, using approximate Riemann solvers, flux vector splitting, finite element and finite volume methods, total variation diminishing schemes (Zhao et al. 1996; Anastasiou & Chan 1997; Unami et al. 1999; Yoshioka et al. 2014). The finite volume scheme developed by Unami et al. (2009) is used here to operate on unstructured triangular
meshes. While, hydraulic model tests were conducted at one of experimental stations in Japan (35 29 23 N 135 21 58 E) according to the hydraulic similitude for distorted models.
Figure 4-1: The Jordan Rift Valley including the Dead Sea and LP with key hydraulic structures.
Figure 4-2: The Lisan Peninsula including the RWH system.
4.2 Governing Equations and Methods

The conservative form of 2D SWEs in x-y Cartesian coordinates describes the conservation laws of mass and horizontal momentum as

\[
\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + \frac{\partial \mathbf{G}}{\partial y} = \mathbf{S}
\]  

(4.1)

where \( t \) is the time; \( \mathbf{U} \) is the state vector; \( \mathbf{F} \) and \( \mathbf{G} \) are flux vectors; and \( \mathbf{S} \) is the source term vector. The components of the vectors are as follows.

\[
\mathbf{U} = \begin{pmatrix} U_0 \\ U_1 \\ U_2 \end{pmatrix} = \begin{pmatrix} h \\ p \\ q \end{pmatrix}
\]  

(4.2)
\[ \mathbf{F} = \begin{pmatrix} F_0 \\ F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} p \\ \frac{p^2 + gh^2}{h} \\ \frac{p}{h} \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} G_0 \\ G_1 \\ G_2 \end{pmatrix} = \begin{pmatrix} q \\ \frac{pq}{h} \\ \frac{p^2 + gh^2}{h} \end{pmatrix} \] (4.3)

\[ \mathbf{S} = \begin{pmatrix} S_0 \\ S_1 \\ S_2 \end{pmatrix} = \begin{pmatrix} r_e \\ -gh \frac{\partial z_b}{\partial x} - gh n^2 \frac{p^2 + q^2}{h^{10/3}} \\ -gh \frac{\partial z_b}{\partial y} - gh n^2 \frac{p^2 + q^2}{h^{10/3}} \end{pmatrix} \] (4.4)

where \( h \) is the water depth; \( p \) and \( q \) are unit width discharges in the \( x \) and \( y \) directions, respectively; \( g \) is the acceleration due to gravity; \( n \) is the Manning’s roughness coefficient; \( r_e \) is the effective rainfall intensity; and \( z_b \) is the channel bed surface elevation.

### 4.3 Finite Volume Scheme

The numerical scheme for the 2D SWEs on unstructured triangular meshes is based on the finite volume method discretizing the equations as

\[ \frac{d}{dt} \int_{\Omega_i} \mathbf{U} d\Omega_i + \int_{\Gamma_i} (\mathbf{Fn}_x + \mathbf{Gn}_y) d\Gamma_i = \int_{\Gamma_i} \mathbf{S} d\Gamma_i \] (4.5)

where \( \Omega_i \) is the domain of the \( i \)th cell of the mesh; \( \Gamma_i \) is the boundary of \( \Omega_i \); and \( (n_x, n_y)^T \) is the outward unit normal vector on \( \Gamma_i \). Equation (4.5) can be rewritten in the form of ordinary differential equation as

\[ \frac{d\mathbf{U}_i}{dt} = - \frac{1}{A_i} \int_{\Gamma_i} (\mathbf{Fn}_x + \mathbf{Gn}_y) d\Gamma_i + \mathbf{S}_i \] (4.6)

where \( A_i \) is the area of the \( i \)th cell; \( \mathbf{U}_i \) is \( \mathbf{U} \) allocated to the \( i \)th cell; and \( \mathbf{S}_i \) is \( \mathbf{S} \) evaluated in the \( i \)th cell.

#### 4.3.1 Reconstruction of Water Surface Elevation

The water surface elevation \( \eta_i \) at the \( s \)th vertex of the \( i \)th cell is reconstructed from \( \mathbf{U}_i \). Firstly, the cell uniform \( h_0 \) is calculated by
\[ h_b = \left\{ \frac{n^2 (p_i^2 q_i^2)}{I} \right\}^{\gamma_{10}} \quad (4.7) \]

where \( I = \sqrt{\left( \frac{\partial z_h}{\partial x} \right)^2 + \left( \frac{\partial z_h}{\partial y} \right)^2} \) is the channel bed slope in the cell, and then reconstruction is performed by

\[ \eta_{is} = \omega(z_{is} + h_i) + (1 - \omega)(z_i + h_i) \quad (4.8) \]

where \( z_{is} = z_b \) at the \( s \)th vertex of the \( i \)th cell; \( z_i \) is the average of \( z_{is} \) for \( s = 0, 1, \text{and } 2 \); and \( \omega \) is the interpolation parameter set as

\[ \omega = \max \left[ \min \left( \frac{\delta z + h_0 - h_i}{\delta z}, 1 \right), 0 \right] \quad (4.9) \]

where \( \delta z = \max \left( z_{i0}, z_{i1}, z_{i2} \right) - z_i \). The water depth \( h_s \) at the \( s \)th vertex of the \( i \)th cell is given by \( \eta_{is} - z_{is} \).

### 4.3.2 Flux Splitting Scheme

A flux splitting scheme is used for the evaluation of the flux vectors at the triangular cell faces. On the cell interface \( \Gamma \), the flux vectors in the \( i \)th cell is split as

\[
F_i = \left( \begin{array}{c}
p_i \\
\frac{gh_{it}^2}{2} \\
0
\end{array} \right) + \frac{1+w}{2} \left( \begin{array}{c}
0 \\
\frac{p_i^2}{h_i} \\
\frac{p_i q_i}{h_i}
\end{array} \right) + \frac{1-w}{2} \left( \begin{array}{c}
0 \\
\frac{p_i^2}{h_i} \\
\frac{p_i q_i}{h_i}
\end{array} \right)
\quad (4.10)
\]

and

\[
G_i = \left( \begin{array}{c}
q_i \\
0 \\
\frac{gh_{it}^2}{2}
\end{array} \right) + \frac{1+w}{2} \left( \begin{array}{c}
0 \\
\frac{p_i q_i}{h_i} \\
\frac{q_i^2}{h_i}
\end{array} \right) + \frac{1-w}{2} \left( \begin{array}{c}
0 \\
\frac{p_i q_i}{h_i} \\
\frac{q_i^2}{h_i}
\end{array} \right)
\quad (4.11)
\]

where
\[ h_{ir}^2 = \frac{h_{io}^2 + h_{oj} h_{oj} + h_{oj}^2}{3}, \]  

and

\[ |S_i| = \frac{1}{2} \sum_{j=1}^{v(i)} l_{s(i,j)} \]  

where \( j_1 = \text{mod}(j + 1, 3) \); and \( j_2 = \text{mod}(j + 2, 3) \); and \( w \) is the weight determined by

\[ w = \max\left[ \min\left( \text{Fr}_i, 1 \right), -1 \right] \]

where the Froude number \( \text{Fr}_i \) in the normal direction of \( \Gamma \) is defined as

\[ \text{Fr}_i = \frac{p_i \Delta y - q_i \Delta x}{\sqrt{gh_i^2 \sqrt{\Delta x^2 + \Delta y^2}}} \]

where \((\Delta x, \Delta y)^T\) is the vector originating at the \( j_1 \)th vertex and ending at the \( j_2 \)th vertex.

### 4.3.3 Time Integration with Total Variation Diminishing

Temporal integration of (4.6) is carried out using the fourth-order Runge-Kutta method with a specified time increment \( \Delta t \). The boundary integrals appearing in (4.6) are consistently controlled at each step of the fourth-order Runge-Kutta method, so that the total variation diminishing properties are fulfilled. However, when the water depth in a cell becomes less than \( \varepsilon = 1.81 \times 10^{-5} \) m, it is replaced by \( \varepsilon \). If the revised water depth after the temporal integration of \( \Delta t \) is negative in a cell, then the unit width discharges are reset as 0 while the water depth is reset as the machine epsilon.

### 4.4 Hydraulic Model Testing and Similitude

In general, hydraulic model testing is recommended to predict phenomena occurring in the prototype. However, the hydraulic similitude deduced from the common governing equations such as the 2D SWEs exclusively determines compatibility of the flow fields between the model and the prototype. Considering homogeneity of dimensions among the terms in each equation of the 2D SWEs (4.1), the model must be scaled to the prototype according to the ratios prescribed in Table 4-1. The acceleration due to gravity \( g = 9.8 \) m/s\(^2\) and the Manning’s roughness coefficient \( n = 0.015 \) s/m\(^{1/3}\) are assumed to be identical in the model.
and the prototype, since the differences in the altitudes, the latitudes, and the construction materials slightly affect these two parameters.

Table 4-1: The ratio between prototype and model

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Scale</th>
<th>Scale in distorted model</th>
<th>Scale when $g_r = 1$ and $n_r = 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Horizontal length</td>
<td>$L_r$</td>
<td>$L_r$</td>
<td>$L_r$</td>
</tr>
<tr>
<td>Vertical length</td>
<td>$H_r$</td>
<td>$H_r$</td>
<td>$L_{r}^{3/4}$</td>
</tr>
<tr>
<td>Time</td>
<td>$t_r$</td>
<td>$g_r^{-1/2}H_r^{-1/2}L_r$</td>
<td>$L_{r}^{5/8}$</td>
</tr>
<tr>
<td>Velocity</td>
<td>$V_r$</td>
<td>$g_r^{1/2}H_r^{1/2}$</td>
<td>$L_{r}^{3/8}$</td>
</tr>
<tr>
<td>Discharge</td>
<td>$Q_r$</td>
<td>$g_r^{1/2}H_r^{3/2}L_r$</td>
<td>$L_{r}^{17/8}$</td>
</tr>
<tr>
<td>Acceleration due to gravity</td>
<td>$g_r$</td>
<td>$g_r$</td>
<td>1</td>
</tr>
<tr>
<td>Manning’s roughness coefficient</td>
<td>$n_r$</td>
<td>$g_r^{-1/2}L_r^{3/2}H_r^{2/3}$</td>
<td>1</td>
</tr>
</tbody>
</table>

4.5 Design and Construction
4.5.1 Initial Numerical Design and Model Testing

The level surveys and hydrological considerations revealed that the hydraulic structure should have a width of 1.6 m and supercritical flows should be dominant. Construction at the site was initiated in early June of 2014 to determine horizontal routes of the gutter and the conveyance channel so that excavation works would be the minimum. There appeared a 67 degrees bend at the downstream part of the conveyance channel, and initial numerical design was performed to decide whether the bend should be shaped as a single curve or as a continuation of straight segments. Figure 4-4 shows computational meshes for the structure.
with those two different bend shapes as well as measurement points in model testing. The height of side wall would be lower at the left bank of the bend part to function as a spillway. The outflow discharge from the spillway was treated as a boundary condition of the 2D SWE, assuming that the flow on the top of the wall was critical with the total head identical to \( \eta_{is} \) at the relevant boundary node. With a hypothetical hydrograph flowing into the gutter and with a dry initial condition, computation was executed on each mesh. The reservoir was added to the mesh during the course of computation considering the momentum balance at the downstream end of the conveyance channel. The computational results are presented in Figure 4-5 for the prescribed inflow discharge \( Q_m \), the discharge \( Q_o \) flowing into the reservoir from the downstream and of the conveyance channel as a free fall, and the discharge \( Q_s \) outflowing from the spillway. It is predicted that the discharge which is guided to the reservoir is significantly more in the case of curved bend than in that of straight bend, though the releasing ability of the spillway is almost the same. Therefore, the bend shape was decided to be curved. The heights of the side walls were determined to be higher than the computed maximum water levels.
Figure 4-4: Computational meshes for the structure with different bend shapes: curved (left panel) and straight (right panel). The black lines indicate channel bottom contours with 0.5 m interval. The red points in the left panel indicate the corresponding measurement points in model testing.

Figure 4-5: Charts of discharges computed for a hypothetical 110 minutes flood event.
In August of 2014, a distorted model of the structure with the curved bend was built in an open-air plot of the experimental station, as shown in Figure 4-6. A reservoir constructed with sand bags was attached to the upstream side of the gutter to supply water to the model. The horizontal scale was decided as \( L_r = 1/8 \) due to the constraint of available land, resulting in the vertical scale \( H_r = 1/4.757 \) under the assumption of \( g_r = n_r = 1 \). The scales for the other parameters became \( t_r = 1/3.668 \), \( V_r = 1/2.181 \), and \( Q_r = 1/83.00 \), accordingly. The model testing was conducted in early September of 2014. The upstream reservoir was filled with water which was pumped up from the sea and a freshwater reservoir beside, and then the overflowing water entered from the gutter to the conveyance channel. The water depths were measured using a point gauge at sixteen (16) points including the left side (L), the center (C), and the right side (R) points on each of five (5) cross sections S1 through S5, as well as one (1) point P0 on the gutter, as depicted in the left panel of Figure 4-4. Traveling time of tracer dye was measured along the middle reach of the conveyance channel to estimate the cross-sectional average velocity and then the value of \( n \) at steady states. The results for different discharges confirmed that \( n = 0.0154 \pm 0.0006 \ s^{1/3} \). The maximum discharge actually attained 8.93 L/s, which was equivalent to 0.741 m\(^3\)/s in the prototype. Eyewitnesses confirmed dominance of supercritical flows with minor water stagnation at the right side of the curved bend. A series of supercritical hydraulic jumps was observed in the curved bend, as indicated in Reinauer and Hager (1997), implying that the total head of the outflow over the spillway was higher than \( \eta_s \).
4.5.2 Construction

The prototype of the structure was completed in late September of 2014. The channel bottoms are made of reinforced concrete over compacted crusher run, while the side walls are of concrete blocks whose sizes have been chosen according to the design. However, minor changes were made during the construction processes due to workability constraints and insufficient overseeing. Another bend of 14 degrees at the upstream side of the conveyance channel was shaped as a curve as well, and the slope of channel became not steep enough in some sections. Therefore, the side walls were raised with additional concrete blocks in the sections where occurrence of subcritical flow was expected. Figure 4-7 shows the constructed structure including the reservoir with the capacity of 1,000 m³ (a water surface area of 500 m² by 2.0 m deep).
4.6 Performance Evaluation

4.6.1 Comparison between Experimental and Computational Results

Based on the results of model testing, the boundary condition for the spillway in the numerical scheme was revised to evaluate the total head as that in the adjacent cell instead of \( \eta_i \). Then, the revised numerical method was validated in terms of comparison with the results of the model testing. The experimental water depths and computational water depths as well as the relative error at the measurement points in the steady state for the inflow discharge of 0.741 m\(^3\)/s are shown in Table 4-2. The values of the experimental water depths have been converted to the prototype scale based on Table 4-1. Locations of the measurement points including P0 on the gutter, and the cross sections S1-S5 along the conveyance channel are depicted in the left panel of Figure 4-4, while L, C, and R refer to the left side, center, and right side of each cross section. Discrepancy can be seen at the inflow point P0 and also in cross section S5 located downstream of the spillway. Further researches are necessary for better prescription of the boundary conditions.
Table 4-2: Experimental and computational water depths.

<table>
<thead>
<tr>
<th>Point</th>
<th>Experimental water depth (m)</th>
<th>Computational water depth (m)</th>
<th>Relative error</th>
</tr>
</thead>
<tbody>
<tr>
<td>P0</td>
<td>0.298</td>
<td>0.404</td>
<td>0.26</td>
</tr>
<tr>
<td>S1-L</td>
<td>0.351</td>
<td>0.338</td>
<td>0.04</td>
</tr>
<tr>
<td>S1-C</td>
<td>0.220</td>
<td>0.256</td>
<td>0.14</td>
</tr>
<tr>
<td>S1-R</td>
<td>0.171</td>
<td>0.180</td>
<td>0.05</td>
</tr>
<tr>
<td>S2-L</td>
<td>0.175</td>
<td>0.209</td>
<td>0.16</td>
</tr>
<tr>
<td>S2-C</td>
<td>0.212</td>
<td>0.218</td>
<td>0.03</td>
</tr>
<tr>
<td>S2-R</td>
<td>0.205</td>
<td>0.208</td>
<td>0.01</td>
</tr>
<tr>
<td>S3-L</td>
<td>0.186</td>
<td>0.199</td>
<td>0.07</td>
</tr>
<tr>
<td>S3-C</td>
<td>0.207</td>
<td>0.204</td>
<td>0.01</td>
</tr>
<tr>
<td>S3-R</td>
<td>0.201</td>
<td>0.199</td>
<td>0.01</td>
</tr>
<tr>
<td>S4-L</td>
<td>0.179</td>
<td>0.192</td>
<td>0.07</td>
</tr>
<tr>
<td>S4-C</td>
<td>0.202</td>
<td>0.196</td>
<td>0.03</td>
</tr>
<tr>
<td>S4-R</td>
<td>0.165</td>
<td>0.190</td>
<td>0.13</td>
</tr>
<tr>
<td>S5-L</td>
<td>0.106</td>
<td>0.130</td>
<td>0.18</td>
</tr>
<tr>
<td>S5-C</td>
<td>0.104</td>
<td>0.136</td>
<td>0.24</td>
</tr>
<tr>
<td>S5-R</td>
<td>0.091</td>
<td>0.129</td>
<td>0.29</td>
</tr>
</tbody>
</table>

4.6.2 Measurement System of the Prototype

An SR50 Sonic Ranging Sensor (Campbell Scientific Inc.) has been installed at the downstream reach of the prototype, in order to monitor the water level in vicinity of the point 0.03 m right side from the center line at the cross section 2.45 m upstream from the downstream end of the conveyance channel. Measurement and logging of the water levels shall be done every 1 minute during every rainfall event and succeeding 24 hours. The numerical method, with another mesh revised for the actual prototype, was applied for determining the relation between the discharge $Q_o$ at the downstream end of the conveyance channel and the measured water depth $h_{SR50}$, as shown in Figure 4-8. A regression formula was obtained as

$$Q_o = -4655h_{SR50}^4 + 1967h_{SR50}^4 - 317.7h_{SR50}^3 + 31.27h_{SR50}^2 + 1.440h_{SR50}.$$  \hspace{1cm} (4.16)
4.6.3 Prediction of Flow Fields during an Extreme Event

To evaluate the performance of the structure during flood events, unsteady flows were computed on the revised mesh shown in Figure 4-9, where departure from the initial design in the channel bed elevation can be seen. The inflow hydrograph was similar as in the initial numerical design but the discharge during the first leveling-off stage was changed from 0.500 m\(^3\)/s to 0.741 m\(^3\)/s to be comparable with the model testing. From the dry initial condition, temporal evolution of flow fields was simulated with a constant time step \( \Delta t = 0.002 \) s. The reservoir was hydraulically connected to the downstream end of the conveyance channel at \( t = 2220 \) s (37 min). As shown in Figure 4-10, some zones of water stagnation had already appeared mostly inside of the curves, though the flow field was dominantly supercritical. In this and henceforth Figures, red arrows and yellow arrows for velocities represent the supercritical and the subcritical flows, respectively. Water depths and Froude numbers are indicated using colors. Overflowing discharges from different parts of the walls including the spillway are figured out. As the water level of the reservoir rose, the subcritical zone in the downstream of the conveyance channel expanded due to backwater effect mainly along the right bank where the channel bed elevation was lower. Figure 4-11 shows the flow field at \( t = 2820 \) s (47 min). Reverse flows are recognized in that subcritical zone, involving large
horizontal circulation rather than hydraulic bore traveling upstream. After the inflow discharge was raised to 1.000 m$^3$/s, huge transcritical flows occurred in the mainstream, as shown in Figure 4-12 depicting the flow field at $t = 5400$ s (90 min). Relevant lift of the side walls has been done as mentioned earlier, however, silting in the subcritical reaches is the practical problem.
Figure 4-9: Computational mesh for the actually constructed structure. The black lines indicate channel bottom contours with 0.5 m interval.
Figure 4-10: Computed flow field in the actual structure immediately before the reservoir is hydraulically connected.
Figure 4-11: Computed flow field in the actual structure with the backwater effect from the reservoir.
Figure 4-12: Computed flow field in the actual structure with high inflow discharge.

4.7 Conclusions

The RWH system developed in LP of the Dead Sea is a pilot scheme for agricultural development in Jordan Rift Valley, where aridity and salinity are the major problems. The
hydraulic structure to divert ephemeral flood flows to the reservoir is the key component of
the RWH system, and its design and construction processes have been discussed. Commonly
based on the 2D SWEs, numerical experiments and hydraulic model testing were
implemented to complement each other for better design. The finite volume scheme with
several skillful techniques was applied for numerical reproduction of flow fields occurring
in the structure. The distorted hydraulic model had 1/8 horizontal and 1/1 roughness scales
to the prototype and achieved the hydraulic similitude deduced from the 2D SWEs. After the
actual construction of the structure at the site, its performance was evaluated in terms of
prediction of flow fields during a hypothetical extreme event. The computational results
showed appropriate amount of water guided to the reservoir as well as the occurrence of
transcritical flows which may cause silting problems. Ongoing research is being done to
determine the inflow discharges, which is the runoff discharges from the catchment area. The
water level measurement with the regression formula (4.16) as well as time series data of
rainfall and other weather items will serve for analyzing the runoff processes and for
management of the whole RWH system.
CHAPTER 5

5  Optimal Operational Rules for Rainwater Harvesting Systems

5.1  Introduction

RWH technology is drastically improving agricultural productivity in arid and semi-arid regions, where irregular precipitation and prolonged dry spells are the major constraints (Pachpate et al. 2009). It also alleviates inadequate access to clean drinking water under conditions of poverty (De Moraes & Rocha 2013). Mathematical methods are expected to advance development of RWH technology in the modern world. Unami et al. (2010) modeled irregularity of rainfall intensity as well as duration of dry spells using stochastic differential equations. Unami et al. (2013) discussed applicability of the stochastic control theory to establishing operational strategies for dry season irrigation with a RWH micro-dam.

A generic RWH system for irrigation purposes includes a catchment area and a command area. Rainwater storage tanks (RWSTs) such as RWH micro-dams may be installed in order to balance the demand and supply of water. Kahinda & Taigbenu (2011) classified RWH systems as in situ RWH when the command area includes the catchment area and as ex situ RWH when the catchment area excludes the command area. The water is conveyed to and within the command area most commonly by gravity. However, under such a situation that availability of arable land is limited and the cost of lifting water is affordable, it may be a feasible option to develop a RWH system where the catchment area includes the command area.

No matter how small the scale of the RWH system, understanding hydrological processes in the catchment area is imperative for rational management. Guerra et al. (1990) analyzed the hydrological processes in terraced rice fields with RWSTs having capacities of 1,000-4,000 m$^3$ in central Luzon, Philippines. Ngigi et al. (2005) examined rainfall-runoff relationships in the catchments of RWSTs with 30-100 m$^3$ capacities to evaluate the RWH system in the Laikipia district of Kenya. Panigrahi et al. (2007) experimentally studied water balance in a rainfed rice-mustard cropping system consisting of a small RWST with a
capacity of 61 m$^3$ and a command area of 800 m$^2$. Makurira et al. (2007) analyzed water distribution in a community-driven smallholder irrigation scheme including several RWSTs whose capacities range from 200 to 1,600 m$^3$. Chang et al. (2011) proposed an optimal design scheme for roof RWH systems under mixed uncertainties stemming from irregular precipitation and water demand. Conventional hydrology considers runoff processes in a catchment area as the input-output relationship generating the runoff discharge from the precipitation (Nápoles-Rivera et al. 2013). Karczewska & Lizama (2009) presented stochastic Volterra equations applicable to such precipitation-runoff models. Unami & Kawachi (2005) identified runoff processes in a tank irrigated paddy fields area as transfer functions in the frequency domain.

An alternative novel approach is to model the output runoff exclusively, so that occurrence of recharge events filling the RWST of a RWH system is well reproduced. In this context, the Langevin equation is employed here for representing a virtual water flow index to be coupled with a water balance equation of the RWST to construct a mathematical model for the RWH system. The model parameters are identified from time series data of recharge events and dry spells. Then, the stochastic control theory is further applied to this model, to constitute a Markov control system determining the optimal irrigation strategies during a dry spell. The feed-back rule is obtained as a solution of the HJB equation, which is a second order partial differential equation with advection terms. A computational method including finite element and finite difference schemes is presented and applied to resolution of the HJB equation. Indeed, few researchers are tackling HJB equations with computational approaches. Boulbrachene & Haiour (2001) presented error estimate for finite element approximation of steady HJB equations with Dirichlet boundary conditions, and Boulbrachene & Dumont (2009) dealt with the case of Neumann boundary conditions establishing a quasi-optimal error estimate. Boulbrachene & Chentouf (2004) discussed the piecewise linear approximation of HJB equations with non-coercive operators. Kumar & Muthuraman (2004) illustrated optimal strategies for real world problems by numerically solving a class of singular stochastic control problems. They combined the finite element method for partial differential equations with a policy update procedure based on the principle of smooth pasting,
in order to iteratively solve HJB equations. A dominant advection term of the HJB equation requires fine meshes for accurate solutions.

Validity of the methodology proposed here is demonstrated in terms of operation of optimal irrigation strategies for the RWH system presented in Chapter 4. Currently, the lowland area occupying the eastern part of LP is irrigated and cultivated with different crops such as tomato, cucumber, and maize. The sources of irrigation water are local groundwater wells and modernized pipelines. Since the saline groundwater being continuation of the Dead Sea and the freshwater aquifer recharged in the eastern highlands meet at the lowland area, very limited quantity and quality of irrigation water is available from the wells. The surface water once collected at Wala Dam and Mujib Dam is conveyed to the pipelines via the Mujib Diversion Weir (MDW), which is also responsible for stable supply of industrial and urban water in the most part of the country (Jordan Valley Authority 2004). Irrigation water is distributed to different farms according to rotation schedules, and combined use of those two sources of irrigation water is commonly practiced. The farmlands in the lowland area are being expanded toward the dried up lands as well as toward the feet of hills. The formulated stochastic control problem is applied to synthesizing optimal irrigation strategies, which are represented in terms of rule curves prescribing water withdrawal limits. Rule curves for optimal operation of large irrigation dams have been well discussed in the literature. Senga (1991) defined rule curves for Japanese irrigation dams to anticipate future droughts, asserting their advantages over conventional management strategies. Moghaddasi et al. (2013) proposed rule curves for long-lead reservoir operation, considering regional optimal allocation of water among different crops and irrigation units. Khan et al. (2012) improved conventional rule curves for the Tarbela Reservoir, Pakistan, to minimize irrigation deficits allowing for sediment evacuation. Eum et al. (2012) developed an integrated reservoir management system to adapt operational strategies to changing climate conditions, generating the optimal reservoir rule curves for historic, dry, and wet climate scenarios. Bashiri-Atrabi et al. (2015) used the harmony search algorithm for reservoir operation optimization to minimize the water supply deficit and flood damages downstream of a reservoir. Genetic Algorithms (GAs) have been applied for determining rule curves set at
different reliability levels (Jothiprakash & Shanthi 2006), based on floods and water shortages scenarios (Suiadee & Tingsanchali 2007), or constrained with a penalty strategy (Ngoc et al. 2014). However, there is no attempt dealing with a RWH system in the framework of stochastic calculus so far. It is demonstrated that the computational results generating the rule curves as a key tool strongly support the decision making process in practical irrigation water management in the RWH system.

5.2 A Primitive Optimal Control Problem for an Irrigation Tank

In order to outline mathematical issues intrinsic to dynamic programming, a simple deterministic problem is firstly considered in this section.

The concept of viscosity solutions is a clue to understanding nonlinear partial differential equations such as HJB equations (Crandall & Lions 1992). Botkin et al. (2011) formulated a mathematical model describing competitive ice formation inside and outside of a living cell. They developed a stable finite difference scheme to compute the value function as a viscosity solution to the corresponding HJB equation, and then designed the optimal controls to produce cooling protocols in the model. Emami-Mehrgani et al. (2016) derived an optimal policy for manufacturing systems to justify maintenance activities and optimize production rate using a set of HJB equations, and proved that the value function is the unique viscosity solution to the HJB equations.

Here, a primitive problem to make a decision on the intake discharge from an irrigation tank is considered, where the optimal strategies are obtained as the viscosity solution of the HJB equation. Using the comparison theorem (Fleming & Soner 2006) it is shown that a heuristic solution, which is approximated by a numerical solution, is indeed the viscosity solution.

5.2.1 Formulation of a Primitive Problem

A primitive problem for an irrigation tank without any inflow and outflow is formulated as follows.
Let the irrigation period be \((0, T]\). The storage volume \(x\) of the irrigation tank having a capacity \(V\) and the intake discharge \(u\) are governed by
\[
\frac{dx}{dt} = -u, \quad u = 0 \quad \text{if} \quad x = 0 \quad (5.1)
\]
where \(t\) is the time. The optimal control strategy for the intake discharge \(u\) maximizes the performance index
\[
J^u = J^u(s, x) = \int_s^T \left[-|u - q|\right] dt \quad (5.2)
\]
where \(s\) is the current time, and \(q > 0\) is the water demand which is equal to the target intake discharge. The optimal intake discharge \(u^*\) achieves the supremum \(\Phi(s, x)\) of \(J^u(s, x)\) as
\[
\Phi(s, x) = J^u(s, x) \geq J^u(s, x) \quad (5.3)
\]
for any admissible \(u\). This \(\Phi = \Phi(s, x)\) is referred to as the value function.

### 5.2.2 HJB Equation

The HJB equation governing \(\Phi(s, x)\) and \(u^*\) in \((0, V)\) is
\[
\frac{\partial \Phi}{\partial s} - u^* \frac{\partial \Phi}{\partial x} - |u^* - q| = \sup_{u \in U} \left\{ \frac{\partial \Phi}{\partial s} - u \frac{\partial \Phi}{\partial x} - |u - q| \right\} = 0 \quad (5.4)
\]
where \(U\) is the admissible set of \(u\). Here, \(U\) is set as
\[
U = [0, \infty). \quad (5.5)
\]
The terminal condition
\[
\Phi(T, x) = 0 \quad (5.6)
\]
and the boundary condition
\[
\Phi(s, 0) = -(T - s) q \quad (5.7)
\]
are imposed as well.

### 5.2.3 Heuristic and Numerical Solutions

A continuous solution \(\Phi(s, x)\) to the HJB equation (5.4) with (5.6) and (5.7), as well as a resulting class of \(u^*\), is heuristically obtained as
\begin{equation}
\begin{cases}
\Phi = 0 & \text{if } x \geq q(T-s) \\
u^* = q^* & \\
\Phi = x - q(T-s) & \text{if } x < q(T-s) \\
0 \leq u^* \leq q & 
\end{cases}
\tag{5.8}
\end{equation}

which satisfies
\begin{equation}
\frac{\partial \Phi}{\partial s} - q \frac{\partial \Phi}{\partial x} = 0 .
\tag{5.9}
\end{equation}

The optimal intake discharge $u^*$ is arbitrary when the water stored in the irrigation tank is in short. A finite difference scheme is developed to compute (5.9) with (5.6) and (5.7). For discretization in the $x$ direction, the first-order upwind finite difference scheme is used. Then, the fully implicit finite difference scheme is employed for backward discretization in the $s$ direction. The numerical solution well approximates the heuristic solution (5.8), as shown in Figure 5-1 where $q = 1$.

![Figure 5-1: Numerical solution for the HJB equation (5.4).](image)

### 5.2.4 Viscosity Solution

Let $Q$ be $(0,T] \times (0,V)$, and $L$ denotes the line defined by $x = q(T-s)$. Not being differentiable on $L$, (5.8) is not a solution of (5.9) in $Q$ in the classical sense. A weak formulation using the concept of viscosity solutions is widely applied to the degenerate HJB
equations appearing in dynamics programming problems, in order to guarantee uniqueness of continuous solutions. Indeed, (5.8) is a viscosity solution of (5.9) in \( Q \). To show this, it is proved that (5.8) is a viscosity sub-solution and also is a viscosity super-solution. Being a viscosity sub-solution implies that

\[
\frac{\partial w}{\partial s} - q \frac{\partial w}{\partial x} \leq 0 \quad \text{at } (\bar{s}, \bar{x})
\]

(5.10)

for any continuously differentiable test function \( w \) defined over \( Q \) such that

\[
w \geq \Phi \quad \text{in } Q, \quad w = \Phi \quad \text{at } (\bar{s}, \bar{x}).
\]

(5.11)

At points \((\bar{s}, \bar{x})\) not on \( L \), the partial derivatives of \( \Phi \) and those of \( w \) accord if \( \bar{s} < T \), and \( \partial \Phi / \partial s \geq \partial w / \partial s \) if \( \bar{s} = T \). While, the directional derivative of \( w \) along \( L \) must satisfy

\[
\frac{\partial w}{\partial s} - q \frac{\partial w}{\partial x} = 0
\]

(5.12)

which attains (5.10). Being a viscosity super-solution implies that

\[
\frac{\partial w}{\partial s} - q \frac{\partial w}{\partial x} \geq 0 \quad \text{at } (\bar{s}, \bar{x})
\]

(5.13)

for any continuously differentiable test function \( w \) defined over \( Q \) such that

\[
w \leq \Phi \quad \text{in } Q, \quad w = \Phi \quad \text{at } (\bar{s}, \bar{x}).
\]

(5.14)

At points \((\bar{s}, \bar{x})\) not on \( L \), the partial derivatives of \( \Phi \) and those of \( w \) accord if \( \bar{s} < T \), and \( \partial \Phi / \partial s \leq \partial w / \partial s \) if \( \bar{s} = T \). While, there is no continuously differentiable \( w \) defined over \( Q \) such that (5.14) on \( L \). This can be easily shown with proof by contradiction. Therefore, \( \Phi \) is a viscosity super-solution.

Uniqueness of the viscosity solution follows from the comparison theorem, whose assumptions are satisfied because \( \Phi \) complies with the uniformly continuous terminal and boundary conditions, the domain \( Q \) is bounded, and the Hamiltonian, the second term of (5.9), is a continuous function of \( \partial \Phi / \partial x \).

### 5.3 Mathematical Model and Control System for the Constructed RWH System

A mathematical model for controlling the constructed RWH system is developed in this section. A stochastic control problem is formulated to deal with the RWHT which may dry up or spill over during an irrigation period having a fixed terminal time. Operation of the system, which is dry season irrigation, is assumed to continue while there is water in the
RWHT and while the soil moisture was between the wilting point and saturation. The identification procedure for the values of model parameters is as described in Chapter 2 and then Chapter 3, which is applied to the actual data processed as prescribed in Chapter 4. Computational methods are proposed to numerically solve the partial differential equations appearing in the methodology.

### 5.3.1 Model Description

The dynamics of the storage volume $X_t$ of the reservoir at time $t$ is governed by the water balance equation

$$dX_t = (Q_{in} - Q_{out} - u) dt$$

where $Q_{in}$ is the inflow discharge, $Q_{out}$ is the outflow discharge due to evaporation, seepage, and overflow from spillway, and $u$ is the intake discharge as a control variable. A virtual variable $Y_t$ referred to as the water flow index is considered to model the dynamics of the inflow discharge $Q_{in}$ as well as the evaporation rate. The one-dimensional Langevin equation is assumed to govern $Y_t$ as

$$dY_t = -rY_t dt + \sqrt{2D} dB_t$$

where $r$ is a reversion coefficient, $D$ is a diffusion coefficient, and $B_t$ is the standard Brownian motion (Øksendal 2007). A monotone function is assumed to define the relationship between $Y_t$ and $Q_{in}$, while $X_t$ as well as $Y_t$ determine the uncontrollable outflow discharge. $\varphi^+(y)$, where $y$ is the virtual water flow index as an independent variable, is assumed to define the relationship between $Y_t$ and $Q_{in}$ as $Q_{in} = \varphi^+(Y_t)$. A common example for $\varphi^+(y)$ is

$$\varphi^+(y) = 0 \lor C (y - K) \land Q_{\text{max}}$$

where $\lor$ denotes the maximum, $\land$ denotes the minimum, $K$ is a threshold for inflow events, $C$ is a positive constant, and $Q_{\text{max}}$ is the maximum probable inflow discharge determined by the hydraulic structure to introduce runoff water from the catchment basin to the reservoir. While, $X_t$ as well as $Y_t$ determine the outflow discharge due to evaporation and seepage with another function $\varphi^-(x, y)$. The storage volume $X_t$ of the water harvesting
reservoir is assumed not to exceed its capacity $V$ almost surely, because of the wellfunctioning spillway. Therefore, (5.15) is rewritten as
\[ \frac{dX}{dt} = a(X, Y, u) dt \] (5.18)
with
\[ a(x, y, u) = \begin{cases} 
0 \land (\varphi^+ (y) - \varphi^- (x, y) - u) & \text{if } x = V \\
\varphi^+ (y) - \varphi^- (x, y) - u & \text{if } 0 < x < V \\
0 \lor (\varphi^+ (y) - \varphi^- (x, y) - u) & \text{if } x = 0 
\end{cases} \] (5.19)
Assuming that the maximum capacity of the intake facility such as a pump is set as the target discharge $Q_{trg}$, the admissible set $U$ is prescribed as
\[ U = \begin{cases} 
[0, Q_{trg}] & \text{if } X_t > 0 \\
[0, (0 \lor (\varphi^+ - \varphi^-))] & \text{if } X_t = 0 
\end{cases} \] (5.20)
and the abbreviation
\[ u_{\text{max}} = \sup_{u \in U} u \] (5.21)
shall be used.

5.3.2 Performance Index and HJB Equation System
Let the irrigation period $(0, T]$. The storage volume of the reservoir is assumed not to exceed $V$ almost surely. The performance index $J^u(s, x, y)$ at the time $t = s$ with the storage volume $X_s = x$ and the water flow index $Y_s = y$ is defined as
\[ J^u(s, x, y) = \mathbb{E}^{s, x, y} \left[ -\int_s^T f(t, u, y) dt \right] \] (5.22)
where $\mathbb{E}^{s, x, y}$ represents the expectation with respect to the probability law of the stochastic processes starting at the point $(x, y)$, and $f$ is the penalty function. The choice of $u$ is optimized to attain the maximum $J^u(s, x, y)$. The intake discharge $u$ is constrained in the set of admissible control $U$. It is also assumed that $u$ is a Markov control whose choice at the time $t$ depends on the current $X_t$ and $Y_t$ only. The maximum $\Phi = \Phi(s, x, y)$ of $J^u(s, x, y)$ is referred to as the value function. Then, the HJB equation
\[ \frac{\partial \Phi}{\partial s} + a(x, y, u^*) \frac{\partial \Phi}{\partial x} - ry \frac{\partial \Phi}{\partial y} + D \frac{\partial^2 \Phi}{\partial y^2} = f(s, u^*, y) \]

\[ = \sup_{u \in U} \left\{ \frac{\partial \Phi}{\partial s} + a(x, y, u) \frac{\partial \Phi}{\partial x} - ry \frac{\partial \Phi}{\partial y} + D \frac{\partial^2 \Phi}{\partial y^2} - f(s, u, y) \right\} = 0 \]  

(5.23)

governs the value function \( \Phi \) and the optimal control \( u^* \) attaining \( \Phi \) in the \( s \)-x-y-domain \( G = [0, T] \times [0, V] \times \mathbb{R} \). The value function \( \Phi \) should be understood as a viscosity solution of the HJB equation (5.23) with the terminal condition

\[ \Phi(T, x, y) = 0 \]  

(5.24)

No boundary condition is imposed in the \( x \)-direction, because of the special treatment specified in (5.19).

5.3.3 Computational Method for the HJB Equation

The HJB equation (5.23) with (5.24) is numerically solved to obtain \( \Phi \) and \( u^* \), which defines the optimal irrigation strategies. Transformation of the independent variable \( y \) to \( z \) with \( z = \tan^{-1} \left( \frac{r}{\sqrt{2D}y} \right) \) makes the computational domain bounded. Indeed, \( \mathbb{R} \) is mapped to \((-\pi/2, \pi/2)\). The HJB equation is a degenerate elliptic partial differential equation

\[ \frac{\partial \Phi}{\partial s} + \Delta Q \frac{\partial \Phi}{\partial x} - r \sin z \cos z \left( 1 + \cos^2 z \right) \frac{\partial \Phi}{\partial z} + \frac{r}{2} \cos^4 z \frac{\partial^2 \Phi}{\partial z^2} - \psi(u^*) = 0 \]  

(5.25)

with the Neumann boundary condition

\[ \frac{\partial \Phi}{\partial z} = 0 \]  

(5.26)

at \( z = \pm \frac{\pi}{2} \). The weak form of (5.25) in the \( z \) direction with (5.26) is written as
\[
\int_{-\pi/2}^{\pi/2} w \left( \frac{\partial \Phi}{\partial s} + \Delta Q \frac{\partial \Phi}{\partial x} - \bar{p}(z) \frac{\partial \Phi}{\partial z} + \frac{r}{2} \cos^4 z \frac{\partial^2 \Phi}{\partial z^2} - \psi(u^*) \right) dz \\
= \frac{d}{ds} \int_{-\pi/2}^{\pi/2} w \Phi dz \\
+ \int_{-\pi/2}^{\pi/2} \left( -rw \bar{p}(z) \frac{\partial \Phi}{\partial z} - \frac{r}{2} \left( \frac{\partial w}{\partial z} \cos^4 z - 4w \sin z \cos^3 z \right) \frac{\partial \Phi}{\partial z} \right) dz \\
+ \left[ \frac{w}{2} r \cos^4 z \frac{\partial \Phi}{\partial z} \right]_{-\pi/2}^{\pi/2} + \int_{-\pi/2}^{\pi/2} w \left( \Delta Q \frac{\partial \Phi}{\partial x} - \psi(u^*) \right) dz \\
= \frac{d}{ds} \int_{-\pi/2}^{\pi/2} w \Phi dz - \int_{-\pi/2}^{\pi/2} \left( rw \sin^3 z \cos z + \frac{r}{2} \cos^4 z \frac{\partial w}{\partial z} \right) \frac{\partial \Phi}{\partial z} dz \\
+ \int_{-\pi/2}^{\pi/2} w \left( \Delta Q \frac{\partial \Phi}{\partial x} - \psi(u^*) \right) dz \\
= 0
\]

where \( \bar{p}(z) = \sin z \cos z \left( 1 + \cos^2 z \right) \), and \( w \) is any weight in \( H^1((-\pi/2, \pi/2)) \). The \( z \)-domain \((-\pi/2, \pi/2)\) is divided into \( n_z \) sub-domains of equal length \( \Delta z = \pi/n_z \). The \( x \)-domain \([0,V]\) is also divided into \( n_x \) sub-domains of equal length \( \Delta x = V/n_x \), and the unknown \( \Phi \) is attributed to each node \( \{ x = i\Delta x, z = k\Delta z \} \) as \( \Phi_{i,k} \). The finite element scheme with the upwind scheme proposed by Unami et al. (2010) is applied to (5.27). A weight \( w_k \) to be substituted into \( w \) of (5.27) is associated with the generic \( k \)th node in the \( z \)-domain as

\[
w_k = \begin{cases} 
\left( \frac{z - z_{k-1}}{\Delta z} \right)^{\text{exp}(Pe_l)} & \left( z_{k-1} < z \leq z_k \right) \\
\left( \frac{z_{k+1} - z}{\Delta z} \right)^{\text{exp}(Pe_r)} & \left( z_k < z \leq z_{k+1} \right) \\
0 & \text{(Otherwise)}
\end{cases}
\]

where \( Pe_l \) and \( Pe_r \) are local Peclet numbers defined in the left and the right elements of the \( k \)th node, respectively. This setting of weights achieves very strong upwinding when \(|z|\) is large and stabilizes computation with minimum numerical dissipation and satisfactorily handles the degenerating coefficients. For discretization in the \( x \) direction, the first-order upwind finite difference scheme is used. Then, the resulting system of ordinary differential equations is numerically solved in the \( s \) direction using the Runge-Kutta method to update the value of each \( \Phi_{i,k} \).
5.3.4 Computed Rule Curves

The optimal control $u^*$ at any point in $G$ is obtained as

$$u^* = \arg \min_{u \in U} \psi(u)$$

(5.29)

where $\psi$ is the characteristic function defined by

$$\psi(u) = \psi(u; t, x, y, \Phi) = u \frac{\partial \Phi}{\partial x} + f(t, u, y).$$

(5.30)

Then, the HJB equation is rewritten as

$$\frac{\partial \Phi}{\partial s} + \Delta Q \frac{\partial \Phi}{\partial x} - ry \frac{\partial \Phi}{\partial y} + D \frac{\partial^2 \Phi}{\partial y^2} - \psi(u^*) = 0$$

(5.31)

where

$$\Delta Q = \begin{cases} 
    u^* \wedge \Phi^+ - \Phi^- & \text{if } x = V \\
    \Phi^- - \Phi^- & \text{if } 0 < x < V \\
    u^* \vee \Phi^+ - \Phi^- & \text{if } x = 0
\end{cases}$$

(5.32)

A typical penalty function is of

$$f(t, u, y) = \begin{cases} 
    \gamma(y) & \text{if } Q_{\text{ug}}(t) > 0 \text{ and } Q_{\text{ug}}(t) \neq u \\
    0 & \text{if } Q_{\text{ug}}(t) = 0 \text{ or } Q_{\text{ug}}(t) = u
\end{cases}$$

(5.33)

where $\gamma$ is a positive weight depending on the water flow index $y$, and $Q_{\text{ug}} \in U$ is a positive target discharge as a function of the time $t$. When $Q_{\text{ug}} \leq u_{\text{max}}$

$$u^* = \begin{cases} 
    Q_{\text{ug}} & \text{if } Q_{\text{ug}} \frac{\partial \Phi}{\partial x} < \gamma \\
    0 & \text{if } \gamma < Q_{\text{ug}} \frac{\partial \Phi}{\partial x}
\end{cases}$$

(5.34)

and

$$\psi = \gamma \wedge Q_{\text{ug}} \frac{\partial \Phi}{\partial x},$$

(5.35)

otherwise

$$u^* = \begin{cases} 
    u_{\text{max}} & \text{if } \frac{\partial \Phi}{\partial x} < 0 \\
    0 & \text{if } \frac{\partial \Phi}{\partial x} > 0
\end{cases}, \quad 0 \leq u^* \leq u_{\text{max}}$$

(5.36)

and

$$0 \leq u^* \leq u_{\text{max}}$$

(5.37)
and
\[ \psi = \gamma \wedge \left( u_{\text{max}} \frac{\partial \Phi}{\partial x} + \gamma \right). \]  

(5.37)

where \( \gamma = 0 \), and \( Q_{\text{ng}} = 0 \) is set for non-irrigation hours, and
\[ \gamma = \frac{1}{1 + \exp(y \pm K)} \]  

(5.38)

and \( Q_{\text{ng}} = 1 \) for irrigating hours. The HJB equation (5.31) with (5.35) and (5.37) is reduced to either
\[ \frac{\partial \Phi}{\partial s} + \Delta Q \frac{\partial \Phi}{\partial x} - ry \frac{\partial \Phi}{\partial y} + D \frac{\partial^2 \Phi}{\partial y^2} = \gamma \]  

(5.39)

or
\[ \frac{\partial \Phi}{\partial s} + (\Delta Q - Q_{\text{ng}}) \frac{\partial \Phi}{\partial x} - ry \frac{\partial \Phi}{\partial y} + D \frac{\partial^2 \Phi}{\partial y^2} = 0. \]  

(5.40)

for irrigation hours. However, (5.39) and (5.40) are identical during non-irrigation hours as
\[ \frac{\partial \Phi}{\partial s} + \Delta Q \frac{\partial \Phi}{\partial x} - ry \frac{\partial \Phi}{\partial y} + D \frac{\partial^2 \Phi}{\partial y^2} = 0. \]  

(5.41)

Optimal operational rules of the RWH system is derived to be applied for the optimal irrigation strategy using the concept of optimal control. Field water requirement under the drip irrigation system commonly practiced in the region is 3 mm/day (Vallentin 2003). A water harvesting system (Sharifi et al. 2015) is designed and constructed in Jordan which is introduced in Chapter 4. Using the observed data of the study site model parameters are identified and are summarized in Table 5-1.
Table 5-1: Model parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>(K)</td>
<td>2.2446</td>
</tr>
<tr>
<td>(D)</td>
<td>1.0000</td>
</tr>
<tr>
<td>(r)</td>
<td>0.0012651 (1/min)</td>
</tr>
<tr>
<td>(V)</td>
<td>1,000.0 (m^3)</td>
</tr>
<tr>
<td>(T)</td>
<td>241,920 (min)</td>
</tr>
<tr>
<td>(C)</td>
<td>7.07 (in winter)</td>
</tr>
</tbody>
</table>

Duration of irrigation period \(T\) is 168 days, starting with 126 days of winter and will continue 42 days in summer. It is assumed that the irrigation is started in winter and will be continued in summer. There is no inflow to the RWH system during the summer, while the water demand is increasing. Field water requirement is assumed as 3 mm/day for drip tube. The initial irrigation schedule applied two times per week, every Sunday and Wednesday, 6:00 am to 8:00 am with the target discharge \(Q_{tg}\) of 120 m^3/day. The outflow discharge from the reservoir \(\varphi(x, y)\) is mainly due to evaporation. Water consumption index (WCI) and \(\Delta Q\) (5.32) as the value of \(y\) is plotted in Figure 5-2 for summer and winter. \(\Delta Q\) in summer when there is no inflow to the reservoir is assumed as evaporation from the reservoir surface of 500 m^2 during the non-irrigation times, while in winter during the rainfall events there is flood inflow to the reservoir. Maximum probable inflow discharge to the reservoir from catchment area is 60 m^3/min, and is the result of studying the behavior of the hydraulic structure. Using the computational method described in subsection 5.3.3, HJB equation is numerically solved.
Firstly, the scheme is validated for different $n_x$ dividing the $x$ domain to $(0,V_{\text{max}})$, different $n_y$ and number of time steps per minute. Numerical oscillation is reduced with fine meshing. The computational results presented here are for $n_x = 120$ dividing the $x$ domain to $(0,V_{\text{max}})$, $n_y = 120$, and $\Delta t = 1/120$ min. In the numerical solution to the HJB equation during irrigation hours, the boundary between the two zones of (5.39) and (5.40) appears as the threshold indicating the storage volume below which no irrigation water should be withdrawn. The threshold constituting a surface is referred to as a rule curve. The rule curves are plotted in Figures 5-3 through 5-29. Figures 5-3 through 5-21 show the rule curves during the winter with recharge event to the reservoir and less water demand. While during the summer the threshold is higher due to the high water demand and no recharge event to the reservoir, these result are depicted in Figures 5-22 through 5-29. At the end of the irrigation season by approaching the terminal time the threshold is decreasing.
Figure 5-3: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 4

Figure 5-4: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 8
Figure 5-5: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 11

Figure 5-6: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 29
Figure 5-7: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 32

Figure 5-8: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 36
Figure 5-9: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 39

Figure 5-10: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 57
Figure 5-11: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 60

Figure 5-12: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 64
Figure 5-13: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 67

Figure 5-14: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 85
Figure 5-15: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 88

Figure 5-16: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 92
Figure 5-17: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 95

Figure 5-18: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 113
Figure 5-19: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 116

Figure 5-20: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 120
Figure 5-21: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 123

Figure 5-22: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 127
Figure 5-23: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 130

Figure 5-24: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 134
Figure 5-25: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 137

Figure 5-26: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 155
Figure 5-27: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 158

Figure 5-28: Numerical solution to the HJB equation: rule curves for optimal water intake strategy during winter, day 162
5.4 Conclusions

In this chapter at first a primitive problem to make a decision on the intake discharge from an irrigation tank is considered, where the optimal strategies are obtained as the viscosity solution of the HJB equation. The optimal control problem considered to be primitive in application but involves the rigor of viscosity solution. The HJB equation is indeed the advection equation. Using the comparison theorem, it is shown that a heuristic solution, which is approximated by a numerical solution, is indeed the viscosity solution.

Then the developed stochastic model for RWH systems provides a base for deducing optimal irrigation strategies. Numerically solving the HJB equation, the optimal irrigation strategies are obtained in terms of rule curves. Using the observed data of the RWH system which is developed in LP of Jordan Rift Valley RWH system is optimally operated using the rule curves.
CHAPTER 6

6 Conclusions and Future Research

6.1 Conclusions

This thesis studied mathematical and numerical modeling framework for assessing hydrological events, deducing optimal control strategies, and addressing their real world applications for better water management. This thesis focused in particular on stochastic modeling and optimal control of reservoirs as well as rainfed agriculture.

In the first part of the thesis, I present a stochastic process model consisting of a stochastic differential equation governing a zero-reverting Ornstein-Uhlenbeck (OU) process to model the alternation of dry and wet spells. This is a novel approach toward scientifically understanding the subsistent rainfed agriculture. This part attempts to comprehend alternation of dry and wet spells in rainy seasons of West African savanna. The model with the parameter values identified from the observed data series is applicable to a variety of problems. I investigate the optimality of rainfed agriculture in the context of stochastic control theory considering the level of drought severity as the zero-reverting OU process. Then I introduce the HJB equation governing the optimal control to identify the set of cost functions optimizing rainfed agriculture in terms of an inverse problem. Application of the methodology included identification of model parameters, composition of benefit functions and water stress coefficient, and computational solution of the HJB equation. These were demonstrated with data observed at the study site located in the coastal savanna agro-ecological zone of Ghana. The two annual crops, maize and okra, were different in terms of the growth period, the benefit function, and tolerance to water stress. The solutions of the inverse problem for these different cases is studied in Chapter 3.

In the rest of the thesis, I introduce a RWH system in arid area Jordan. The RWH system developed in LP of the Dead Sea is a pilot scheme for agricultural development in Jordan Rift Valley, where aridity and salinity are the major problems. Firstly, I focus on hydraulic design and actual construction processes of the hydraulic structure. Then I employ numerical
and model experimental approaches complimentarily to design and validate details of structure dimensions. The two-dimensional (2D) shallow water equations (SWEs) have been used to describe the shallow free surface flows such as in the structure. After the actual construction of the structure at the site, its performance was evaluated in terms of prediction of flow fields during a hypothetical extreme event. The computational results showed appropriate amount of water guided to the reservoir as well as the occurrence of transcritical flows which may cause silting problems (Chapter 4).

The optimal control problem which is considered at the beginning of Chapter 5 is primitive in application but involves the rigor of viscosity solution. Heuristic and numerical solutions of HJB equation is presented and then it is shown that a heuristic solution, which is approximated by a numerical solution, is indeed the viscosity solution.

The Langevin equation is also employed for representing a virtual water flow index to be coupled with a water balance equation of the RWST to construct a mathematical model for the RWH system. The model parameters are identified from time series data of recharge events and dry spells. Then, the stochastic control theory is further applied to this model, to constitute a Markov control system determining the optimal irrigation strategies during a dry spell. The feed-back rule is obtained as a solution of the HJB equation, which is a second order partial differential equation with advection terms. A computational method including is presented and applied to resolution of the HJB equation. Validity of the methodology proposed at the end of Chapter 5 is demonstrated in terms of operation of optimal irrigation strategies for the RWH system in Jordan.

6.2 Suggestions for Future Research

This thesis developed a mathematical and numerical modeling framework focusing in particular on stochastic modeling and optimal control of reservoirs as well as rainfed agriculture. The proposed modeling framework in thesis has some elements that can be improved in future studies.

Applicability of the presented model could be researched for different study areas with different climates. For adaptation to different social and hydrological conditions, different
Coefficient values, source terms, and boundary conditions should be set for the HJB equation systems.

For the management of the whole RWH system, the water level measurement as well as time series data of rainfall and other weather items should be served for analyzing the runoff processes.

Another topic worth further effort is studying the viscosity solution of HJB equation on different problems under stochastic environment. A weak formulation using the concept of viscosity solutions is widely applied to the degenerate HJB equations appearing in dynamics programming problems, in order to guarantee uniqueness of continuous solutions, hence the development of numerical schemes directly constituting viscosity solutions is also an interesting topic. Here, I formulated a primitive problem for an irrigation tank without any inflow and outflow; however using the concept of viscosity solution, existence and uniqueness of solutions to the adapted HJB equation systems, as well as optimal operational rules, should be established for better management strategies.
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